Some Families of $q$-Sums and $q$-Products

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Abstract. In this paper, we introduce two new binary operations, the one called $q$-sum and defined on the set of all real numbers and the other called $q$-product and defined on a subset of real numbers, which have potential importance in the study of $q$-numbers. The set of $q$-numbers of all real numbers, for example, is a field when these operations are restricted to it. Also, we introduce new $q$-exponential and $q$-logarithm and show some relations for them. Finally, we give some remarks on the well-known $q$-gamma, $q$-exponential, and $q$-beta functions.

1. Introduction and Preliminaries

Throughout this paper, let $q$ denote a fixed real number with $0 < q < 1$. We define a binary operation $\oplus$, called $q$-sum, on the set of real numbers $\mathbb{R}$ as follows: for real numbers $x$ and $y$, $x \oplus y = x + y + (q - 1)xy$ (1.1)

Then $\mathbb{R}$ equipped with the operation $\oplus$ is a commutative monoid with 0 the identity. Note here that the operation is associative, since

$$(x \oplus y) \oplus z = x \oplus (y \oplus z) = x + y + z + (q - 1)(xy + xz + yz) + (q - 1)^2xyz.$$ 

It is easy to see by induction that, for $x_1, \cdots, x_n \in \mathbb{R}$, we have $x_1 \oplus \cdots \oplus x_n = \sum_{k=1}^{n} (q - 1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$ (1.2)

In particular, $x \oplus \cdots \oplus x = \mathcal{P}_n^q x = \sum_{k=1}^{n} \binom{n}{k}(q - 1)^{k-1}x^k = \frac{1 + (q - 1)x}{q - 1}$.

Let $\Sigma_q = \{x \in \mathbb{R} | x < 1 - q\}$. (1.3)

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We define another binary operation \( \otimes \), called \( q \)-product, on \( \Sigma_q \) as follows: for \( x, y \in \Sigma_q \),

\[
x \otimes y = \frac{1}{q-1}(q^{x+y}-1),
\]

where

\[
[x, y]_q = \frac{\log(1 + (q-1)x) \log(1 + (q-1)y)}{(\log q)^2}.
\]

Then \( \mathbb{R} \) equipped with the operation \( \otimes \) is a commutative monoid with 1 the identity. Note here that \( \{x, y\}_q \) is defined for \( x, y \in \Sigma_q, x \otimes y \in \Sigma_q \) if \( x, y \in \Sigma_q \), and that the operation is associative, since

\[
[x \otimes y, z]_q = [x, y \otimes z]_q = \frac{\log(1 + (q-1)x) \log(1 + (q-1)y) \log(1 + (q-1)z)}{(\log q)^3},
\]

which we denote by \( [x, y, z]_q \).

In general, for \( x_1, x_2, \ldots, x_n \in \Sigma_q \), by \( \{x_1, x_2, \ldots, x_n\}_q \) we denote

\[
[x_1, x_2, \ldots, x_n]_q = \frac{\prod_{i=1}^n \log(1 + (q-1)x_i)}{(\log q)^n}.
\]

Then

\[
x_1 \otimes x_2 \otimes \cdots \otimes x_n = \frac{1}{q-1}(q^{\sum_{i=1}^n x_i} - 1).
\]

So, in particular, \( x \otimes \cdots \otimes x = x^{\otimes n} = \frac{1}{q-1}(q^{\sum_{i=1}^n x_i} - 1) \), with

\[
[x, \cdots, x]_q = \prod_{i=1}^n \left( \frac{\log(1 + (q-1)x)}{\log q} \right)^n.
\]

We now restrict the \( q \)-sum and \( q \)-product to sets of \( q \)-numbers. The \( q \)-number \([x]_q\) of the real number \( x \) is as usual defined as

\[
[x]_q = \frac{q^x - 1}{q - 1}.
\]

Then we see that \( \lim_{q \to 1} [x]_q = x \). For any subset \( X \) of \( \mathbb{R} \), let \([X]_q\) be the subset of \( \mathbb{R} \) given by

\[
[X]_q = \{[x]_q | x \in X\},
\]

which may be called the \( q \)-numbers of \( X \).

Now, it is easy to see that, for real numbers \( x \) and \( y \),

\[
[x]_q \oplus [y]_q = [x + y]_q,
\]

\[
[x]_q \otimes [y]_q = [xy]_q.
\]

Note here that \([x]_q \in \Sigma_q\), for any real number \( x \).

Thus we obtain the following proposition.

