Filomat 31:6 (2017), 1755–1767
DOI 10.2298/FIL1706755N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An N-order Iterative Scheme for a Nonlinear Wave Equation Containing a Nonlocal Term

Le Thi Phuong Ngoc[a], Bui Minh Tri[b], Nguyen Thanh Long[b]

[a] *University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam.*
[b] *Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University Ho Chi Minh City, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam.*

Abstract. In this paper, we consider an initial - boundary value problem for a nonlinear wave equation containing a nonlocal term. Using a high order iterative scheme, the existence of a unique weak solution is proved. Furthermore, the sequence established here converges to a unique weak solution at a rate of order N ($N \geq 2$).

1. Introduction

In this paper, we consider the following initial - boundary value problem for a nonlinear wave equation

$$u_{tt} - u_{xx} = f\left(x, t, u, \|u(t)\|^2\right),\ x \in \Omega = (0,1),\ 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and the nonlinear term $f\left(x,t,u,\|u(t)\|^2\right)$ contains a nonlocal term

$$\|u(t)\|^2 = \int_0^1 u^2(x,t)dx.$$

Eq. (1.1) constitutes a case, relatively simpler, of a more general equation, namely

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t,\|u\|^2,\|u_x\|^2)u_x\right) = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2),\ x \in \Omega = (0,1),\ 0 < t < T, \tag{1.4}$$

it has its origin in the nonlinear vibration of an elastic string (Kirchhoff [5]), for which the associated equation is

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right)u_{xx},$$

2010 *Mathematics Subject Classification*. 35L20; 35L70; 35Q72
Keywords. Nonlinear wave equation containing a nonlocal term, Faedo - Galerkin method, the convergence of order N
Received: 13 April 2015; Accepted: 01 October 2015
Communicated by Naseer Shahzad
Email addresses: ngoc1966@gmail.com (Le Thi Phuong Ngoc), tri.hcmuns@gmail.com (Bui Minh Tri), longnt2@gmail.com (Nguyen Thanh Long)

here u is the lateral deflection, ρ is the mass density, h is the cross section, L is the length, E is Young's modulus and P_0 is the initial axial tension. In [2], Carrier also established a model of the type

$$u_{tt} = \left(P_0 + P_1 \int_0^L u^2(y,t)dy\right) u_{xx},$$

where P_0 and P_1 are constants.

In [11], Medeiros has studied Eq. (1.4) with $f = f(u) = -bu^2$, where b is a given positive constant, and Ω is a bounded open set of $\mathbb{R}^3$. In [4], Hosoya and Yamada also have considered Eq. (1.4) with $f = f(u) = -\delta |u|^\alpha u$, where $\delta > 0$, $\alpha \geq 0$ are given constants.

In [3], Ficken and Fleishman established the unique global existence and stability of solutions for the equation

$$u_{xx} - u_{tt} - 2\alpha u_t - \beta u = \varepsilon u^3 + \gamma, \ \varepsilon > 0.$$

Rabinowitz [14] proved the existence of periodic solutions for

$$u_{xx} - u_{tt} - 2\alpha u_t = f(x,t,u,u_x,u_t),$$

where ε is a small parameter and f is periodic in time.

In [8], Long and Diem have studied the linear recursive scheme associated with the nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,u_x,u_t), \ 0 < x < 1, \ 0 < t < T,$$

associated with (1.3) and the following mixed conditions

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0,$$

where $h_0 > 0$, $h_1 \geq 0$ are given constants. Afterwards, this result has been extended in [9], [10] to the nonlinear wave equation with the Kirchhoff - Carrier operator. In [10], the following equation

$$u_{tt} - \mu(t, \|u\|^2, \|u_x\|^2) u_{xx} = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2), \ 0 < x < 1, \ 0 < t < T,$$

associated with the mixed homogeneous conditions was studied. By the linear recursive scheme and by a standard argument, existence of a local solution was proved. On the other hand, an asymptotic expansion was established.

In [12], [15], a high order iterative scheme was established in order to get a convergent sequence at a rate of order N ($N \geq 1$) to a local unique weak solution of a nonlinear Kirchhoff – Carrier wave equation as follows

$$u_{tt} - \mu(t, \|u(t)\|^2, \|u_x(t)\|^2) \frac{\partial}{\partial x}(A(x) u_x) = f(x,t,u), \ 0 < x < 1, \ 0 < t < T,$$

associated with the mixed homogeneous conditions.

Based on the above problems, we consider Prob. (1.1) – (1.3). With the assumption $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$ and some other conditions, we shall establish a high order iterative scheme in order to get a convergent sequence at a rate of order N to a local unique weak solution of Prob. (1.1) – (1.3). By the fact that, we associate with Eq. (1.1) a recurrent sequence $\{u_m\}$ defined by

$$\frac{\partial^2 u_m}{\partial t^2} - \frac{\partial^2 u_m}{\partial x^2} = \sum_{i+j \leq N-1} \frac{1}{i!j!} D_3^i D_4^j f\left(x,t,u_{m-1},\|u_{m-1}\|^2\right)(u_m - u_{m-1})^i \left(\|u_m\|^2 - \|u_{m-1}\|^2\right)^j,$$

$0 < x < 1$, $0 < t < T$, where u_m satisfying (1.2), (1.3) for all $m \geq 1$ and the first term $u_0 = 0$. This result is a relative generalization of [8] - [10], [12], [13], [15].

