Filomat 31:6 (2017), 1611–1618
DOI 10.2298/FIL1706611K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Families of q-Sums and q-Products

Taekyun Kim[a], Dae San Kim[b], Won Sang Chung[c], Hyuck In Kwon[d]

[a]*Department of Mathematics, Kwangwon University, Seoul 139-701, Republic of Korea*
[b]*Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[c]*Department of Physics, Gyeongsang National University, Jinju 660,701, Republic of Korea*
[d]*Department of Mathematics, Kwangwon University, Seoul 139-701, Republic of Korea*

Abstract. In this paper, we introduce two new binary operations, the one called q-sum and defined on the set of all real numbers and the other called q-product and defined on a subset of real numbers, which have potential importance in the study of q-numbers. The set of q-numbers of all real numbers, for example, is a field when these operations are restricted to it. Also, we introduce new q-exponential and q-logarithm and show some relations for them. Finally, we give some remarks on the well-known q-gamma, q-exponential, and q-beta functions.

1. Introduction and Preliminaries

Throughout this paper, let q denote a fixed real number with $0 < q < 1$. We define a binary operation $\oplus$, called q-sum, on the set of real numbers $\mathbb{R}$ as follows: for real numbers x and y,

$$x \oplus y = x + y + (q-1)xy \tag{1.1}$$

Then $\mathbb{R}$ equipped with the operation $\oplus$ is a commutative monoid with 0 the identity. Note here that the operation is associative, since

$$(x \oplus y) \oplus z = x \oplus (y \oplus z) = x + y + z + (q-1)(xy + xz + yz) + (q-1)^2 xyz.$$

It is easy to see by induction that, for $x_1, \cdots, x_n \in \mathbb{R}$, we have

$$x_1 \oplus \cdots \oplus x_n = \sum_{k=1}^{n} (q-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}. \tag{1.2}$$

In particular, $\underbrace{x \oplus \cdots \oplus x}_{n-times} = \oplus_{i=1}^{n} x = \sum_{k=1}^{n} \binom{n}{k} (q-1)^{k-1} x^k = \frac{(1+(q-1)x)^n - 1}{q-1}$.

Let

$$\Sigma_q = \{x \in \mathbb{R} | x < \frac{1}{1-q}\}. \tag{1.3}$$

2010 *Mathematics Subject Classification*. Primary 5A30
Keywords. q-sum, q-product, q-exponential, q-logarithm
Received: 12 April 2015; Accepted: 18 April 2015
Communicated by Hari M. Srivastava
Email addresses: tkkim@kw.ac.kr (Taekyun Kim), dskim@sogang.ac.kr (Dae San Kim), mimip4444@hanmail.net (Won Sang Chung), sura@kw.ac.kr (Hyuck In Kwon)

We define another binary operation $\otimes$, called q-product, on $\sum_q$ as follows: for $x, y \in \sum_q$,

$$x \otimes y = \frac{1}{q-1}(q^{\{x,y\}_q} - 1), \tag{1.4}$$

where

$$\{x, y\}_q = \frac{\log(1 + (q-1)x)\log(1 + (q-1)y)}{(\log q)^2}. \tag{1.5}$$

Then $\mathbb{R}$ equipped with the operation $\otimes$ is a commutative monoid with 1 the identity. Note here that $\{x, y\}_q$ is defined for $x, y \in \sum_q$, $x \otimes y \in \sum_q$ if $x, y \in \sum_q$, and that the operation is associative, since

$$\{x \otimes y, z\}_q = \{x, y \otimes z\}_q = \frac{\log(1 + (q-1)x)\log(1 + (q-1)y)\log(1 + (q-1)z)}{(\log q)^3}, \tag{1.6}$$

which we denote by $\{x, y, z\}_q$.

