Filomat 31:6 (2017), 1583–1593
DOI 10.2298/FIL1706583D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Hölder's Means and Absolute Normalized Norms on $\mathbb{R}^2$

**Mina Dinarvand**[a]

[a] *Faculty of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran*

**Abstract.** Recently, Cui and Lu have introduced the constant $H_p(X)$ ($p \in \mathbb{R}$) of a Banach space $X$ by using Hölder's means. In this paper, we determine and estimate the new constant under the absolute normalized norms on $\mathbb{R}^2$ by means of their corresponding continuous convex functions on $[0,1]$. Furthermore, the exact values of the constant are calculated in some concrete Banach spaces. In particular, we calculate the precise values of the constants $A_2(X)$ and $T(X)$ and the Gao constant $E(X)$ in these concrete spaces.

## 1. Introduction

There are many geometric constants defined on Banach spaces which play an important role in the description of various geometric structures of Banach spaces, and have been investigated in many papers. In particular, many authors have calculated and estimated constants for Banach spaces. It has been shown that these constants are very useful in geometric theory of Banach spaces, which enable us to classify several important concepts of Banach spaces, such as uniformly non-squareness and uniform normal structure. On the other hand, calculation of the constant for some concrete spaces attracted the interest of several authors, and many papers on this topic have appeared.

Recently, geometrical properties of absolute normalized norms have been studied by many authors. In [18] Saito, Kato and Takahashi calculated and estimated the von Neumann-Jordan constant of $\mathbb{C}^2$ with absolute normalized norms. The results obtained in [18] also hold on $\mathbb{R}^2$. In [14] Mitani and Saito computed the James constant for absolute normalized norms on $\mathbb{R}^2$. For further works and results concerning geometrical constants of $\mathbb{R}^2$ or $\mathbb{C}^2$ equipped with an absolute normalized norm, the reader can see, e.g., [8, 16, 17, 24].

By using Hölder's means, the constant $H_p(X)$ of a Banach space $X$ was introduced by Cui and Lu recently. The main purpose of this paper is to determine the new constant $H_p(X)$ under the absolute normalized norms on $\mathbb{R}^2$. Thereafter, as an application, we calculate the exact values of this new constant in some concrete Banach spaces. In particular, the precise values Baronti-Casini-Papini's constant $A_2(X)$, Alonso-Llorens-Fuster's constant $T(X)$ and Gao's constant $E(X)$ are calculated in these concrete spaces.

2010 *Mathematics Subject Classification*. Primary 46B20; Secondary 46B25
*Keywords*. absolute normalized norm, convex function, Hölder's mean, Gao constant, Lorentz sequence space, Cesàro sequence space
Received: 10 April 2015; Accepted: 02 September 2015
Communicated by Dragan S. Djordjević
*Email address:* dinarvand_mina@yahoo.com (Mina Dinarvand)

## 2. Preliminaries

We commence by reviewing some notations and basic concepts which are needed in the sequel.

Throughout this paper we assume that $X$ stands for a real nontrivial Banach space, that is, $\dim(X) \geq 2$. We denote the unit sphere and the closed unit ball of a Banach space $X$ by $S_X$ and $B_X$, respectively.

It can be recalled that the modulus of convexity of a Banach space $X$ is defined for $\varepsilon \in [0,2]$ as

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, \ \|x-y\| = \varepsilon\right\}.$$

The function $\delta_X(\varepsilon)$ is continuous on $[0,2)$, increasing on $[0,2]$, and strictly increasing on $[\epsilon_0(X), 2]$, where $\epsilon_0(X) = \sup\{\varepsilon \in [0,2] \ : \ \delta_X(\varepsilon) = 0\}$ is the so-called characteristic (or coefficient) of convexity of $X$.

The James constant of $X$ was defined by Gao and Lau [11] as

$$J(X) = \sup\left\{\min\left(\|x+y\|, \|x-y\|\right) \ : \ x, y \in S_X\right\}.$$

We recall that Hölder's means (also called power means) between two positive numbers $a$ and $b$ are defined by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \quad \text{for } p \neq 0,$$

$$M_0(a,b) = \lim_{p \to 0} M_p(a,b) = \sqrt{ab}.$$

In particular, the arithmetic mean $A := M_1$ and the geometric mean $G := M_0$ are well known. We should note that Hölders means are symmetric and positively homogeneous, that is,

$$M_p(a,b) = M_p(b,a) \qquad \text{and} \qquad M_p(ta, tb) = tM_p(a,b) \qquad t \geq 0.$$

For two real numbers $p \leq q$,

$$\min(a,b) \leq M_p(a,b) \leq M_q(a,b) \leq \max(a,b),$$

where " = " holds only for the case $a = b$.

