Filomat 31:6 (2017), 1721–1728
DOI 10.2298/FIL1706721S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasi-Uniformities and Quotients of Paratopological Groups

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Abstract. For a subgroup H of a paratopological group G we prove that the quotient topology of the coset space G/H is induced by a rotund quasi-uniformity and the quotient topology of the semiregularization $(G/H)_{sr}$ of G/H is induced by a normal quasi-uniformity. In particular, $(G/H)_{sr}$ is a Tychonoff space provided that G/H is Hausdorff. The previous results are applied in order to show that every Hausdorff Lindelöf paratopological group is ω-admissible. We also show that, if G is an ω-admissible paratopological group, then so are the reflections $T_i(G)$ (i=0,1,2,3), $Reg(G)$ and $Tych(G)$.

1. Introduction

For a function $f\colon X \to Y$ defined on a quasi-uniform space $(X, \mathcal{U})$ with values in a set Y the *quotient quasi-uniformity* on Y is the largest quasi-uniformity making the map f quasi-uniformly continuous. In general, the quotient quasi-uniformity does not induce the quotient topology (see [7, 8]) and even a uniform quotient of a metrizable space can fail to be metrizable (see, for instance, [10]). These facts serve to illustrate the delicate nature of (quasi)-uniformities on quotient spaces and the intrinsic interest of their study.

The aim of this paper is to study quasi-uniformities on coset spaces G/H where H is a subgroup of a paratopological group G. Among other things, in the first section we show that the coset space G/H has a natural rotund uniformity which induces the quotient topology. The same is proved for the semiregularization $(G/H)_{sr}$ of G/H by means of a normal quasi-uniformity. As a consequence of this result, we show that $(G/H)_{sr}$ is a Tychonoff space provided that G/H is Hausdorff. In the second section, we apply these results in order to show that every Hausdorff Lindelöf paratopological group is ω-admissible. We also show that, if G is an ω-admissible paratopological group, then so are the reflections $T_i(G)$ (i=0,1,2,3), $Reg(G)$ and $Tych(G)$.

Now we introduce the basic notions used in this paper.

A *paratopological* (*semitopological*) group is a group with a topology such that multiplication on the group is jointly (separately) continuous. If G is a semitopological group with identity e, the symbol $\mathcal{N}(e)$ denotes the family of open neighborhoods of e in G.

Let X be a space with topology τ. Then the family

$$\{Int\,\overline{U} : U \in \tau\}$$

2010 *Mathematics Subject Classification*. Primary 54H11; Secondary 22A30, 54D10, 54D15, 54E15

Keywords. Axioms of separation, paratopological groups, quasi-uniformity, quotient topology, semiregularization

Received: 29 March 2015; Revised: 09 October 2015; Accepted: 10 October 2015

Communicated by Ljubiša D.R. Kočinac

The first author is supported by CONACyT of Mexico (Grant number 232464). The second author is supported by the Spanish Ministry of Science and Education (Grant MTM2011-23118) and by Universitat Jaume I, Spain (Grant P1-1B2014-35)

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constitutes a base for a coarser topology σ on X. The space $X_{sr} = (X, \sigma)$ is called the *semiregularization* of X.

Given two subsets U and V of $X \times X$, the symbol $U \circ V$ stands for the set $\{(x,z) \mid$ there exists $y \in X$ such that $(x,y) \in U$ and $(y,z) \in V\}$. A quasi-uniformity on a set X is a family $\mathcal{U}$ of subsets of $X \times X$ which satisfies the following conditions:

i) $\Delta = \{(x,x) : x \in X\} \subseteq U$ for every $U \in \mathcal{U}$;

ii) $U \cap V \in \mathcal{U}$, for each $U, V \in \mathcal{U}$;

iii) if $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$;

iv) for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

If in addition we have that $V^{-1} = \{(y,x) \mid (x,y) \in V\}$ belongs to $\mathcal{U}$ for all $V \in \mathcal{U}$, then the quasi-uniformity is called a uniformity.

A family $\mathcal{B} \subset \mathcal{U}$ is called a *base* for the quasi-uniformity $\mathcal{U}$ if for every $V \in \mathcal{U}$, there exists $W \in \mathcal{B}$ such that $W \subseteq V$. A base $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ is *multiplicative* if for every $U, V \in \mathcal{B}$, we have $U \circ V \in \mathcal{B}$.