In general, for $x_1, x_2, \cdots, x_n \in \sum_q$, by $\{x_1, x_2, \cdots, x_n\}_q$ we denote

$$\{x_1, x_2, \cdots, x_n\}_q = \frac{\Pi_{i=1}^n \log(1 + (q-1)x_i)}{(\log q)^n}. \tag{1.7}$$

Then

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n = \frac{1}{q-1}(q^{\{x_1, x_2, \cdots, x_n\}_q} - 1). \tag{1.8}$$

So, in particular, $\underbrace{x \otimes \cdots \otimes x}_{n-times} = x^{\otimes n} = \frac{1}{q-1}(q^{\{x,x,\cdots,x\}_q} - 1)$, with

$$\{x, \cdots, x\}_q = \Pi_{i=1}^n (\frac{\log(1 + (q-1)x)}{\log q})^n. \tag{1.9}$$

We now restrict the q-sum and q-product to sets of q-numbers. The q-number $[x]_q$ of the real number x is as usual defined as

$$[x]_q = \frac{q^x - 1}{q-1}. \tag{1.10}$$

Then we see that $\lim_{q \to 1-}[x]_q = x$. For any subset X of $\mathbb{R}$, let $[X]_q$ be the subset of $\mathbb{R}$ given by

$$[X]_q = \{[x]_q | x \in \mathbb{R}\}, \tag{1.11}$$

which may be called the q-numbers of X.

Now, it is easy to see that, for real numbers x and y,

$$[x]_q \oplus [y]_q = [x + y]_q, \tag{1.12}$$

$$[x]_q \otimes [y]_q = [xy]_q. \tag{1.13}$$

Note here that $[x]_q \in \sum_q$, for any real number x.
Thus we obtain the following proposition.

Proposition 1.1. *Let A, R, F be respectively a subgroup of the additive group of $\mathbb{R}$, a subring of $\mathbb{R}$, and a subfield of $\mathbb{R}$. Then $([A]_q, \oplus, 0)$ is a group, $([R]_q, \oplus, \otimes, 0, 1)$ is an integral domain and $([F]_q, \oplus, \otimes, 0, 1)$ is a field, In particular, $([\mathbb{Z}]_q, \oplus, 0)$ is a cyclic group generated by 1, and $([\mathbb{R}]_q, \oplus, \otimes, 0, 1)$ is a field.*

In this paper, we introduce two new binary operations, the one called q-sum and defined on the set of all real numbers and the other called q-product and defined on a subset of real numbers, which have potential importance in the study of q-numbers. The set of q-numbers of all real numbers, for example, is a field when these operations are restricted to it. Also, we introduce q-exponential and q-logarithm and show some relations for them. Finally, we give some remarks on the well-known q-gamma, q-exponential, and q-beta functions.

As the related works, one is referred to [2,5,6,7] in connection with q-analysis(especially q-series and q-polynomials) and to [8,9] in connection with summation-integral operators.

2. Main Results

Theorem 2.1. *Let $f(x)$ be a real-valued function defined on $[\mathbb{N}]_q$ with $f([1]_q) = f(1) = 1$. Then we have*

$$\oplus_{r=1}^{n} f([r]_q) = 1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k} f([i_j]_q).$$

Proof. We proceed the proof by induction on n. For $n = 1$, the claim is equivalent to $f([1]_q) = 1$ which holds. Assume that the claim holds for n and let us prove it for $n + 1$.

$$\begin{aligned}
\oplus_{r=1}^{n+1} f([r]_q) &= (\oplus_{r=1}^{n} f([r]_q)) \oplus f([n+1]_q) \qquad (2.1)\\
&= (1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k} f([i_j]_q)) \oplus f([n+1]_q)\\
&= (1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k} f([i_j]_q)) + f([n+1]_q)\\
&+ (q-1)(1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k} f([i_j]_q))f([n+1]_q)\\
&= (1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k} f([i_j]_q)) + qf([n+1]_q)\\
&\quad + q\sum_{k=2}^{n}(q-1)^{k-1}\sum_{1=i_1<i_2<\cdots<i_k<i_{k+1}=n+1}\Pi_{j=1}^{k+1} f([i_j]_q)\\
&= 1 + q\sum_{k=2}^{n+1}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k<n+1}\Pi_{j=1}^{k} f([i_j]_q)\\
&\quad + q\sum_{k=3}^{n+1}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k=n+1}\Pi_{j=1}^{k} f([i_j]_q) + qf([n+1]_q)\\
&= 1 + q\sum_{k=2}^{n+1}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k<n+1}\Pi_{j=1}^{k} f([i_j]_q)\\
&\quad + q\sum_{k=2}^{n+1}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k=n+1}\Pi_{j=1}^{k} f([i_j]_q)\\
&= 1 + q\sum_{k=2}^{n+1}(q-1)^{k-2}\sum_{1=i_1<\cdots<i_k\le n+1}\Pi_{j=1}^{k} f([i_j]_q).
\end{aligned}$$

□

Let us take $f(x) = x$. Then we obtain the following corollary.