2. The High Order Iterative Method

First, we denote the usual function spaces used in this paper by the notations $L^p = L^p(0,1)$, $H^m = H^m(0,1)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \le p \le \infty$ for the Banach space of real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\, dt\right)^{1/p} < +\infty \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup}\,\|u(t)\|_X \ \text{ for } \ p=\infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)$, $f = f(x,t,u,z)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$, $D_4 f = \frac{\partial f}{\partial z}$ and $D^\alpha f = D_1^{\alpha_1}...D_4^{\alpha_4} f$, $\alpha = (\alpha_1,...,\alpha_4) \in \mathbb{Z}_+^4$, $|\alpha| = \alpha_1 + ... + \alpha_4 = k$, $D^{(0,0,0,0)} f = f$.
We then have the following lemma, the proof of which can be found in [1].
Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

(i) $\|v\|_{C^0([0,1])} \le \sqrt{2}\,\|v\|_{H^1}$, *for all $v \in H^1$,*

(ii) $\|v\|_{C^0([0,1])} \le \|v_x\|$, *for all $v \in H_0^1$.*

Now, we make the following assumptions:

(H_1) $\quad \tilde{u}_0 \in H^2 \cap H_0^1$ and $\tilde{u}_1 \in H_0^1$,

(H_2) $\quad f \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)$ with $f(0,t,0,z) = f(1,t,0,z) = 0$, $\forall t, z \ge 0$.

Fix $T^* > 0$. For each $M > 0$ given, we define two constants $K_0(M,f)$, $K_M(f)$ as follows

$$\begin{cases} K_0(f,M) = \sup\{\left|f(x,t,u,z)\right| : 0 \le x \le 1,\ 0 \le t \le T^*,\ |u| \le M,\ 0 \le z \le M^2\}, \\ K_M(f) = \sum_{|\alpha|\le N} K_0(D^\alpha f, M). \end{cases}$$

For every $T \in (0,T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T;H_0^1\cap H^2) : v_t \in L^\infty(0,T;H_0^1) \text{ and } v_{tt} \in L^2(Q_T), \\ \qquad\qquad \text{with } \|v\|_{L^\infty(0,T;H_0^1\cap H^2)},\ \|v_t\|_{L^\infty(0,T;H_0^1)},\ \|v_{tt}\|_{L^2(Q_T)} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}, \end{cases} \tag{2.1}$$

with $Q_T = (0,1)\times(0,T)$. We shall choose as first term $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{2.2}$$

and associate with problem (1.1) – (1.3) the following variational problem:
Find $u_m \in W_1(M,T)$ $(m \ge 1)$ so that

$$\begin{cases} \langle u_m''(t), v\rangle + \langle u_{mx}(t), v_x\rangle = \langle F_m(t), v\rangle \ \ \forall v \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{2.3}$$

where

$$F_m(x,t) = \sum_{i+j\le N-1} \tfrac{1}{i!j!} D_3^i D_4^j f[u_{m-1}](u_m - u_{m-1})^i \left(\|u_m(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j, \tag{2.4}$$

here we use the following notations $f[u] = f\left(x,t,u,\|u(t)\|^2\right)$, $D_i f[u] = D_i f\left(x,t,u,\|u(t)\|^2\right)$, $i = 1,2,3,4.$

Then, we have the following theorem.

Theorem 2.2. *Let* (H_1), (H_2) *hold. Then there exist a constant* $M > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$ *and a constant* $T > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, f *such that, for* $u_0 \equiv 0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (2.3), (2.4).

Proof. The proof consists of several steps.

Step 1: *The Faedo - Galerkin approximation* (introduced by Lions [7]).

Let us consider a special basis of H_0^1, formed by the eigenfunctions w_j of the operator $-\Delta = -\frac{\partial^2 u}{\partial x^2}$:

$$-\Delta w_j = \lambda_j^2 w_j,\ w_j \in H_0^1 \cap H^2,\ w_j(x) = \sqrt{2}\sin(j\pi x),\ \lambda_j = j\pi,\ j = 1,2,3... \tag{2.5}$$

Put

$$u_m^{(k)}(t) = \textstyle\sum_{j=1}^k c_{mj}^{(k)}(t) w_j, \tag{2.6}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \left\langle u_{mx}^{(k)}(t), w_{jx}\right\rangle = \langle F_m^{(k)}(t), w_j\rangle,\ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{2.7}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H_0^1 \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H_0^1, \end{cases} \tag{2.8}$$

$$F_m^{(k)}(x,t) = \sum_{i+j\le N-1} D^{ij} f[u_{m-1}](u_m^{(k)} - u_{m-1})^i \left(\left\|u_m^{(k)}(t)\right\|^2 - \|u_{m-1}(t)\|^2\right)^j, \tag{2.9}$$

with the notations $D^{ij} f = \frac{1}{i!j!} D_3^i D_4^j f = \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial u^i \partial z^j}$, $i + j \le N$, $D^{00} f = f$.

Let us suppose that u_{m-1} satisfies (2.2). Then we have the following lemma.

Lemma 2.3. *Let* (H_1), (H_2) *hold. For fixed* $M > 0$ *and* $T > 0$, *then, the system* (2.7) - (2.9) *has a unique solution* $u_m^{(k)}(t)$ *on an interval* $[0, T_m^{(k)}] \subset [0,T]$.