In a recent paper [6], Cui and Lu have introduced the constant

$$H_p(X) = \sup\left\{M_p\left(\|x+y\|, \|x-y\|\right) \ : \ x, y \in S_X\right\},$$

for a real number $p$, by considering Hölder's means of $\|x+y\|$ and $\|x-y\|$. It is obvious that the constant $H_p(X)$ include some known constants, such as Baronti-Casini-Papini's constant $A_2(X)$ (see [2]), Alonso-Llorens-Fuster's constant $T(X)$ (see [1]) and Gao's constant $E(X)$ (see [10]). These constants are defined by $A_2(X) = H_1(X)$, $T(X) = H_0(X)$ and $E(X) = 2H_2^2(X)$. According to the above, the following constants:

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} \ : \ x, y \in S_X\right\},$$

$$T(X) = \sup\left\{\sqrt{\|x+y\| \, \|x-y\|} \ : \ x, y \in S_X\right\},$$

$$E(X) = \sup\left\{\|x+y\|^2 + \|x-y\|^2 \ : \ x, y \in S_X\right\},$$

will be considered in this paper.

From [6], we have $\sqrt{2} \leq H_p(X) \leq 2$ for any Banach space $X$. Clearly, if $p \leq q$, then $H_p(X) \leq H_q(X)$. It is known that $H_p(X) < 2$ if and only if $X$ is uniformly non-square, that is, there exists $\delta > 0$ such that $x, y \in S_X$ and $\|x-y\| \geq 2(1-\delta)$ imply $\|x+y\| \leq 2(1-\delta)$. It has been shown in [6] that if $p \leq 2$, then $H_p(\ell_r) = \max\{2^r, 2^{1-\frac{1}{r}}\}$ for any $r \geq 1$, and if $2 \leq p \leq r$, then $H_p(\ell_r) = 2^{1-\frac{1}{r}}$.

We recall some properties of absolute normalized norms on $\mathbb{R}^2$. The readers interested in this topic are referred to [3, 19, 24]. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if

$$\|(x,y)\| = \|(|x|,|y|)\| \quad \text{for all } (x,y)\in\mathbb{R}^2,$$

and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. An absolute normalized norm on $\mathbb{R}^2$ is a norm that is both absolute and normalized. A simple example of absolute normalized norm is usual $\ell_r$-norms $\|\cdot\|_r$ $(1\le r\le\infty)$ on $\mathbb{R}^2$ defined by

$$\|(x,y)\|_r = \begin{cases} \left(|x|^r+|y|^r\right)^{\frac{1}{r}}, & 1\le r<\infty,\\ \max\{|x|,|y|\}, & p=\infty.\end{cases}$$

Let $N_\alpha$ be the set of all absolute normalized norms on $\mathbb{R}^2$, and let $\Psi$ be the set of all continuous convex functions $\psi$ on $[0,1]$ such that $\psi(0)=\psi(1)=1$ and $\max\{1-t,t\}\le\psi(t)\le 1$ for all $t\in[0,1]$. For $\varphi,\psi\in\Psi$, we denote $\varphi\le\psi$ if $\varphi(t)\le\psi(t)$ for all $0\le t\le 1$. As in [3], it is well known that $N_\alpha$ and $\Psi$ can be identified by a one to one correspondence $\psi\to\|\cdot\|_\psi$ with the equation

$$\psi(t) = \|(1-t,t)\|_\psi \quad (0\le t\le 1). \tag{1}$$

Indeed, for any $\|\cdot\|\in N_\alpha$, the function $\psi(t)=\|(1-t,t)\|$ belongs to $\Psi$. Conversely, for all $\psi\in\Psi$, we define the norm $\|\cdot\|_\psi$ as

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\,\psi\left(\frac{|y|}{|x|+|y|}\right), & (x,y)\ne(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

Then $\|\cdot\|_\psi\in N_\alpha$, and $\|\cdot\|_\psi$ satisfies (1).

We recall that an absolute normalized norm $\|\cdot\|$ on $\mathbb{R}^2$ is symmetric in the sense that $\|(x,y)\|=\|(y,x)\|$ for all $x,y\in\mathbb{R}$ if and only if the corresponding function $\psi$ is symmetric with respect to $t=\frac{1}{2}$, that is, $\psi(1-t)=\psi(t)$ for all $t\in[0,1]$ (see [18]).

For $1\le r\le\infty$, the function $\psi_r$ corresponding to the $\ell_r$-norms $\|\cdot\|_r$ on $\mathbb{R}^2$ are given by

$$\psi_r(t) = \begin{cases} \left\{(1-t)^r+t^r\right\}^{\frac{1}{r}}, & 1\le r<\infty,\\ \max\{1-t,t\}, & r=\infty.\end{cases}$$

Then $\psi_r\in\Psi$ and, as is easily seen, the $\ell_r$-norm $\|\cdot\|_r$ is associated with $\psi_r$. Also, the above correspondence enables us to get many non-$\ell_r$-norms on $\mathbb{R}^2$. One of the properties of these norms is stated in the following result.

**Proposition 2.1.** ([18]) *Let $\varphi,\psi\in\Psi$ and $\varphi\le\psi$. Put $N=\max_{0\le t\le 1}\frac{\psi(t)}{\varphi(t)}$, then*

$$\|\cdot\|_\varphi \le \|\cdot\|_\psi \le N\|\cdot\|_\varphi.$$

## 3. Absolute Normalized Norm

First, let us show some basic properties related to the constant $H_p(X)$.