Corollary 2.2. *For $n \in \mathbb{N}$, we have*

$$\left[\binom{n+1}{2}\right]_q = \oplus_{r=1}^{n}[r]_q = 1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k}[i_j]_q.$$

In the special case, $f(x) = x^2$, we obtain the following corollary.

Corollary 2.3. *For $n \in \mathbb{N}$, we have*

$$\oplus_{r=1}^{n}[r]_q^2 = 1 + q\sum_{k=2}^{n}(q-1)^{k-2}\sum_{1=i_1<i_2<\cdots<i_k\le n}\Pi_{j=1}^{k}[i_j]_q^2.$$

From (1.7) and (1.8), we have, for $x_1, \cdots, x_n \in \sum_q (cf.(1.3))$,

$$\otimes_{i=1}^{n}x_i = \frac{1}{q-1}\left(q^{\frac{\Pi_{i=1}^{n}\log(1+(q-1)x_i)}{(\log q)^n}} - 1\right), \tag{2.2}$$

and, for any $x_1, \cdots, x_n \in \mathbb{R}$,

$$\otimes_{i=1}^{n}[x_i]_q = [x_1x_2\cdots x_n]_q = \left[\Pi_{i=1}^{n}x_i\right]_q. \tag{2.3}$$

Thus, from (2.2) and (2.3), we have, for $x \in \sum_q$,

$$x^{\otimes n} = \underbrace{x\otimes\cdots\otimes x}_{n-times} = \frac{1}{q-1}\left(q^{\left(\frac{\log(1+(q-1)x)}{\log q}\right)^n} - 1\right), \tag{2.4}$$

and, for any $x \in \mathbb{R}$,

$$[x]_q^{\otimes n} = \underbrace{[x]_q\otimes\cdots\otimes[x]_q}_{n-times} = [x^n]_q. \tag{2.5}$$

Provided that $x + y \in \sum_q$, from (2.4) and (2.5) we have

$$(x+y)^{\otimes n} = \frac{1}{q-1}\left(q^{\left(\frac{\log(1+(q-1)(x+y))}{\log q}\right)^n} - 1\right), \tag{2.6}$$

and, for any $x, y \in \mathbb{R}$,

$$([x]_q \oplus [y]_q)^{\otimes n} = \left[(x+y)^n\right]_q. \tag{2.7}$$

Let us define a q-analogue of exponential function on $\mathbb{R}$ as follows:

$$e_q(x) = \lim_{n\to\infty}(1\oplus\left[\frac{x}{n}\right]_q)^{\otimes n}. \tag{2.8}$$

Then, by (2.7) and (2.8), we get

$$e_q(x) = \lim_{n\to\infty}\left[(1+\frac{x}{n})^n\right]_q = [e^x]_q. \tag{2.9}$$

From (1.13) and (2.9), we have

$$\begin{aligned} e_q(x) \otimes e_q(y) &= [e^x]_q \otimes [e^y]_q \\ &= [e^{x+y}]_q = e_q(x+y), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} e_q(x \oplus y) &= e_q(x + y + (q-1)xy) = [e^{x+y+(q-1)xy}]_q \\ &= [e^{x+y}e^{(q-1)xy}]_q = [e^{x+y}]_q \otimes [e^{(q-1)xy}]_q \\ &= [e^x]_q \otimes [e^y]_q \otimes [e^{(q-1)xy}]_q \\ &= e_q(x) \otimes e_q(y) \otimes e_q((q-1)xy). \end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following proposition.

Proposition 2.4. *For $x, y \in \mathbb{R}$, we have*

$$e_q(x \oplus y) = e_q(x) \otimes e_q(y) \otimes e_q((q-1)x),$$

and

$$e_q(x) \otimes e_q(y) = e_q(x+y).$$

Let us define a q-logarithm on $\sum_q^+ = \sum_q \cap \mathbb{R}^+ = \{x \in \mathbb{R} | 0 < x < \frac{1}{1-q}\}$ as follows:

$$\log_q x = \log\left(\frac{\log(1+(q-1)x)}{\log q}\right). \tag{2.12}$$

Then, by (2.12), we get

$$\log_q [x]_q = \log x \ (x > 0), \ \ \log_q(e_q(x)) = x \ (x \in \mathbb{R}). \tag{2.13}$$

It is easy to show that, for $x, y \in \sum_q^+$,

$$\log_q(x \otimes y) = \log_q x + \log_q y. \tag{2.14}$$

Note here that $x \otimes y \in \sum_q^+$, if $x, y \in \sum_q^+$. Therefore, by (2.13) and (2.14), we obtain the following proposition.