Proof of Lemma 2.3. The system of Eqs. (2.7) - (2.9) is rewritten in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j^2 c_{mi}^{(k)}(t) = \left\langle F_m^{(k)}(t), w_j\right\rangle,\ 1 \le j \le k, \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)},\ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}. \end{cases} \tag{2.10}$$

and it is equivalent to the system of integral equations

$$c_{mj}^{(k)}(t) = \alpha_j^{(k)} \cos(\lambda_j t) + \tfrac{1}{\lambda_j}\beta_j^{(k)} \sin(\lambda_j t) + \tfrac{1}{\lambda_j}\int_0^t \sin(\lambda_j(t-s))\left\langle F_m^{(k)}(s), w_j\right\rangle ds, \tag{2.11}$$

for $1 \le j \le k$. Omitting the indexs m, k, it is written as follows

$$c = L[c], \tag{2.12}$$

where $L[c] = (L_1[c], ..., L_k[c])$, $c = (c_1, ..., c_k)$,

$$\begin{cases} L_j[c](t) = q_j(t) + N_j[c](t), \\ q_j(t) = \alpha_j \cos(\lambda_j t) + \frac{1}{\lambda_j}\beta_j \sin(\lambda_j t), \\ N_j[c](t) = \frac{1}{\lambda_j}\int_0^t \sin(\lambda_j(t-s)) \left\langle F[c](s), w_j \right\rangle ds, \ 1 \leq j \leq k, \\ F[c](t) = \sum\limits_{i+j \leq N-1} D^{ij} f[u_{m-1}](u(t) - u_{m-1})^i \left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2 \right)^j, \\ u(t) = \sum_{j=1}^k c_j(t) w_j. \end{cases}$$

For every $T_m^{(k)} \in (0, T]$ and $\rho > 0$ that will be chosen later, we put $X = C^0\left([0, T_m^{(k)}]; \mathbb{R}^k\right)$, $S = \{c \in X : \|c\|_X \leq \rho\}$, where $\|c\|_X = \sup\limits_{0 \leq t \leq T_m^{(k)}} |c(t)|_1$, $|c(t)|_1 = \sum_{j=1}^k \left|c_j(t)\right|$, for each $c = (c_1, ..., c_k) \in Y$. Clearly S is a closed nonempty subset in X and we have the operator $L : X \to X$. In what follows, we shall choose $\rho > 0$ and $T_m^{(k)} > 0$ such that $L : S \to S$ is contractive.

(i) First we note that, for all $c = (c_1, ..., c_k) \in S$,

$$\|u(t)\| \leq |c(t)|_1 \leq \|c\|_X \leq \rho, \ \|u(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, |c(t)|_1 \leq \sqrt{2}\rho, \tag{2.13}$$

so

$$|N[c](t)|_1 \leq \frac{k}{\lambda_1} \int_0^t \|F(s)\|\, ds.$$

On the other hand, by

$$\begin{aligned} |F[c](x,t)| &\leq K_M(f) \sum_{i+j \leq N-1} \tfrac{1}{i!j!} |u(t) - u_{m-1}|^i \left| \|u(t)\|^2 - \|u_{m-1}(t)\|^2 \right|^j \\ &\leq K_M(f) \sum_{i+j \leq N-1} \tfrac{1}{i!j!} \left(\|u(t)\|_{C^0(\overline{\Omega})} + M \right)^i (\|u(t)\| + \|u_{m-1}(t)\|)^{2j} \\ &\leq K_M(f) \sum_{i+j \leq N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho + M \right)^i (\rho + M)^{2j} \\ &\leq K_M(f) \sum_{i+j \leq N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho + M \right)^{i+2j}, \end{aligned}$$

we have

$$\|N[c]\|_X \leq \tfrac{k}{\lambda_1} T_m^{(k)} K_M(f) \sum_{i+j \leq N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho + M \right)^{i+2j}.$$

Hence, we obtain

$$\|L[c]\|_X \leq |\alpha|_1 + \frac{1}{\lambda_1} \left|\beta\right|_1 + T_m^{(k)} \overline{D}_\rho^{(1)}(\rho, M). \tag{2.14}$$

where

$$\overline{D}_\rho^{(1)}(\rho, M) = \tfrac{k}{\lambda_1} K_M(f) \sum_{i+j \leq N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho + M \right)^{i+2j}. \tag{2.15}$$

(ii) We now prove that

$$\|L[c](t) - L[d](t)\|_X \leq \tfrac{k}{\lambda_1} T_m^{(k)} \overline{D}_\rho^{(2)}(\rho, M)\, \|c - d\|_X, \ \forall c, d \in S, \tag{2.16}$$

where

$$\overline{D}_{\rho}^{(2)}(\rho, M) = K_M(f) \sum_{1\le i+j\le N-1} \frac{1}{i!j!} \left(\sqrt{2}\rho + M\right)^{i+2j-2} \left(\sqrt{2}iM + 2(i+j)\rho\right). \tag{2.17}$$

Proof of (2.16) is as follows.
Let $c, d \in S$, put $u(t) = \sum_{j=1}^{k} c_j(t) w_j$, $u(t) = \sum_{j=1}^{k} c_j(t) w_j$.
For all $t \in [0, T_m^{(k)}]$, we have

$$|L[c](t) - L[d](t)|_1 = |N[c](t) - N[d](t)|_1 \le \frac{k}{\lambda_1} \int_0^t \|F[c](s) - F[d](s)\| \, ds. \tag{2.18}$$

On the other hand

$$\begin{aligned} &F[c](x,t) - F[d](x,t) \\ &= \sum_{1\le i+j\le N-1} D^{ij} f[u_{m-1}](u(t) - u_{m-1})^i \left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j \\ &- \sum_{1\le i+j\le N-1} D^{ij} f[u_{m-1}](v(t) - u_{m-1})^i \left(\|v(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j \\ &= \sum_{1\le i+j\le N-1} D^{ij} f[u_{m-1}] \left[(u(t) - u_{m-1})^i - (v(t) - u_{m-1})^i\right] \left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j \\ &+ \sum_{1\le i+j\le N-1} D^{ij} f[u_{m-1}](v(t) - u_{m-1})^i \left[\left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j - \left(\|v(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j\right]. \end{aligned} \tag{2.19}$$