**Proposition 1.1.** Let \( A, R, F \) be respectively a subgroup of the additive group of \( \mathbb{R} \), a subring of \( \mathbb{R} \), and a subfield of \( \mathbb{R} \). Then \( ([A]_q, \oplus, 0) \) is a group, \( ([R]_q, \otimes, 0, 1) \) is an integral domain and \( ([F]_q, \oplus, 0, 1) \) is a field. In particular, \( ([\mathbb{Z}]_q, \oplus, 0) \) is a cyclic group generated by 1, and \( ([\mathbb{R}]_q, \oplus, 0, 1) \) is a field.
In this paper, we introduce two new binary operations, the one called \(q\)-sum and defined on the set of all real numbers and the other called \(q\)-product and defined on a subset of real numbers, which have potential importance in the study of \(q\)-numbers. The set of \(q\)-numbers of all real numbers, for example, is a field when these operations are restricted to it. Also, we introduce \(q\)-exponential and \(q\)-logarithm and show some relations for them. Finally, we give some remarks on the well-known \(q\)-gamma, \(q\)-exponential, and \(q\)-beta functions.

As the related works, one is referred to [2,5,6,7] in connection with \(q\)-analysis (especially \(q\)-series and \(q\)-polynomials) and to [8,9] in connection with summation-integral operators.

2. Main Results

\textbf{Theorem 2.1.} Let \(f(x)\) be a real-valued function defined on \([\mathbb{N}]_q\) with \(f([1]) = f(1) = 1\). Then we have

\[
\varphi_{r=1}^n f([r]_q) = 1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} f([i_j]_q).
\]

\[
\varphi_{r=1}^{n+1} f([r]_q) = (\varphi_{r=1}^{n} f([r]_q)) \oplus f([n+1]_q)
\]

\[
= (1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} f([i_j]_q) \oplus f([n+1]_q))
\]

\[
= (1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} f([i_j]_q)) + f([n+1]_q)
\]

\[
+ (q-1)(1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} f([i_j]_q)) f([n+1]_q)
\]

\[
= (1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} f([i_j]_q)) + q f([n+1]_q)
\]

\[
+ q \sum_{k=2}^{n+1} (q-1)^{k-1} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n+1} \prod_{j=1}^{k+1} f([i_j]_q)
\]

\[
= 1 + q \sum_{k=2}^{n+1} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} f([i_j]_q)
\]

\[
+ q \sum_{k=2}^{n+1} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} f([i_j]_q) + q f([n+1]_q)
\]

\[
= 1 + q \sum_{k=2}^{n+1} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} f([i_j]_q)
\]

\[
+ q \sum_{k=2}^{n+1} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} f([i_j]_q)
\]

\[
= 1 + q \sum_{k=2}^{n+1} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n+1} \prod_{j=1}^{k} f([i_j]_q).
\]

\[\Box\]
Let us take $f(x) = x$. Then we obtain the following corollary.

**Corollary 2.2.** For $n \in \mathbb{N}$, we have

$$
\left( \binom{n+1}{2} \right)_q = \oplus_{r=1}^{n} [r]_q = 1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \Pi_{j=1}^{k} [i_j]_q.
$$

In the special case, $f(x) = x^2$, we obtain the following corollary.

**Corollary 2.3.** For $n \in \mathbb{N}$, we have

$$
\oplus_{r=1}^{n} [r]_q^2 = 1 + q \sum_{k=2}^{n} (q-1)^{k-2} \sum_{1=i_1 < i_2 < \cdots < i_k \leq n} \Pi_{j=1}^{k} [i_j]_q^2.
$$

From (1.7) and (1.8), we have, for $x_1, \ldots, x_n \in \sum_q$, (1.3),

$$
\otimes_{i=1}^{n} x_i = \frac{1}{q-1} \left( q^{\frac{\log(1+(q-1)x_i)}{\log q}} - 1 \right),
$$

(2.2)

and, for any $x_1, \ldots, x_n \in \mathbb{R}$,

$$
[\prod_{i=1}^{n} x_i]_q = [x_1 x_2 \cdots x_n]_q = [\Pi_{j=1}^{n} x_j]_q.
$$

(2.3)

Thus, from (2.2) and (2.3), we have, for $x \in \sum_q$

$$
x^{\otimes n} = x \otimes \cdots \otimes x = \frac{1}{q-1} \left( q^{\frac{\log(1+(q-1)x)}{\log q}} - 1 \right),
$$

(2.4)

and, for any $x \in \mathbb{R}$,

$$
[x]_q^{\otimes n} = [x]_q \otimes \cdots \otimes [x]_q = [x^n]_q.
$$

(2.5)

Provided that $x + y \in \sum_q$, from (2.4) and (2.5) we have

$$
(x + y)^{\otimes n} = \frac{1}{q-1} \left( q^{\frac{\log(1+(q-1)(x+y))}{\log q}} - 1 \right),
$$