The following theorem can be found in [6].

**Theorem 3.1.** ([6]) *For any Banach space $X$,*

$$H_p(X) = \begin{cases} \sup\left\{\left(\frac{\varepsilon^p+2^p(1-\delta_X(\varepsilon))^p}{2}\right)^{\frac{1}{p}} \ : \ 0\le\varepsilon\le 2\right\}, & p\ne 0,\\ \sup\{\sqrt{2\varepsilon(1-\delta_X(\varepsilon))} \ : \ 0\le\varepsilon\le 2\}, & p=0.\end{cases}$$

**Corollary 3.2.** *Let H be a Hilbert space. Then* $H_p(H) = \max\{\sqrt{2}, 2^{1-\frac{1}{p}}\}$.

*Proof.* Since $H$ is a Hilbert space, it follows that $\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$. By using Theorem 3.1, we have

$$H_p(H) = \sup\left\{\left(\frac{\varepsilon^p + 2^p\left(1 - \frac{\varepsilon^2}{4}\right)^{\frac{p}{2}}}{2}\right)^{\frac{1}{p}} \;:\; 0 \le \varepsilon \le 2\right\},$$

where $p \neq 0$.

Now, by letting $f(\varepsilon) = \left(\frac{\varepsilon^p + 2^p\left(1 - \frac{\varepsilon^2}{4}\right)^{\frac{p}{2}}}{2}\right)^{\frac{1}{p}}$, we can consider the following two cases, (i) and (ii). By calculating, we obtain

(i) If $p \le 2$, then $f_{\max} = f(\sqrt{2}) = \sqrt{2}$, and if $p = 0$, we can also get the same result.

(ii) If $p \ge 2$, then $f_{\max} = f(2) = 2^{1-\frac{1}{p}}$.

Therefore, $H_p(H) = \max\{\sqrt{2}, 2^{1-\frac{1}{p}}\}$. □

**Remark 3.3.** *For any Banach space X, we have*

$$H_p(X) \ge H_p(H) \ge \max\{\sqrt{2}, 2^{1-\frac{1}{p}}\}.$$

*Note that the first inequality holds because* $\delta_X(\varepsilon) \le \delta_H(\varepsilon)$.

**Theorem 3.4.** *For any Banach space X,* $H_p(X) \le M_p(J(X), 2)$.

*Proof.* Let $p \neq 0$. Since

$$\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{\frac{1}{p}} \le \left(\frac{\min^p\left(\|x+y\|, \|x-y\|\right) + 2^p}{2}\right)^{\frac{1}{p}},$$

it follows that $H_p(X) \le \left(\frac{J(X)^p + 2^p}{2}\right)^{\frac{1}{p}}$. If $p = 0$, we can also get the same result. Hence, we have $H_p(X) \le M_p(J(X), 2)$. □

Next, we give a simple method to determine and estimate the constant $H_p(X)$ of a class of absolute normalized norms on $\mathbb{R}^2$. For a norm $\|\cdot\|$ on $\mathbb{R}^2$, we write $H_p(\|\cdot\|)$ for $H_p(\mathbb{R}^2, \|\cdot\|)$.

The following Proposition is a direct result of [6, Theorem 4.1].

**Proposition 3.5.** *Let X be a Banach space. Then*

$$H_p(X) = \sup\left\{\frac{M_p\left(\|x+y\|, \|x-y\|\right)}{\max(\|x\|, \|y\|)} \;:\; x, y \in X,\ \|x\| + \|y\| \neq 0\right\}.$$

**Proposition 3.6.** *Let* $\varphi \in \Psi$ *and* $\psi(t) = \varphi(1-t)$. *Then* $H_p(\|\cdot\|_\varphi) = H_p(\|\cdot\|_\psi)$.

*Proof.* For any $x = (a, b) \in \mathbb{R}^2$ and $a \neq 0, b \neq 0$, put $\widetilde{x} = (b, a)$. Then

$$\|x\|_\varphi = (|a| + |b|)\,\varphi\left(\frac{|b|}{|a| + |b|}\right) = (|b| + |a|)\,\psi\left(\frac{|a|}{|b| + |a|}\right) = \|\widetilde{x}\|_\psi.$$

Consequently, we have

$$\begin{aligned} H_p(\|\cdot\|_\varphi) &= \sup\left\{\frac{M_p\Big(\|x+y\|_\varphi, \|x-y\|_\varphi\Big)}{\max(\|x\|_\varphi, \|y\|_\varphi)} \ : \ x, y \in X,\ \|x\|_\varphi + \|y\|_\varphi \neq 0\right\} \\ &= \sup\left\{\frac{M_p\Big(\|\widetilde{x}+\widetilde{y}\|_\psi, \|\widetilde{x}-\widetilde{y}\|_\psi\Big)}{\max(\|\widetilde{x}\|_\psi, \|\widetilde{y}\|_\psi)} \ : \ \widetilde{x}, \widetilde{y} \in X,\ \|\widetilde{x}\|_\psi + \|\widetilde{y}\|_\psi \neq 0\right\} \\ &= H_p(\|\cdot\|_\psi). \end{aligned}$$