Proposition 2.5. *We have the following identities:*

$$\log_q(e_q(x)) = x \ (x \in \mathbb{R}), \ \ \log_q(x \otimes y) = \log_q x + \log_q y \qquad (x, y \in \Sigma_q^+).$$

3. Further Remarks

In this section, we use the following notations:

$$(a+b)_q^n = \Pi_{j=0}^n (a + q^j b), \ if \ n \in \mathbb{Z}_+ \tag{3.1}$$

$$(1+a)_q^t = \frac{(1+a)_q^\infty}{(1+q^t a)_q^\infty}, \ if \ t \in \mathbb{C}, \ (see[1,4]), \tag{3.2}$$

where q is a fixed real number with $0 < q < 1$.
The q-integral is defined as

$$\int_0^x f(t)d_qt = (1-q)x\sum_{k=0}^{\infty} f(q^kx)q^k, \ (see[3,4]). \tag{3.3}$$

The q-gamma function is defined as

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1}E_q(-qx)d_qx, \ (t>0), \tag{3.4}$$

where $E_q(x)$ is one of the q-analogues of exponential fucntion which is defined by

$$E_q(x) = (1+(1-q)x)_q^{\infty} = \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{[k]_q!}x^k, \tag{3.5}$$

where$[k]_q! = [k]_q[k-1]_q\cdots[2]_q[1]_q$.

As is well known, another q-exponential function is defined by

$$e_q(x) = \frac{1}{(1-(1-q)x)_q^{\infty}} = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}, \ (see[1,4]). \tag{3.6}$$

Thomae and Jackson have shown the q-beta function as follows:

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}, \ (t,s>0). \tag{3.7}$$

The q-integral representation, which is a q-analogue of Euler's formula, is given by

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}d_qx. \tag{3.8}$$

From (3.4), we note that

$$\Gamma_q(t) = \frac{(1-q)_q^{t-1}}{(1-q)^{t-1}}, \ [t]_q\Gamma(t) = \Gamma_q(t+1), \ (see[3,4]). \tag{3.9}$$

From (3.4), we have

$$\begin{aligned}
\Gamma_q(\frac{1}{2}) &= \int_0^{\frac{1}{1-q}} x^{-\frac{1}{2}}E_q(-qx)d_qx \\
&= \sum_{n=0}^{\infty} q^n\left(\frac{q^n}{1-q}\right)^{-\frac{1}{2}} E_q\left(-\frac{q^{n+1}}{1-q}\right) \\
&= \sqrt{1-q}\sum_{n=0}^{\infty} q^{\frac{n}{2}}(1-q^{n+1})_q^{\infty} \\
&= \sqrt{1-q}(1-q)_q^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n}{2}}}{(1-q)_q^n} \\
&= \sqrt{1-q}(1-q)_q^{\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n}{2}}}{[n]_q!}\left(\frac{1}{1-q}\right)^n \\
&= \sqrt{1-q}E_q\left(\frac{q}{q-1}\right)e_q\left(\frac{q^{\frac{1}{2}}}{1-q}\right).
\end{aligned} \tag{3.10}$$

Therefore, by (3.10), we obtain the following proposition.

Proposition 3.1. *For* $0 < q < 1$, *we have*

$$\Gamma_q(\frac{1}{2}) = \sqrt{1-q}E_q\left(\frac{q}{q-1}\right)e_q\left(\frac{q^{\frac{1}{2}}}{1-q}\right).$$

Note that

$$\lim_{q\to 1-}\sqrt{1-q}E_q\left(\frac{q}{q-1}\right)e_q\left(\frac{q^{\frac{1}{2}}}{1-q}\right) = \sqrt{\pi}.$$

We note that

$$(1-x)_q^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k, \tag{3.11}$$

and

$$\frac{1}{(1-x)_q^n} = \sum_{k=0}^{n}\begin{bmatrix} k+n-1 \\ k \end{bmatrix}_q x^k, \tag{3.12}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ (*see* [1, 3, 4]).