(2.6)

and, for any $x, y \in \mathbb{R}$,

$$
([x]_q \oplus [y]_q)^{\otimes n} = [(x + y)^n]_q.
$$

(2.7)

Let us define a $q$-analogue of exponential function on $\mathbb{R}$ as follows:

$$
e_q(x) = \lim_{n \to \infty} (1 \oplus \left[ \frac{x}{n} \right]_q)^{\otimes n}.
$$

(2.8)

Then, by (2.7) and (2.8), we get

$$
e_q(x) = \lim_{n \to \infty} \left[ 1 + \frac{x^n}{n} \right]_q = [e^x]_q.
$$

(2.9)
From (1.13) and (2.9), we have
\[ e_q(x) \otimes e_q(y) = [e^{x+y}]_q = e_q(x+y), \tag{2.10} \]
and
\[ e_q(x \oplus y) = e_q(x + y + (q-1)xy) = [e^{x+y+q-1}xy]_q \]
\[ = [e^x]_q \otimes [e^y]_q \otimes [e^{q-1}xy]_q \]
\[ = e_q(x) \otimes e_q(y) \otimes e_q((q-1)xy). \tag{2.11} \]

Therefore, by (2.10) and (2.11), we obtain the following proposition.

**Proposition 2.4.** For \( x, y \in \mathbb{R} \), we have
\[ e_q(x \oplus y) = e_q(x) \otimes e_q(y) \otimes e_q((q-1)x), \]
and
\[ e_q(x) \otimes e_q(y) = e_q(x + y). \]

Let us define a \( q \)-logarithm on \( \Sigma_q^+ = \Sigma_q \cap \mathbb{R}^+ = \{ x \in \mathbb{R} | 0 < x < \frac{1}{1-q} \} \) as follows:
\[ \log_q x = \log_\left( \frac{\log(1 + (q-1)x)}{\log q} \right). \tag{2.12} \]

Then, by (2.12), we get
\[ \log_q[x]_q = \log x (x > 0), \quad \log_q(e_q(x)) = x (x \in \mathbb{R}). \tag{2.13} \]

It is easy to show that, for \( x, y \in \Sigma_q^+ \),
\[ \log_q(x \otimes y) = \log_q x + \log_q y. \tag{2.14} \]

Note here that \( x \otimes y \in \Sigma_q^+ \), if \( x, y \in \Sigma_q^+ \). Therefore, by (2.13) and (2.14), we obtain the following proposition.

**Proposition 2.5.** We have the following identities:
\[ \log_q(e_q(x)) = x (x \in \mathbb{R}), \quad \log_q(x \otimes y) = \log_q x + \log_q y \quad (x, y \in \Sigma_q^+). \]

### 3. Further Remarks

In this section, we use the following notations:
\[ (a + b)_q^n = \prod_{i=0}^{n-1}(a + q^ib), \text{ if } n \in \mathbb{Z}_+ \] \( (3.1) \)
\[ (1 + a)_q^t = \frac{(1 + a)_q^n}{(1 + q^n)(1 + q^{n-1})}, \text{ if } t \in \mathbb{C}, \text{ (see[1,4])}. \quad (3.2) \]
where $q$ is a fixed real number with $0 < q < 1$.
The $q$-integral is defined as
\[
\int_0^\infty f(t)d_qt = (1 - q)\sum_{n=0}^\infty f(q^n)t^n, \quad (\text{see}[3, 4]).
\] (3.3)

The $q$-gamma function is defined as
\[
\Gamma_q(t) = \int_0^\infty x^{t-1}E_q(-qx)d_qx, \quad (t > 0),
\] (3.4)

where $E_q(x)$ is one of the $q$-analogues of exponential function which is defined by
\[
E_q(x) = (1 + (1 - q)x)^\infty = \sum_{n=0}^\infty \frac{q^n}{[n]_q!} x^n,
\] (3.5)

where $[k]_q! = [k]_q[k - 1]_q \cdots [2]_q[1]_q$.

As is well known, another $q$-exponential function is defined by
\[
e_q(x) = \frac{1}{(1 - (1 - q)x)_q^\infty} = \sum_{n=0}^\infty \frac{x^n}{[n]_q!}, \quad (\text{see}[1, 4]).
\] (3.6)

Thomae and Jackson have shown the $q$-beta function as follows:
\[
B_q(t, s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t + s)}, \quad (t, s > 0).
\] (3.7)

The $q$-integral representation, which is a $q$-analogue of Euler’s formula, is given by
\[
B_q(t, s) = \int_0^1 x^{t-1}(1 - qx)_q^{s-1}d_qx.
\] (3.8)