□

We now consider the constant $H_p(X)$ of a class of absolute normalized norms on $\mathbb{R}^2$. Now, let us put

$$N_1 = \max_{0\le t\le 1} \frac{\psi_r(t)}{\psi(t)} \qquad \text{and} \qquad N_2 = \max_{0\le t\le 1} \frac{\psi(t)}{\psi_r(t)}.$$

**Theorem 3.7.** *Let $\psi \in \Psi$ and $\psi \le \psi_r$ $(2 \le r < \infty)$. If the function $\Phi(t) := \frac{\psi_r(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $r \ge p$, then*

$$H_p(\|\cdot\|_\psi) = \frac{1}{\psi\left(\frac{1}{2}\right)}.$$

*In particular,*

$$A_2(\|\cdot\|_\psi) = T(\|\cdot\|_\psi) = \frac{1}{\psi\left(\frac{1}{2}\right)} \qquad \textit{and} \qquad E(\|\cdot\|_\psi) = \frac{2}{\psi^2\left(\frac{1}{2}\right)}.$$

*Proof.* By applying Proposition 2.1, we have $\|\cdot\|_\psi \le \|\cdot\|_r \le N_1\|\cdot\|_\psi$. Let $x, y \in X$, $(x, y) \neq (0, 0)$, where $X = \mathbb{R}^2$. Thus, we obtain

$$\begin{aligned} M_p\Big(\|x+y\|_\psi, \|x-y\|_\psi\Big) &\le M_p\Big(\|x+y\|_r, \|x-y\|_r\Big) \\ &\le H_p(\|\cdot\|_r) \max\Big\{\|x\|_r, \|y\|_r\Big\} \\ &\le H_p(\|\cdot\|_\psi) N_1 \max\Big\{\|x\|_\psi, \|y\|_\psi\Big\}, \end{aligned}$$

which, from the equivalent definition of $H_p(X)$, implies that

$$H_p(\|\cdot\|_\psi) \le H_p(\|\cdot\|_r) N_1.$$

Note that $r \ge p$ and the function $\Phi(t) = \frac{\psi_r(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$, i.e., $N_1 = \frac{\psi_r(\frac{1}{2})}{\psi(\frac{1}{2})}$. Hence, we have

$$H_p(\|\cdot\|_\psi) \le H_p(\|\cdot\|_r) N_1 = \frac{1}{\psi\left(\frac{1}{2}\right)}. \tag{2}$$

On the other hand, by taking the points $x = (a, a)$ and $y = (a, -a)$ where $\frac{1}{a} = 2\psi\left(\frac{1}{2}\right)$, we obtain $\|x\|_\psi = \|y\|_\psi = 1$ and

$$M_p\Big(\|x+y\|_\psi, \|x-y\|_\psi\Big) = 2a = \frac{1}{\psi\left(\frac{1}{2}\right)}. \tag{3}$$

Therefore, by applying (2) and (3), we have

$$H_p(\|\cdot\|_\psi) = \frac{1}{\psi\left(\frac{1}{2}\right)}.$$

□

**Theorem 3.8.** *Let $\psi \in \Psi$ and $\psi \geq \psi_r$ $(1 \leq r \leq 2)$. If the function $\Phi(t) := \frac{\psi(t)}{\psi_r(t)}$ attains its maximum at $t = \frac{1}{2}$ and $p \leq 2$, then*

$$H_p(\|\cdot\|_\psi) = 2\psi\Big(\frac{1}{2}\Big).$$

*In particular,*

$$A_2(\|\cdot\|_\psi) = T(\|\cdot\|_\psi) = 2\psi\Big(\frac{1}{2}\Big) \qquad \text{and} \qquad E(\|\cdot\|_\psi) = 8\psi^2\Big(\frac{1}{2}\Big).$$

*Proof.* By applying Proposition 2.1, we have $\|\cdot\|_r \leq \|\cdot\|_\psi \leq N_2\|\cdot\|_r$. Let $x, y \in X$, $(x, y) \neq (0, 0)$, where $X = \mathbb{R}^2$. Hence, we obtain

$$\begin{aligned} M_p\Big(\|x+y\|_\psi, \|x-y\|_\psi\Big) &\leq N_2 M_p\Big(\|x+y\|_r, \|x-y\|_r\Big) \\ &\leq H_p(\|\cdot\|_r) N_2 \max\Big\{\|x\|_r, \|y\|_r\Big\} \\ &\leq H_p(\|\cdot\|_\psi) N_2 \max\Big\{\|x\|_\psi, \|y\|_\psi\Big\}, \end{aligned}$$

which, from the equivalent definition of $H_p(X)$, implies that

$$H_p(\|\cdot\|_\psi) \leq H_p(\|\cdot\|_r) N_2.$$

Note that $p \leq 2$ and the function $\Phi(t) = \frac{\psi(t)}{\psi_r(t)}$ attains its maximum at $t = \frac{1}{2}$, i.e., $N_2 = \frac{\psi(\frac{1}{2})}{\psi_r(\frac{1}{2})}$. Thus, we have

$$H_p(\|\cdot\|_\psi) \leq H_p(\|\cdot\|_r) N_2 = 2\psi\Big(\frac{1}{2}\Big). \tag{4}$$