We also note that $a^i - b^i = (a-b) \sum_{\nu=0}^{i-1} a^{\nu} b^{i-1-\nu}$ for all $a, b \in \mathbb{R}$, $i = 1, 2, \ldots$, we deduce from (2.13) that

$$\begin{aligned} \left|(u(t) - u_{m-1})^i - (v(t) - u_{m-1})^i\right| &= |u(t) - v(t)| \left|\sum_{\nu=0}^{i-1} (u(t) - u_{m-1})^{\nu} (v(t) - u_{m-1})^{i-1-\nu}\right| \\ &\le |u(t) - v(t)| \sum_{\nu=0}^{i-1} |u(t) - u_{m-1}|^{\nu} |v(t) - u_{m-1}|^{i-1-\nu} \\ &\le \sqrt{2} \|c - d\|_X \sum_{\nu=0}^{i-1} \left(\sqrt{2}\rho + M\right)^{\nu} \left(\sqrt{2}\rho + M\right)^{i-1-\nu} \\ &= \sqrt{2} i \left(\sqrt{2}\rho + M\right)^{i-1} \|c - d\|_X \,. \end{aligned} \tag{2.20}$$

Similarly

$$\begin{aligned} &\left|\left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j - \left(\|v(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j\right| \\ &\quad = \left|\|u(t)\|^2 - \|v(t)\|^2\right| \left|\sum_{\nu=0}^{j-1} \left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right)^{\nu} \left(\|v(t)\|^2 - \|u_{m-1}(t)\|^2\right)^{j-1-\nu}\right| \\ &\quad \le \left|\|u(t)\|^2 - \|v(t)\|^2\right| \sum_{\nu=0}^{j-1} \left|\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right|^{\nu} \left|\|v(t)\|^2 - \|u_{m-1}(t)\|^2\right|^{j-1-\nu} \\ &\quad \le 2\rho \|c - d\|_X \sum_{\nu=0}^{j-1} (\rho + M)^{2\nu} (\rho + M)^{2(j-1-\nu)} \\ &\quad = 2j\rho (\rho + M)^{2j-2} \|c - d\|_X \,. \end{aligned} \tag{2.21}$$

It implies that

$$
\begin{aligned}
&|F[c](x,t) - F[d](x,t)| \\
&\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \left|(u(t)-u_{m-1})^i - (v(t)-u_{m-1})^i\right| \left|\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right|^j \\
&+K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} |v(t)-u_{m-1}|^i \left|\left(\|u(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j - \left(\|v(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j\right| \\
&\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \sqrt{2}i\left(\sqrt{2}\rho+M\right)^{i-1} \|c-d\|_X (\rho+M)^{2j} \\
&+K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho+M\right)^{i} 2j\rho(\rho+M)^{2j-2} \|c-d\|_X \\
&\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \sqrt{2}i\left(\sqrt{2}\rho+M\right)^{i-1+2j} \|c-d\|_X \\
&+K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho+M\right)^{i+2j-2} 2j\rho \|c-d\|_X \\
&\le K_M(f) \|c-d\|_X \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \left(\sqrt{2}\rho+M\right)^{i+2j-2} \left(\sqrt{2}iM + 2(i+j)\rho\right) \\
&= \overline{D}_\rho^{(2)}(\rho,M)\|c-d\|_X,
\end{aligned}
\tag{2.22}
$$

where $\overline{D}_\rho^{(2)}(\rho,M)$ defined as in (2.17).

It follows from (2.18), (2.22), that (2.16) holds.

Choosing $\rho > |\alpha|_1 + \frac{1}{\lambda_1}\left|\beta\right|_1$ and $T_m^{(k)} \in (0,T]$ such that

$$0 < T_m^{(k)}\overline{D}_\rho^{(1)}(\rho,M) \le \rho - |\alpha|_1 - \tfrac{1}{\lambda_1}\left|\beta\right|_1 \text{ and } \tfrac{k}{\lambda_1} T_m^{(k)}\overline{D}_\rho^{(2)}(\rho,M) < 1. \tag{2.23}$$

Therefore, it follows from (2.14), (2.16) and (2.23) that $L : S \to S$ is contractive. We deduce that L has a unique fixed point in S, i.e., the system (2.7) – (2.9) has a unique solution $u_m^{(k)}(t)$ on an interval $[0, T_m^{(k)}]$. The proof of Lemma 2.3 is complete. □

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k.

Step 2: *A priori estimates.* Put

$$
\begin{cases}
S_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds, \\
p_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2, \\
q_m^{(k)}(t) = \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2,
\end{cases}
\tag{2.24}
$$

Then, it follows from (2.7) and (2.24) that

$$
\begin{aligned}
S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right\rangle ds + 2\int_0^t \left\langle F_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\right\rangle ds \\
&\quad + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds = S_m^{(k)}(0) + \textstyle\sum_{j=1}^3 J_j.
\end{aligned}
\tag{2.25}
$$

We shall estimate step by step all the terms J_1, J_2, J_3 and $S_m^{(k)}(0)$.