Suppose that $\mathcal{U}$ is a quasi-uniformity on a set X. Then for each $x \in X$ and $U \in \mathcal{U}$, we put $B(x,U) = \{y \in X : (x,y) \in U\}$. If $A \subseteq X$ and $U \in \mathcal{U}$, then $B(A,U) = \bigcup_{x \in A} B(x,U)$.

A quasi-uniformity $\mathcal{U}$ induces a topology $\tau_{\mathcal{U}}$ on X as follows: the family $\{B(x,U) : U \in \mathcal{U}\}$ is a neighborhood base at each point $x \in X$.

A quasi-uniformity is *rotund* if $\mathcal{U}$ has a multiplicative base $\mathcal{B}$ such that $B(\overline{A}, W) \subseteq \overline{B(A, UW)}$ for each $A \subseteq X$ and $U, W \in \mathcal{B}$ (see [4]). For topological notions not defined here the reader can consult [5] and for paratopological groups [1].

2. Quasi-Uniformities and Quotients of Paratopological Groups

In this section we study quasi-uniformities on the coset space G/H, where H is a subgroup of a paratopological group G. We start with a lemma which is straightforward.

Lemma 2.1. *Let H be a subgroup of a paratopological group G and p the quotient function from G onto the quotient space given by the right cosets G/H (respectively, by the left cosets). Then $\overline{p(U)} = p(\overline{HU})$ for each open subset of G (respectively, $\overline{p(U)} = p(\overline{UH})$) for each open subset U of G.*

The following theorem tells us how to generate the quotient topology of G/H by means of a rotund quasi-uniformity.

Theorem 2.2. *Let H be a subgroup of a paratopological group G and G/H the quotient space given by the right cosets. Then the topology of G/H is induced by a rotund quasi-uniformity.*

Proof. Let $p\colon G \to G/H$ be the quotient function. Take $U \in \mathcal{N}(e)$ and put

$$\varepsilon_U = \{(p(x), p(y)) \in G/H \times G/H : y \in xU\}.$$

We claim that the family $\mathcal{B} = \{\varepsilon_U : \mathcal{N}(e)\}$ is a multiplicative base of a quasi-uniformity $\mathcal{U}$ on G/H. In fact, it is apparent that the diagonal of G/H is contained in ε_U for each $U \in \mathcal{N}(e)$ and that $\varepsilon_{U \cap V} \subseteq \varepsilon_U \cap \varepsilon_V$ for every $U, V \in \mathcal{N}(e)$.

Let us now show that $\varepsilon_U \circ \varepsilon_V = \varepsilon_{UV}$ for each $U, V \in \mathcal{N}(e)$. Take $(a,c) \in \varepsilon_U \circ \varepsilon_V$. Thus, there exists $b \in G/H$ such that $(a,b) \in \varepsilon_U$ and $(b,c) \in \varepsilon_V$. Therefore, we can find $(w,x),(y,z) \in G \times G$ such that $p(w) = a$, $p(x) = b$, $p(y) = b$, $p(z) = c$, $x \in wU$ and $z \in yV$. It follow that $hx = y$ for some $h \in H$. Also, $z \in yV = hxV \subseteq hwUV$. Since $p(hw) = p(w) = a$, $p(z) = c$ and $z \in hwUV$, we have that $(a,c) \in \varepsilon_{UV}$.

Now, take $(a,c) \in \varepsilon_{UV}$. Then, there exists $(w,z) \in G \times G$ such that $p(w) = a$, $p(z) = c$ and $z \in wUV$. So $z = wuv$ with $u \in U$ and $v \in V$. Put $x = zv^{-1}$ and $b = p(x)$. We conclude that $x = zv^{-1} = wu \in wU$ and

$z \in zv^{-1}V = xV$. It follows that $(a,b) \in \varepsilon_U$ and $(b,c) \in \varepsilon_V$. We have thus proved that $(a,c) \in \varepsilon_U \circ \varepsilon_V$. This proves the claim.