On the other hand, by taking the points $x = (1, 0)$ and $y = (0, 1)$, we obtain $\|x\|_\psi = \|y\|_\psi = 1$ and

$$M_p\Big(\|x+y\|_\psi, \|x-y\|_\psi\Big) = 2\psi\Big(\frac{1}{2}\Big). \tag{5}$$

Therefore, by applying (4) and (5), we have

$$H_p(\|\cdot\|_\psi) = 2\psi\Big(\frac{1}{2}\Big).$$

□

## 4. Some Concrete Banach Spaces

As an application of Theorems 3.7 and 3.8, we present various examples by easy arguments which enable us to calculate the exact values of the constant $H_p(X)$ for some concrete Banach spaces.

To show the results, we make use of the following useful lemma.

**Lemma 4.1.** ([6]) *Let $\|\cdot\|$ and $|\cdot|$ be two equivalent norms on a Banach space $X$. If*

$$a|\cdot| \leq \|\cdot\| \leq b|\cdot| \quad (b \geq a > 0),$$

*then*

$$\frac{a}{b} H_p(|\cdot|) \leq H_p(\|\cdot\|) \leq \frac{b}{a} H_p(|\cdot|).$$

*Moreover, if $\|\cdot\| = a|\cdot|$, then $H_p(|\cdot|) = H_p(\|\cdot\|)$.*

**Example 4.2.** Let $X = \mathbb{R}^2$ endowed with the norm

$$\|x\| = \max\left\{\|x\|_2, \lambda\|x\|_1\right\} \quad \Big(\frac{1}{\sqrt{2}} \le \lambda \le 1\Big).$$

Then

$$H_p(\|\cdot\|) = 2\lambda \quad (p \le 2).$$

In particular,

$$A_2(\|\cdot\|) = T(\|\cdot\|) = 2\lambda \qquad \text{and} \qquad E(\|\cdot\|) = 8\lambda^2.$$

*Proof.* It is easy to see that $\|\cdot\| = \max\{\|\cdot\|_2, \lambda\|\cdot\|_1\} \in N_\alpha$ and so its corresponding function is

$$\psi(t) = \|(1-t,t)\| = \max\{\psi_2(t), \lambda\} \ge \psi_2(t).$$

Thus, we have

$$\Phi(t) := \frac{\psi(t)}{\psi_2(t)} = \max\left\{1, \frac{\lambda}{\psi_2(t)}\right\}.$$

Note that $\psi_2(t)$ attains minimum at $t = \frac{1}{2}$, from which it follows that $\Phi(t)$ attains maximum at $t = \frac{1}{2}$. Therefore, by applying Theorem 3.8, we have

$$H_p(\|\cdot\|) = 2\psi\Big(\frac{1}{2}\Big) = 2\lambda.$$

□

**Example 4.3.** Let $X = \mathbb{R}^2$ endowed with the norm

$$\|x\| = \max\left\{\|x\|_2, \lambda\|x\|_\infty\right\} \quad (1 \le \lambda \le \sqrt{2}).$$

Then

$$H_p(\|\cdot\|) = \sqrt{2}\lambda \quad (p \le 2).$$

In particular,

$$A_2(\|\cdot\|) = T(\|\cdot\|) = \sqrt{2}\lambda \qquad \text{and} \qquad E(\|\cdot\|) = 4\lambda^2.$$

*Proof.* It is obvious that the norm $\|\cdot\| = \max\{\|\cdot\|_2, \lambda\|\cdot\|_\infty\}$ is absolute, but not normalized since $\|(1,0)\| = \|(0,1)\| = \lambda$. Now, let us put

$$|\cdot| = \frac{\|\cdot\|}{\lambda} = \max\left\{\frac{\|\cdot\|_2}{\lambda}, \|\cdot\|_\infty\right\},$$

which implies that $|\cdot| \in N_\alpha$ and its corresponding function is

$$\psi(t) = |(1-t,t)| = \max\left\{\frac{\psi_2(t)}{\lambda}, \psi_\infty(t)\right\} \le \psi_2(t).$$

Thus, we have

$$\Phi(t) := \frac{\psi_2(t)}{\psi(t)} = \min\left\{\lambda, \frac{\psi_2(t)}{\psi_\infty(t)}\right\}.$$

Consider the increasing continuous function $f(t) := \frac{\psi_2(t)}{\psi_\infty(t)}$ $(0 \le t \le \frac{1}{2})$. Since $f(0) = 1$ and $f(\frac{1}{2}) = \sqrt{2}$, there exists a unique point $0 \le t_0 \le \frac{1}{2}$ such that $f(t_0) = \lambda$. Hence, from the fact that $\Phi(t)$ is symmetric with respect to $t = \frac{1}{2}$, we get

$$\Phi(t) = \begin{cases} \frac{\psi_2(t)}{\psi_\infty(t)}, & t \in [0,t_0] \cup [1-t_0,1], \\ \lambda, & t \in [t_0, 1-t_0]. \end{cases}$$