From (3.4), we note that
\[
\Gamma_q(t) = \frac{(1 - q)^{t-1}}{(1 - q)^{t-1}}, \quad [n]_q!\Gamma(t) = \Gamma_q(t + 1), \quad (\text{see}[3, 4]).
\] (3.9)

From (3.4), we have
\[
\Gamma_q\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}}E_q(-qx)d_qx
\] 
\[= \sum_{n=0}^\infty q^n\left(\frac{q^n}{1 - q}\right)^{\frac{1}{2}} E_q\left(-\frac{q^{n+1}}{1 - q}\right)
\] 
\[= \sqrt{1 - q}\sum_{n=0}^\infty q^{\frac{n}{2}}(1 - q^{n+1})_q^\infty
\] 
\[= \sqrt{1 - q}(1 - q)^{\frac{1}{2}}\sum_{n=0}^\infty \frac{q^{\frac{n}{2}}}{(1 - q)_q^n}
\] 
\[= \sqrt{1 - q}(1 - q)^{\frac{1}{2}} \sum_{n=0}^\infty \frac{q^{\frac{n}{2}}}{[n]_q!}\left(\frac{1}{1 - q}\right)^n
\] 
\[= \sqrt{1 - q}E_q\left(\frac{q}{q - 1}\right)\epsilon_q\left(\frac{q^{\frac{1}{2}}}{1 - q}\right).
\] (3.10)
Therefore, by (3.10), we obtain the following proposition.

**Proposition 3.1.** For $0 < q < 1$, we have

$$\Gamma_q\left(\frac{1}{2}\right) = \sqrt{1 - q}E_q\left(\frac{q}{q - 1}\right)e_q\left(\frac{q^2}{1 - q}\right).$$

Note that

$$\lim_{q \to 1} \sqrt{1 - q}E_q\left(\frac{q}{q - 1}\right)e_q\left(\frac{q^2}{1 - q}\right) = \sqrt{\pi}.$$

We note that

$$(1 - x)^n_q = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q (-1)^k q^k x^k,$$

and

$$\frac{1}{(1 - x)^n_q} = \sum_{k=0}^{n} \left[\begin{array}{c} k + n - 1 \\ k \end{array}\right]_q x^k,$$

where $[\begin{array}{c} n \\ k \end{array}]_q = \frac{[\begin{array}{c} n \end{array}]}{[\begin{array}{c} k \end{array}][\begin{array}{c} n - k \end{array}]}$ (see [1, 3, 4]).

From (3.12), we have

$$\frac{1}{(1 - x)^n_q} = \sum_{k=0}^{\infty} \left[\begin{array}{c} 1 + k - 1 \\ n \end{array}\right]_q x^k = \sum_{k=0}^{\infty} \left[\begin{array}{c} k - \frac{1}{2} \\ n \end{array}\right]_q x^k$$

$$= \sum_{k=0}^{\infty} \frac{[k - \frac{1}{2}]_q [k - \frac{3}{2}]_q \cdots [\frac{1}{2}]_q}{[k]_q!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{[k - \frac{1}{2}]_q [k - \frac{3}{2}]_q \cdots [\frac{1}{2}]_q \Gamma_q\left(\frac{1}{2}\right)}{[k]_q! \Gamma_q\left(\frac{1}{2}\right)} x^k$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma_q(1 + k - \frac{1}{2})}{[k]_q! \Gamma_q\left(\frac{k}{2}\right)} x^k = \sum_{k=0}^{\infty} \frac{\Gamma_q(k + \frac{1}{2})}{[k]_q! \Gamma_q\left(\frac{k}{2}\right)} x^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{[k]_q! B_q(k, \frac{1}{2})}.$$  \hfill (3.13)

It is not difficult to show that

$$\frac{[k - \frac{1}{2}]_q \cdots [\frac{1}{2}]_q}{[k]_q!} = \frac{[2k]_q!}{([k]_q!)^2 ([2]_{\frac{1}{2}})^{2k}}$$

$$= \frac{\Gamma_q(2k + 1)}{\Gamma_q(2k + 1)^2 ([2]_{\frac{1}{2}})^{2k}}.$$

Thus, by (3.12) and (3.14), we get
\[
\frac{1}{(1 - x)_q^{\frac{1}{2}}} = \sum_{k=0}^{\infty} \frac{\Gamma_q(2k + 1)}{(\Gamma_q(k + 1))^2([2]_q^{1})^{2k}} x^k.
\]

Therefore, by (3.13) and (3.15), we obtain the following proposition.

**Proposition 3.2.** For \( k \geq 0 \), we have

\[
[k]_q B_q(k, \frac{1}{2}) = \frac{(\Gamma_q(k + 1))^2([2]_q^{1})^{2k}}{\Gamma_q(2k + 1)}.
\]

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**References**