From (3.12), we have

$$\begin{aligned}
\frac{1}{(1-x)_q^{\frac{1}{2}}} &= \sum_{k=0}^{\infty}\begin{bmatrix} \frac{1}{2}+k-1 \\ k \end{bmatrix}_q x^k = \sum_{k=0}^{\infty}\begin{bmatrix} k-\frac{1}{2} \\ k \end{bmatrix}_q x^k \\
&= \sum_{k=0}^{\infty}\frac{[k-\frac{1}{2}]_q[k-\frac{3}{2}]_q\cdots\left[\frac{1}{2}\right]_q}{[k]_q!}x^k \\
&= \sum_{k=0}^{\infty}\frac{[k-\frac{1}{2}]_q[k-\frac{3}{2}]_q\cdots\left[\frac{1}{2}\right]_q\Gamma_q\left[\frac{1}{2}\right]}{[k]_q!\Gamma_q\left(\frac{1}{2}\right)}x^k \\
&= \sum_{k=0}^{\infty}\frac{\Gamma_q(1+k-\frac{1}{2})}{[k]_q!\Gamma_q\left(\frac{1}{2}\right)}x^k = \sum_{k=0}^{\infty}\frac{\Gamma_q(k+\frac{1}{2})}{\Gamma_q(k+1)\Gamma_q\left(\frac{1}{2}\right)}x^k \\
&= \sum_{k=0}^{\infty}\frac{1}{[k]_q}\frac{1}{B_q(k,\frac{1}{2})}.
\end{aligned} \tag{3.13}$$

It is not difficult to show that

$$\begin{aligned}
&\frac{[k-\frac{1}{2}]_q[k-\frac{3}{2}]_q\cdots\left[\frac{1}{2}\right]_q}{[k]_q!} = \frac{[2k]_{q^{\frac{1}{2}}}!}{([k]_q!)^2([2]_{q^{\frac{1}{2}}})^{2k}} \\
&= \frac{\Gamma_{q^{\frac{1}{2}}}(2k+1)}{(\Gamma_q(k+1))^2([2]_{q^{\frac{1}{2}}})^{2k}}.
\end{aligned} \tag{3.14}$$

Thus, by (3.12) and (3.14), we get

$$\frac{1}{(1-x)_q^{\frac{1}{2}}}=\sum_{k=0}^{\infty}\frac{\Gamma_{q^{\frac{1}{2}}}(2k+1)}{(\Gamma_q(k+1))^2([2]_{q^{\frac{1}{2}}})^{2k}}x^k. \tag{3.15}$$

Therefore, by (3.13) and (3.15), we obtain the following proposition.

Proposition 3.2. *For $k \geq 0$, we have*

$$[k]_q B_q(k,\frac{1}{2})=\frac{(\Gamma_q(k+1))^2([2]_{q^{\frac{1}{2}}})^{2k}}{\Gamma_{q^{\frac{1}{2}}}(2k+1)}.$$

Acknowledgements

The authors would like to express their sincere gratitude to Professor Toufik Mansour for his valuable comments. Our thanks also go to the referee for his valuable suggestions.

References

[1] R. Agarwal and V. Gupta, *On q-analogue of complex summation-integral type operators in compact disks*, J. Ineq. Appl. (2012), 2012:111,13pp.
[2] W. S. Chung, T. Kim and T. Mansour, *The q-deformed gamma function and q-deformed polygamma function*, Bull. Korean Math. Soc., **51** (2014), 1155–1161.
[3] N. K. Govil and V. Gupta, *q-Beta-Szász-Stancu operators*, Adv, Stud. Contemp, Math, **22** (2012), 117-123.
[4] V. Gupta and T. Kim, *On the rate of approximation by q-modified beta operators*, J.Math. Anal, Appl., **377** (2011), 471-480.
[5] D. S. Kim and T. Kim, *q-Bernoulli polynomials and q-umbral calculus*, Sci. China Math., **57** (2014), 1867–1874.
[6] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inform. Sci. **5** (2011), 390–444.
[7] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[8] H. M. Srivastava, Z. Finta and V. Gupta, *Direct results for a certain family of summation-integral type operators*, Appl. Math. Comput. **1**90 (2007), 449–447.
[9] H. M. Srivastava and V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Modelling **37** (2003), 1307–1315.