Now fix $A \subseteq G/H$ and $U \in \mathcal{N}(e)$. Put $C = p^{-1}(A)$. We claim that $p(CU) = B(A,\varepsilon_U)$. Indeed, take $c \in C$ and $u \in U$. Put $a = p(c)$ and $b = p(cu)$. Hence $(a,b) \in \varepsilon_U$. So $p(cu) \in B(a,\varepsilon_U) \subseteq B(A,\varepsilon_U)$. For the other inclusion, choose $b \in B(A,\varepsilon_U)$. Hence $b \in B(a,\varepsilon_U)$ for some $a \in A$. So $(a,b) \in \varepsilon_U$. It follows that there exist $x,y \in G$ such that $p(x) = a$, $p(y) = b$ and $y \in xU$. We conclude that $b \in p(xU) \subseteq p(CU)$. We have thus proved that $p(CU) = B(A,\varepsilon_U)$. In particular, if $a \in G/H$, then $B(a,\varepsilon_U) = p(xU)$ for each $x \in G$ such that $p(x) = a$. Therefore, $\mathcal{U}$ induces the quotient topology on G/H.

Let us show that $\mathcal{U}$ is rotund. Take $A \subseteq G/H$ and $U,W \in \mathcal{N}(e)$. Put $C = p^{-1}(A)$. Since p is open and continuous, $p^{-1}(\overline{A}) = \overline{p^{-1}(A)} = \overline{C}$.

We have that $B(\overline{A},\varepsilon_U) = p(\overline{C}U) \subseteq p(\overline{CU}) = \overline{p(CU)} \subseteq \overline{p(CWU)} = \overline{B(A,\varepsilon_W\varepsilon_U)}$. This finishes the proof. □

Corollary 2.3. *If H is a subgroup of a paratopological group G such that G/H is regular, then G/H is Tychonoff.*

Proof. It follows from Theorem 2.2 and [2]. □

A quasi-uniformity $\mathcal{U}$ on a set X is *normal* if $\overline{A} \subseteq Int\overline{B(A,U)}$ for any subset $A \subseteq X$ and any entourage $U \in \mathcal{U}$. Here the interior and the closure are taken in $\tau_{\mathcal{U}}$. It is known that a uniformity is always normal (see [2]).

In the following result, if B is a subset of the paratopological group G, we put $\overline{B} = \overline{B}^G$.

Theorem 2.4. *Let H be a subgroup of a paratopological group G. Then the topology of the semiregularization $(G/H)_{sr}$ of the quotient space G/H is induced by a normal quasi-uniformity.*

Proof. Let $p\colon G \to G/H$ be the quotient function, $X = G/H$ and $Y = (G/H)_{sr}$. Take $U \in \mathcal{N}(e)$ and put

$$\varepsilon_U = \{(p(x),p(y)) \in Y \times Y : p(y) \in Int\,\overline{p(xU)}^X\}.$$

Let us show that the family $\{\varepsilon_U : U \in \mathcal{N}(e)\}$ is a base for some quasi-uniformity $\mathcal{U}$ on Y. Clearly, the diagonal $\Delta_Y \subseteq \varepsilon_U$ for each $U \in \mathcal{N}(e)$.

Let us show that $\varepsilon_U \circ \varepsilon_V \subseteq \varepsilon_{UV}$ for each $U,V \in \mathcal{N}(e)$. Take $(a,c) \in \varepsilon_U \circ \varepsilon_V$. Thus, there exists $b \in G/H$ such that $(a,b) \in \varepsilon_U$ and $(b,c) \in \varepsilon_V$. Therefore, we can find $w,x,y,z \in G$ such that $p(w) = a$, $p(x) = b$, $p(y) = b$, $p(z) = c$, $p(x) \in Int\,\overline{p(wU)}^X$ and $p(z) \in Int\,\overline{p(yV)}^X$. We know that $hx = y$ for some $h \in H$. So $p(z) \in Int\,\overline{p(yV)}^X = Int\,\overline{p(hxV)}^X = Int\,\overline{p(xV)}^X$. By Lemma 2.1, $p(x) \in \overline{p(wU)}^X = p(\overline{HwU})$. Then $x \in H\overline{HwU}$. By the continuity of the multiplication in G and the fact that H is subgroup, $x \in H\overline{HwU} \subseteq \overline{HwU}$. Therefore, $p(z) \in Int\,\overline{p(xV)}^X \subseteq Int\,\overline{p(\overline{HwU}V)}^X \subseteq Int\,\overline{p(\overline{HwUV})}^X = Int\,\overline{p(wUV)}^X$. Hence $(a,c) \in \varepsilon_{UV}$.