Note that $\Phi(t)$ attains its maximum at $t = \frac{1}{2}$. Therefore, by applying Theorem 3.7 and Lemma 4.1, we have

$$H_p(\|\cdot\|) = H_p(|\cdot|) = \frac{1}{\psi(\frac{1}{2})} = \sqrt{2}\lambda.$$

□

**Example 4.4.** Let $X = \mathbb{R}^2$ endowed with the norm

$$|x|_\lambda = \left(\|x\|_2^2 + \lambda\|x\|_\infty^2\right)^{\frac{1}{2}} \quad (\lambda > 0).$$

Then

$$H_p(|\cdot|_\lambda) = 2\sqrt{\frac{\lambda+1}{\lambda+2}} \quad (p \le 2).$$

In particular,

$$A_2(|\cdot|_\lambda) = T(|\cdot|_\lambda) = 2\sqrt{\frac{\lambda+1}{\lambda+2}} \qquad \text{and} \qquad E(|\cdot|_\lambda) = \frac{8(\lambda+1)}{\lambda+2}.$$

*Proof.* It is easy to see that the norm $|\cdot|_\lambda = \left(\|\cdot\|_2^2 + \lambda\|\cdot\|_\infty^2\right)^{\frac{1}{2}}$ is absolute, but not normalized because $|(1,0)|_\lambda = |(0,1)|_\lambda = \sqrt{1+\lambda}$. Now, let us put

$$\|\cdot\| = \frac{|\cdot|_\lambda}{\sqrt{1+\lambda}},$$

which implies that $\|\cdot\| \in N_\alpha$ and its corresponding function is

$$\psi(t) = \begin{cases} [(1-t)^2 + t^2/(1+\lambda)]^{\frac{1}{2}}, & t \in [0, \frac{1}{2}], \\ [t^2 + (1-t)^2/(1+\lambda)]^{\frac{1}{2}}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, $\psi(t) \le \psi_2(t)$. Since $\lambda > 0$, it follows that $\Phi(t) := \frac{\psi_2(t)}{\psi(t)}$ is symmetric with respect to $t = \frac{1}{2}$ and then it suffices to consider $\Phi(t)$ for $t \in [0, \frac{1}{2}]$. Note that for any $t \in [0, \frac{1}{2}]$,

$$\Phi^2(t) = \frac{(1+\lambda)[(1-t)^2 + t^2]}{(1+\lambda)(1-t)^2 + t^2} = \frac{u(t)}{v(t)}.$$

Some elementary computation show that $u(t) - v(t) = \lambda t^2$ attains its maximum at $t = \frac{1}{2}$ and $v(t) = (1+\lambda)(1-t)^2 + t^2$ attains its minimum at $t = \frac{1}{2}$. This implies that

$$\Phi^2(t) = \frac{u(t) - v(t)}{v(t)} + 1,$$

attains its maximum at $t = \frac{1}{2}$ and so does $\Phi(t)$. Therefore, by applying Theorem 3.7 and Lemma 4.1, we have

$$H_p(|\cdot|_\lambda) = H_p(\|\cdot\|) = \frac{1}{\psi(\frac{1}{2})} = 2\sqrt{\frac{\lambda+1}{\lambda+2}}.$$

□

In the next example, we calculate the constant $H_p(X)$ in the case when $X$ is a two-dimensional Lorentz sequence space $d^{(2)}(w,q)$ with $2 \le q < \infty$.

Let $w = (w_1, w_2)$ with $w_1 \geq w_2 > 0$. For $2 \leq q < \infty$, the two-dimensional Lorentz sequence space $d^{(2)}(w, q)$ is defined as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x, y)\|_{w,q} = \left\{ w_1 |x|^{*q} + w_2 |y|^{*q} \right\}^{\frac{1}{q}},$$

where $(|x|^*, |y|^*)$ is the rearrangement of $(|x|, |y|)$ satisfying $|x|^* \geq |y|^*$. For more detailed discussion and some results concerning Lorentz sequence spaces, we refer the reader to [12, 15, 22].