The term J_1. Using the inequalities $(a+b)^p \le 2^{p-1}(a^p + b^p)$, for all $a, b \ge 0$, $p \ge 1$ and

$$s^q \le 1 + s^p, \ \forall s \ge 0, \ \forall q \in (0,p], \tag{2.26}$$

we get from (2.9) that

$$
\begin{aligned}
\left|F_m^{(k)}(x,t)\right| &\leq K_M(f) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} \left|u_m^{(k)} - u_{m-1}\right|^i \left|\left\|u_m^{(k)}(t)\right\|^2 - \|u_{m-1}(t)\|^2\right|^j \\
&\leq K_M(f) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} \left(\left|u_m^{(k)}\right| + |u_{m-1}|\right)^i \left(\left\|u_m^{(k)}(t)\right\| + \|u_{m-1}(t)\|\right)^{2j} \\
&\leq K_M(f) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} \left(\sqrt{S_m^{(k)}(t)} + M\right)^i \left(\sqrt{S_m^{(k)}(t)} + M\right)^{2j} \\
&\leq K_M(f) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} \left(\sqrt{S_m^{(k)}(t)} + M\right)^{i+2j} \\
&\leq K_M(f) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} 2^{i+2j-1} \left[\left(\sqrt{S_m^{(k)}(t)}\right)^{i+2j} + M^{i+2j}\right] \\
&\leq K_M(f) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} 2^{i+2j-1} \left[1 + \left(S_m^{(k)}(t)\right)^{N-\frac{3}{2}} + 1 + M^{2N-3}\right] \\
&\leq K_M(f)\left(1 + M^{2N-3}\right) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} 2^{i+2j} \left[1 + \left(S_m^{(k)}(t)\right)^{N-\frac{3}{2}}\right].
\end{aligned} \tag{2.27}
$$

Hence

$$
\begin{aligned}
\left\|F_m^{(k)}(t)\right\| &\leq K_M(f)\left(1 + M^{2N-3}\right) \sum_{i+j\leq N-1} \tfrac{1}{i!j!} 2^{i+2j} \left[1 + \left(S_m^{(k)}(t)\right)^{N-\frac{3}{2}}\right] \\
&\equiv \xi_1(M)\left[1 + \left(S_m^{(k)}(t)\right)^{N-\frac{3}{2}}\right],
\end{aligned} \tag{2.28}
$$

where

$$
\xi_1(M) = K_M(f)\left(1 + M^{2N-3}\right) \sum_{i+j\leq N-1} \frac{1}{i!j!} 2^{i+2j}. \tag{2.29}
$$

Using the inequality

$$
s^q \leq 1 + s^{N_0}, \ \forall s \geq 0, \ \forall q \in (0, N_0], \ N_0 = \max\{N, 2N-3\}, \ N \geq 2, \tag{2.30}
$$

we get from (2.28), (2.30) that

$$
\begin{aligned}
J_1 &= 2\int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right\rangle ds \leq 2\int_0^t \left\|F_m^{(k)}(s)\right\| \left\|\dot{u}_m^{(k)}(s)\right\| ds \\
&\leq 2\xi_1(M) \int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-\frac{3}{2}}\right] \sqrt{S_m^{(k)}(s)} ds \\
&= 2\xi_1(M) \int_0^t \left[\sqrt{S_m^{(k)}(s)} + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds \\
&\leq 4\xi_1(M) \int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N_0}\right] ds \\
&\leq \bar{\xi}_1(M) \int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N_0}\right] ds,
\end{aligned} \tag{2.31}
$$

where $\bar{\xi}_1(M) = 4\xi_1(M)$.

The term J_2. By (2.9), we have

$$\begin{aligned} F_{mx}^{(k)}(t) &= D_1 f[u_{m-1}] + D_3 f[u_{m-1}]\nabla u_{m-1} \\ &\quad + \sum_{1\le i+j\le N-1} \left[D_1 D^{ij} f[u_{m-1}] + D_3 D^{ij} f[u_{m-1}]\nabla u_{m-1}\right](u_m^{(k)} - u_{m-1})^i \\ &\qquad \times \left(\left\|u_m^{(k)}(t)\right\|^2 - \|u_{m-1}(t)\|^2\right)^j \\ &\quad + \sum_{1\le i+j\le N-1} D^{ij} f[u_{m-1}] i (u_m^{(k)} - u_{m-1})^{i-1}(u_{mx}^{(k)} - \nabla u_{m-1}) \\ &\qquad \times \left(\left\|u_m^{(k)}(t)\right\|^2 - \|u_{m-1}(t)\|^2\right)^j . \end{aligned} \tag{2.32}$$

Hence

$$\begin{aligned} \left\|F_{mx}^{(k)}(t)\right\| &\le K_M(f)(1+M) + K_M(f)(1+M) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!}\left(M + \sqrt{S_m^{(k)}(t)}\right)^{i+2j} \\ &\quad + K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} i \left(M + \sqrt{S_m^{(k)}(t)}\right)^{i-1}\left(M + \sqrt{S_m^{(k)}(t)}\right)\left(M + \sqrt{S_m^{(k)}(t)}\right)^{2j} \\ &\le K_M(f)(1+M) + K_M(f)(1+M) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!}\left(M + \sqrt{S_m^{(k)}(t)}\right)^{i+2j} \\ &\quad + K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} i \left(M + \sqrt{S_m^{(k)}(t)}\right)^{i+2j} \\ &\le K_M(f)(1+M) + K_M(f)(1+M) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j-1}\left[M^{i+2j} + \left(\sqrt{S_m^{(k)}(t)}\right)^{i+2j}\right] \\ &\quad + (N-1)K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j-1}\left[M^{i+2j} + \left(\sqrt{S_m^{(k)}(t)}\right)^{i+2j}\right] \\ &\le K_M(f)(1+M) \sum_{i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j-1}\left[M^{i+2j} + \left(\sqrt{S_m^{(k)}(t)}\right)^{i+2j}\right] \\ &\quad + (N-1)K_M(f)(1+M) \sum_{i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j-1}\left[M^{i+2j} + \left(\sqrt{S_m^{(k)}(t)}\right)^{i+2j}\right] \\ &= N(1+M)K_M(f) \sum_{i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j-1}\left[M^{i+2j} + \left(\sqrt{S_m^{(k)}(t)}\right)^{i+2j}\right] \\ &\le N(1+M)K_M(f) \sum_{i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j-1}\left[1 + M^{2N-2} + 1 + \left(S_m^{(k)}(t)\right)^{N-1}\right] \\ &\le N(1+M)K_M(f)\left(1 + M^{2N-2}\right) \sum_{i+j\le N-1} \tfrac{1}{i!j!} 2^{i+2j}\left[1 + \left(S_m^{(k)}(t)\right)^{N-1}\right] \\ &\equiv \xi_2(M)\left[1 + \left(S_m^{(k)}(t)\right)^{N-1}\right], \end{aligned} \tag{2.33}$$