**Example 4.5.** (Lorentz sequence space). Let $X$ be a two-dimensional Lorentz space $d^{(2)}(w, q)$. Then

$$H_p\Big(\|\cdot\|_{w,q}\Big) = 2\Big(\frac{w_1}{w_1 + w_2}\Big)^{\frac{1}{q}} \quad (p \leq q).$$

In particular,

$$A_2\Big(\|\cdot\|_{w,q}\Big) = T\Big(\|\cdot\|_{w,q}\Big) = 2\Big(\frac{w_1}{w_1 + w_2}\Big)^{\frac{1}{q}} \qquad \text{and} \qquad E\Big(\|\cdot\|_{w,q}\Big) = 8\Big(\frac{w_1}{w_1 + w_2}\Big)^{\frac{2}{q}}.$$

*Proof.* It is easy to check that $|\cdot| = (\|\cdot\|_{w,q})/w_1^{\frac{1}{q}} \in N_\alpha$, and its corresponding convex function is given by

$$\psi(t) = \begin{cases} \Big((1-t)^q + \frac{w_2}{w_1} t^q\Big)^{\frac{1}{q}}, & t \in [0, \frac{1}{2}], \\ \Big(t^q + \frac{w_2}{w_1}(1-t)^q\Big)^{\frac{1}{q}}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, $\psi(t) \leq \psi_q(t)$. Since $\Phi(t) := \frac{\psi_q(t)}{\psi(t)}$ is symmetric with respect to $t = \frac{1}{2}$, it then suffices to consider $\Phi(t)$ for $t \in [0, \frac{1}{2}]$. Note that for any $t \in [0, \frac{1}{2}]$,

$$\Phi^q(t) = \frac{\psi_q^q(t)}{\psi^q(t)} = \frac{(1-t)^q + t^q}{(1-t)^q + \frac{w_2}{w_1} t^q} = \frac{u(t)}{v(t)}.$$

Some elementary computation show that $u(t) - v(t) = \Big(1 - \big(\frac{w_2}{w_1}\big)\Big) t^q$ attains its maximum at $t = \frac{1}{2}$ and $v(t) = (1-t)^q + \frac{w_2}{w_1} t^q$ attains its minimum at $t = \frac{1}{2}$. This implies that

$$\Phi^q(t) = \frac{u(t) - v(t)}{v(t)} + 1,$$

attains its maximum at $t = \frac{1}{2}$ and so does $\Phi(t)$. Therefore, by applying Theorem 3.7 and Lemma 4.1, we have

$$H_p\Big(\|\cdot\|_{w,q}\Big) = H_p(|\cdot|) = \frac{1}{\psi(\frac{1}{2})} = 2\Big(\frac{w_1}{w_1 + w_2}\Big)^{\frac{1}{q}}.$$

□

We now compute the constant $H_p(X)$ in the case when $X$ is a two-dimensional Cesàro sequence space $\mathrm{ces}_2^{(2)}$. The Cesàro sequence space was defined by Shue [21] in 1970. It is very useful in the theory of matrix operators and others.

Let $\ell$ be the space of real sequences. For $1 < p < \infty$, the Cesàro sequence space $\mathrm{ces}_p$ is defined by

$$\mathrm{ces}_p = \Big\{ x \in \ell \, : \, \|x\| = \Big\|\big(x(i)\big)\Big\| = \Big( \sum_{n=1}^{\infty} \Big( \frac{1}{n} \sum_{i=1}^{n} \big|x(i)\big| \Big)^p \Big)^{\frac{1}{p}} < \infty \Big\}.$$

The geometry of Cesàro sequence spaces and their generalizations have been extensively studied in [4, 5, 7, 9, 13, 20, 21]. Let us restrict ourselves to the two-dimensional Cesàro sequence space $\mathrm{ces}_p^{(2)}$ which is just $\mathbb{R}^2$ equipped with the norm defined by

$$\big\|(x,y)\big\| = \left(|x|^p + \left(\frac{|x|+|y|}{2}\right)^p\right)^{\frac{1}{p}}$$

for all $(x,y) \in \mathbb{R}^2$.

**Example 4.6.** (Cesàro sequence space). Let $X$ be a two-dimensional Cesàro space $\mathrm{ces}_2^{(2)}$. Then

$$H_p\big(\mathrm{ces}_2^{(2)}\big) = \sqrt{2 + \frac{2\sqrt{5}}{5}}, \quad (p \le 2).$$

In particular,

$$A_2\big(\mathrm{ces}_2^{(2)}\big) = T\big(\mathrm{ces}_2^{(2)}\big) = \sqrt{2 + \frac{2\sqrt{5}}{5}} \qquad \text{and} \qquad E\big(\mathrm{ces}_2^{(2)}\big) = 4 + \frac{4\sqrt{5}}{5}.$$

*Proof.* We first define

$$\big|(x,y)\big| = \left\|\left(\frac{2x}{\sqrt{5}}, 2y\right)\right\|_{\mathrm{ces}_2^{(2)}}$$

for $(x,y) \in \mathbb{R}^2$. It follows that $\mathrm{ces}_2^{(2)}$ is isometrically isomorphic to $(\mathbb{R}^2, |\cdot|)$ and $|\cdot|$ is an absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi(t) = \left(\frac{4(1-t)^2}{5} + \left(\frac{1-t}{\sqrt{5}} + t\right)^2\right)^{\frac{1}{2}}.$$

Indeed, $g : \mathrm{ces}_2^{(2)} \to (\mathbb{R}^2, |\cdot|)$ defined by

$$g(x,y) = \left(\frac{x}{\sqrt{5}}, 2y\right)$$

is an isometric isomorphism. We prove that $\psi \ge \psi_2$. Note that

$$\left(\frac{1-t}{\sqrt{5}} + t\right)^2 \ge \left(\frac{1-t}{\sqrt{5}}\right)^2 + t^2.$$