where

$$\xi_2(M) = N(1+M)K_M(f)\left(1 + M^{2N-2}\right) \sum_{i+j\le N-1} \frac{1}{i!j!} 2^{i+2j}. \tag{2.34}$$

Using the inequality (2.30) we get from (2.33) that

$$\begin{aligned} J_2 &= 2\int_0^t \left\langle F_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\right\rangle ds \leq 2\int_0^t \left\|F_{mx}^{(k)}(s)\right\| \left\|\dot{u}_{mx}^{(k)}(s)\right\| ds \\ &\leq 2\xi_2(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds \\ &= 2\xi_2(M)\int_0^t \left[\sqrt{S_m^{(k)}(s)}+\left(S_m^{(k)}(s)\right)^{N-\frac{1}{2}}\right]ds \\ &\leq 4\xi_2(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds \\ &\equiv \bar{\xi}_2(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right]ds, \end{aligned} \tag{2.35}$$

where $\bar{\xi}_2(M) = 4\xi_2(M)$.

The term J_3. Equation $(2.7)_1$ can be rewritten as follows

$$\langle \ddot{u}_m^{(k)}(t), w_j\rangle - \left\langle \Delta u_m^{(k)}(t), w_j\right\rangle = \langle F_m^{(k)}(t), w_j\rangle, 1 \leq j \leq k. \tag{2.36}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned} J_3 &= \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \leq 2\int_0^t \|\Delta u_m^{(k)}(s)\|^2 ds + 2\int_0^t \|F_m^{(k)}(s)\|^2 ds \\ &\leq 2\int_0^t S_m^{(k)}(s)ds + 2\xi_1^2(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-\frac{3}{2}}\right]^2 ds \\ &\leq 2\int_0^t S_m^{(k)}(s)ds + 4\xi_1^2(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{2N-3}\right] ds \\ &\leq 2\int_0^t S_m^{(k)}(s)ds + 4\xi_1^2(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{2N-3}\right] ds \\ &\leq 2(1+2\xi_1^2(M))\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right] ds \\ &\equiv \bar{\xi}_3(M)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N_0}\right] ds, \end{aligned} \tag{2.37}$$

with $\bar{\xi}_3(M) = 2(1+2\xi_1^2(M))$.

Now, we need an estimate on the term $S_m^{(k)}(0)$. We have

$$S_m^{(k)}(0) = \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{1kx}\|^2 + \|\tilde{u}_{0kx}\|^2 + \|\Delta\tilde{u}_{k0}\|^2. \tag{2.38}$$

By means of the convergences in (2.8), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) \leq M^2/2. \tag{2.39}$$

Finally, it follows from (2.25), (2.31), (2.35), (2.37), (2.39) that

$$S_m^{(k)}(t) \leq \frac{M^2}{2} + T\bar{\xi}(M) + \bar{\xi}(M)\int_0^t \left(S_m^{(k)}(s)\right)^{N_0} ds, \text{ for } 0 \leq t \leq T_m^{(k)} \leq T, \tag{2.40}$$

where

$$\bar{\xi}(M) = \bar{\xi}_1(M) + \bar{\xi}_2(M) + \bar{\xi}_3(M).$$

Then, by solving a nonlinear Volterra integral inequality (2.40) (based on the methods in [6]), the following lemma is proved.

Lemma 2.4. *There exists a constant $T > 0$ independent of k and m such that*

$$S_m^{(k)}(t) \leq M^2 \ \forall t \in [0, T], \text{ for all } k \text{ and } m. \tag{2.41}$$

By Lemma 2.4, we can take constant $T_m^{(k)} = T$ for all k and m. Therefore, we have

$$u_m^{(k)} \in W(M, T), \text{ for all } k \text{ and } m. \tag{2.42}$$

Step 3: *Convergence.* From (2.42), we can extract from $\{u_m^{(k)}\}$ a subsequence $\{u_m^{(k_i)}\}$ such that

$$\begin{cases} u_m^{(k_i)} \to u_m & \text{in} \quad L^\infty(0, T; H_0^1 \cap H^2) \quad \text{weak*}, \\ \dot{u}_m^{(k_i)} \to u'_m & \text{in} \quad L^\infty(0, T; H_0^1) \quad \text{weak*}, \\ \ddot{u}_m^{(k_i)} \to u''_m & \text{in} \quad L^2(Q_T) \quad \text{weak}, \end{cases} \tag{2.43}$$

$$u_m \in W(M, T). \tag{2.44}$$

We can easily check from (2.7), (2.8), (2.43), (2.44) that u_m satisfies (2.3), (2.4) in $L^2(0, T)$, weak.

On the other hand, it follows from $(2.3)_1$ and $u_m \in W(M, T)$ that $u''_m = \Delta u_m + F_m \in L^\infty(0, T; L^2)$, hence $u_m \in W_1(M, T)$ and the proof of Theorem 2.2 is complete. □

Next, we put

$$W_1(T) = \{v \in L^\infty(0, T; H_0^1) : v' \in L^\infty(0, T; L^2)\},$$

then $W_1(T)$ is a Banach space with respect to the norm (see [7]):

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \|v'\|_{L^\infty(0,T;L^2)}.$$

Then, we have the following theorem.

Theorem 2.5. *Let (H_1), (H_2) hold. Then, there exist constants $M > 0$ and $T > 0$ such that*

(i) *Prob.* (1.1) – (1.3) *has a unique weak solution $u \in W_1(M, T)$.*

(ii) *The recurrent sequence $\{u_m\}$ defined by* (2.3), (2.4) *converges at a rate of order N to the solution u strongly in the space $W_1(T)$ in the sense*

$$\|u_m - u\|_{W_1(T)} \leq C\|u_{m-1} - u\|_{W_1(T)}^N, \tag{2.45}$$

for all $m \geq 1$, where C is a suitable constant.

Furthermore, we have the estimation

$$\|u_m - u\|_{W_1(T)} \leq C_T \beta^{N^m}, \tag{2.46}$$

for all $m \geq 1$, where C_T and $0 < \beta < 1$ are positive constants depending only on T.

Proof.

Put $v_m = u_{m+1} - u_m$, it is clear that v_m satisfies the variational problem

$$\begin{cases} \langle v''_m(t), v\rangle + \langle v_{mx}(t), v_x\rangle = \langle F_{m+1}(t) - F_m(t), v\rangle \quad \forall v \in H_0^1, \\ v_m(0) = v'_m(0) = 0, \end{cases} \tag{2.47}$$

where

$$F_m(x, t) = \sum_{i+j \leq N-1} D^{ij} f[u_{m-1}](u_m - u_{m-1})^i \left(\|u_m(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j. \tag{2.48}$$

Taking $v = v'_m$ in (2.47), after integrating in t we get

$$\sigma_m(t) = 2\int_0^t \langle F_{m+1}(s) - F_m(s), v'_m(s)\rangle \, ds, \tag{2.49}$$

with

$$\sigma_m(t) = \|v_m'(t)\|^2 + \|v_{mx}(t)\|^2. \tag{2.50}$$

On the other hand, by using Taylor's expansion for the function $f\left(x, t, u_m, \|u_m(t)\|^2\right)$ around the point $\left(x, t, u_{m-1}, \|u_{m-1}(t)\|^2\right)$ up to order N, we obtain

$$\begin{aligned} f[u_m] - f[u_{m-1}] &= f\left(x, t, u_m, \|u_m(t)\|^2\right) - f\left(x, t, u_{m-1}, \|u_{m-1}(t)\|^2\right) \\ &= \sum_{1\le i+j\le N-1} D^{ij} f[u_{m-1}] v_{m-1}^i \left(\|u_m(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j \\ &\quad + \sum_{i+j=N} D^{ij} f[\eta_m] v_{m-1}^i \left(\|u_m(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j, \end{aligned} \tag{2.51}$$

where

$$[\eta_m] = \left(x, t, u_{m-1} + \theta v_{m-1},\ \theta \|u_m(t)\|^2 + (1-\theta)\|u_{m-1}(t)\|^2\right),\ \ 0 < \theta < 1.$$

Hence, it follows from (2.4), (2.51) that

$$\begin{aligned} F_{m+1}(t) - F_m(t) &= \sum_{1\le i+j\le N-1} D^{ij} f[u_m] v_m^i \left(\|u_{m+1}(t)\|^2 - \|u_m(t)\|^2\right)^j \\ &\quad + \sum_{i+j=N} D^{ij} f[\eta_m] v_{m-1}^i \left(\|u_m(t)\|^2 - \|u_{m-1}(t)\|^2\right)^j. \end{aligned} \tag{2.52}$$

Then we deduce, from (2.52), that

$$\begin{aligned} &\|F_{m+1}(t) - F_m(t)\| \\ &\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \|v_{mx}(t)\|^i \left(\|u_{m+1}(t)\| + \|u_m(t)\|\ \right)^j \left|\|u_{m+1}(t)\| - \|u_m(t)\|\ \right|^j \\ &+ K_M(f) \sum_{i+j=N} \tfrac{1}{i!j!} \|v_{m-1}\|_{W_1(T)}^i \left(\|u_m(t)\| + \|u_{m-1}(t)\|\ \right)^j \left|\|u_m(t)\| - \|u_{m-1}(t)\|\ \right|^j \\ &\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \|v_{mx}(t)\|^{i+j} (2M)^j \\ &+ K_M(f) \sum_{i+j=N} \tfrac{1}{i!j!} \|v_{m-1}\|_{W_1(T)}^i (2M)^j \|v_{m-1}(t)\|^j \\ &\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} \|v_{mx}(t)\|^{i+j-1} (2M)^j \|v_{mx}(t)\| + K_M(f) \sum_{i+j=N} \tfrac{1}{i!j!} (2M)^j \|v_{m-1}\|_{W_1(T)}^{i+j} \\ &\le K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} 2^j M^{i+2j-1} \|v_{mx}(t)\| + K_M(f) \sum_{i+j=N} \tfrac{1}{i!j!} (2M)^j \|v_{m-1}\|_{W_1(T)}^N \\ &\equiv \gamma_T \|v_{mx}(t)\| + \bar{\gamma}_T \|v_{m-1}\|_{W_1(T)}^N, \end{aligned} \tag{2.53}$$

where

$$\gamma_T = K_M(f) \sum_{1\le i+j\le N-1} \tfrac{1}{i!j!} 2^j M^{i+2j-1},\ \ \bar{\gamma}_T = K_M(f) \sum_{i+j=N} \tfrac{1}{i!j!} (2M)^j. \tag{2.54}$$