Consequently,

$$\psi(t) \ge \left((1-t)^2 + t^2\right)^{\frac{1}{2}} = \psi_2(t).$$

Some elementary computation show that $\frac{\psi(t)}{\psi_2(t)}$ attains its maximum at $t = \frac{1}{2}$. Therefore, by applying Theorem 3.8, we have

$$H_p\big(\mathrm{ces}_2^{(2)}\big) = 2\psi\Big(\frac{1}{2}\Big) = \sqrt{2 + \frac{2\sqrt{5}}{5}}.$$

□

## References

[1] J. Alonso, E. Llorens-Fuster, Geometric mean and triangles inscribed in a semicircle in Banach spaces, Journal of Mathematical Analysis and Applications 340 (2008) 1271–1283.
[2] M. Baronti, E. Casini, P. L. Papini, Triangles inscribed in a semicircle, in Minkowski planes, and in normed spaces, Journal of Mathematical Analysis and Applications 252 (2000) 124–146.
[3] F. F. Bonsall, J. Duncan, Numerical Ranges II, London Mathematical Society Lecture Notes Series, Volume 10, Cambridge University Press, London, 1973.
[4] Y. Cui, L. Jie, R. Płuciennik, Local uniform nonsquareness in Cesàro sequence spaces, Commentationes Mathematicae 37 (1997) 47–58.
[5] Y. Cui, H. Hudzik, Some geometric properties related to fixed point theory in Cesàro spaces, Collectanea Mathematica 50(3) (1999) 277–288.
[6] H. Cui, G. Lu, Hölders means and triangles inscribed in a semicircle in Banach spaces, Filomat 26:2 (2012) 371–377.
[7] Y. Cui, C. Meng, R. Płuciennik, Banach-Saks property and property ($\beta$) in Cesàro sequence spaces, Southeast Asian Bulletin of Mathematics 24 (2000) 201–210.
[8] H. Cui, F. Wang, Gao's constants of lorentz sequence spaces, Soochow Journal of Mathematics 33(4) (2007) 707–717.
[9] P. Foralewski, H. Hudzik, A. Szymaszkiewicz, Local rotundity structure of Cesàro-Orlicz sequence spaces, Journal of Mathematical Analysis and Applications 345(1) (2008) 410–419.
[10] J. Gao, A Pythagorean approach in Banach spaces, Journal of Inequalities and Applications 2006(2006), Article ID 94982, 11 pages.
[11] J. Gao, K. S. Lau, On two classes of Banach spaces with uniform normal structure, Studia Mathematica 99 (1991) 41–56.
[12] M. Kato, L. Maligranda, On James and Jordan-von Neumann constants of Lorentz sequence spaces, Journal of Mathematical Analysis and Applications. 258 (2001) 457–465.
[13] L. Maligranda, N. Petrot, S. Suantai, On the James constant and $B$-convexity of Cesàro and Cesàro-Orlicz sequence spaces, Journal of Mathematical Analysis and Applications 326(1) (2007) 312–331.
[14] K. -I. Mitani, K. -S. Saito, The James constant of absolute normalized norms on $\mathbb{R}^2$, Journal of Nonlinear and Convex Analysis 4 (2003) 399–410.
[15] K. -I. Mitani, K. -S. Saito, T. Suzuki, On the calculation of the James constant of Lorentz sequence spaces, Journal of Mathematical Analysis and Applications 343 (2008) 310–314.
[16] H. Mizuguchi, K. -S. Saito, Some geometric constants of absolute normalized norms on $\mathbb{R}^2$, Annals of Functional Analysis 2 (2011) 22–33.
[17] W. Nilsrakoo, S. Saejung, The James constant of normalized norms on $\mathbb{R}^2$, Journal of Inequalities and Applications 2006(2006), Article ID 26265, 12 pages.
[18] K. -S. Saito, M. Kato, Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$, Journal of Mathematical Analysis and Applications 244 (2000) 515–532.
[19] K. -S. Saito, M. Kato, Y. Takahashi, Absolute norms on $\mathbb{C}^2$, Journal of Mathematical Analysis and Applications 252 (2000) 879–905.
[20] W. Sanhan, S. Suantai, Some geometric properties of Cesàro sequence space, Kyungpook Mathematical Journal 43(2) (2003) 191–197.
[21] J. S. Shue, On the Cesàro sequence spaces, Tamkang Journal of Mathematics 1 (1970) 143–150.
[22] T. Suzuki, A. Yamano, M. Kato, The James constant of 2-dimensional Lorentz sequence spaces, Bulletin of the Kyushu Institute of Technology. Pure and applied mathematics 53 (2006) 15–24.
[23] Y. Takahashi, M. Kato, K.-S. Saito, Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, Journal of Inequalities and Applications 7 (2002) 179–186.
[24] Z. Zuo, The Ptolemy constant of absolute normalized norms on $\mathbb{R}^2$, Journal of Inequalities and Applications 2012:107, 12 pages.