Then we deduce, from (2.49), (2.50) and (2.53), that

$$\begin{aligned} \sigma_m(t) &= 2\int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s)\rangle\, ds \le 2\int_0^t \|F_{m+1}(s) - F_m(s)\| \left\|v_m'(s)\right\| ds \\ &\le 2\int_0^t \left(\gamma_T \|v_{mx}(s)\| + \bar{\gamma}_T \|v_{m-1}\|_{W_1(T)}^N\right) \left\|v_m'(s)\right\| ds \\ &\le 2\gamma_T \int_0^t \|v_{mx}(s)\| \left\|v_m'(s)\right\| ds + 2\bar{\gamma}_T \int_0^t \|v_{m-1}\|_{W_1(T)}^N \left\|v_m'(s)\right\| ds \\ &\le T\, \bar{\gamma}_T \|v_{m-1}\|_{W_1(T)}^{2N} + (\gamma_T + \bar{\gamma}_T) \int_0^t \sigma_m(s) ds. \end{aligned} \tag{2.55}$$

By using Gronwall's lemma, we obtain from (2.55) that

$$\|v_m\|_{W_1(T)} \leq 2\sqrt{T\bar{\gamma}_T e^{T(\gamma_T+\bar{\gamma}_T)}}\,\|v_{m-1}\|_{W_1(T)}^N \equiv \mu_T\,\|v_{m-1}\|_{W_1(T)}^N, \tag{2.56}$$

where μ_T is the constant given by

$$\mu_T = 2\sqrt{T\bar{\gamma}_T e^{T(\gamma_T+\bar{\gamma}_T)}}. \tag{2.57}$$

Hence, we obtain from (2.56) that

$$\|u_m - u_{m+p}\|_{W_1(T)} \leq (1-\beta)^{-1}\,(\mu_T)^{\frac{-1}{N-1}}\,\beta^{N^m}, \tag{2.58}$$

for all m and p.

We take $T > 0$ small enough, such that $\beta = (\mu_T)^{\frac{1}{N-1}}\,M < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that $u_m \to u$ strongly in $W_1(T)$.

It is similar to argument used in the proof of Theorem 2.2, we obtain that $u \in W_1(M,T)$ is a unique weak solution of Prob. (1.1) – (1.3). Passing to the limit as $p \to +\infty$ for m fixed, we get the estimate (2.46) from (2.58). This completes the proof of Theorem 2.5. □

Remark. In order to construct a $N-$order iterative scheme, we need the condition $f \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)$. Then, we get a convergent sequence at a rate of order N to a local unique weak solution of problem and the existence follows. This condition of f can be relaxed if we only consider the existence of solution, it is not necessary that $f \in C^1([0,1]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)$, see [10].

Acknowledgements. The authors wish to express their sincere thanks to the referees for the suggestions, remarks and valuable comments.

References

[1] R. A. Adams, Sobolev Spaces, Academic Press, NewYork, 1975.

[2] G.F. Carrier, On the nonlinear vibrations problem of elastic string, Quart. J. Appl. Math. 3 (1945) 157 - 165.

[3] F. Ficken, B. Fleishman, Initial value problems and time periodic solutions for a nonlinear wave equation, Comm. Pure Appl. Math. 10 (1957) 331 - 356.

[4] M. Hosoya, Y. Yamada, On some nonlinear wave equation I: Local existence and regularity of solutions, J. Fac. Sci. Univ. Tokyo. Sect. IA, Math. 38 (1991) 225 - 238.

[5] G. R. Kirchhoff, Vorlesungen über Mathematiche Physik: Mechanik, Teuber, Leipzig, 1876, Section 29.7.

[6] Lakshmikantham V, Leela S, Differential and Integral Inequalities, Vol.1. Academic Press, NewYork, 1969.

[7] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non-linéaires, Dunod-Gauthier -Villars, Paris 1969.

[8] N. T. Long, T. N. Diem, On the nonlinear wave equation $u_{tt} - u_{xx} = f(x,t,u,u_x,u_t)$ associated with the mixed homogeneous conditions, Nonlinear Anal. TMA. 29 (11) (1997) 1217 - 1230.

[9] N.T. Long, A. P. N. Dinh, T. N. Diem, Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator, J. Math. Anal. Appl. 267 (1) (2002) 116 - 134.

[10] N. T. Long, On the nonlinear wave equation $u_{tt} - B(t,\|u\|^2,\|u_x\|^2)u_{xx} = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2)$ associated with the mixed homogeneous conditions, J. Math. Anal. Appl. 306 (1) (2005) 243 - 268.

[11] L. A. Medeiros, On some nonlinear perturbation of Kirchhoff-Carrier operator, Comp. Appl. Math. 13 (1994) 225 - 233.

[12] L. T. P. Ngoc, L. X. Truong, N. T. Long, An N order iterative scheme for a nonlinear Kirchhoff-Carrier wave equation associated with mixed homogeneous conditions Acta Mathematica Vietnamica, 35 (2) (2010) 207 - 227.

[13] E. L. Ortiz, A. P. N. Dinh, Linear recursive schemes associated with some nonlinear partial differential equations in one dimension and the Tau method, SIAM J. Math. Anal. 18 (1987) 452 - 464.

[14] P. H. Rabinowitz, Periodic solutions of nonlinear hyperbolic differential equations, Comm. Pure. Appl. Math. 20 (1967) 145 - 205.

[15] L. X. Truong, L. T. P. Ngoc, N. T. Long, High-order iterative schemes for a nonlinear Kirchhoff - Carrier wave equation associated with the mixed homogeneous conditions, Nonlinear Anal. TMA. 71 (1 - 2) (2009) 467 - 484.