Fix $a \in X$ and $U \in \mathcal{N}(e)$. Choose $x \in G$ such that $p(x) = a$. We claim that $B(a,\varepsilon_U) = Int\,\overline{p(xU)}^X$. Indeed, if $(a,b) \in \varepsilon_U$, then there exist $y,z \in G$ such that $p(y) = a$, $p(z) = b$ and $p(z) \in Int\,\overline{p(yU)}^X$. Since $p(x) = p(y)$, we can find $h \in H$ satisfying $hx = y$. Therefore, $p(z) \in Int\,\overline{p(hxU)}^X$ and $b = p(z) \in Int\,\overline{p(hxU)}^X = Int\,\overline{p(xU)}^X$. Hence $B(a,\varepsilon_U) \subseteq Int\,\overline{p(xU)}^X$.

Conversely, $b \in Int\,\overline{p(xU)}^X$. Take $y \in G$ such that $p(y) = b$. We have that $(a,b) \in \varepsilon_U$. We have thus proved that $B(a,\varepsilon_U) = Int\,\overline{p(xU)}^X$. This shows that $\mathcal{U}$ generates the topology on $Y = (G/H)_{sr}$.

Let us show that the quasi-uniformity $\mathcal{U}$ is normal. Take $A \subseteq Y$, $U \in \mathcal{N}(e)$, and put $C = p^{-1}(A)$. Take an open neighborhood V of e in G such that $V^2 \subseteq U$. Denote by $B_X(A,\varepsilon_U)$ the U-neighborhood of A given by the quasi-uniformity on X as in Theorem 2.2. Since $B_X(A,\varepsilon_V)$ is open in X, we have that

$$\overline{A}^Y \subseteq \overline{Int_X\overline{B_X(A,\varepsilon_V)}^X}^Y = \overline{Int_X\overline{B_X(A,\varepsilon_V)}^X}^X = \overline{B_X(A,\varepsilon_V)}^X \tag{1}$$

The quasi-uniformity in Theorem 2.2 is rotund, so it is normal. Then, we conclude that

$$\overline{B_X(A,\varepsilon_V)}^X \subseteq Int_X\overline{B_X(B_X(A,\varepsilon_V),\varepsilon_V)}^X \subseteq \overline{B_X(A,\varepsilon_U)}^X \subseteq \overline{B_X(A,\varepsilon_U)}^Y \tag{2}$$

It follows from (1), (2) and the inclusion $B_X(A,\varepsilon_U) \subseteq B(A,\varepsilon_U)$ that $\overline{A}^Y \subseteq Int_Y\overline{B(A,\varepsilon_U)}^Y$. This shows that $\mathcal{U}$ is normal. □

Corollary 2.5. *Let H be a subgroup of a paratopological group G such that G/H is Hausdorff. Then $(G/H)_{sr}$ is Tychonoff.*

Proof. It is known that $(X_{sr})_{sr} = X_{sr}$ for every space X. So $(G/H)_{sr}$ is semiregular. By Theorem 2.4, the topology on $(G/H)_{sr}$ is induced by a normal quasi-uniformity. Finally, [2] implies that $(G/H)_{sr}$ is Tychonoff. □

3. Some Results on ω-Admissible Paratopological Groups

According to [14], a paratopological group G with identity e is *ω-admissible* if for every sequence $\{U_n : n \in \omega\}$ of open neighborhoods of e in G, there exists a subgroup H of G such that $H \subseteq \bigcap_{n\in\omega} U_n$ and the quotient space G/H is submetrizable.

In Theorem 3.5, we will prove that every Hausdorff Lindelöf paratopological group is ω-admissible. Before, we need to recall some concepts. Let G be a semitopological group with identity e. A subset V of G is called *ω-good* if there exists a countable family $\gamma \subset \mathcal{N}(e)$ such that for every $x \in V$, we can find $W \in \gamma$ with $xW \subseteq V$. The symbol $\mathcal{N}^*(e)$ denotes the family of ω-good sets of G which contains the identity. The following lemmas are useful.

Lemma 3.1. ([15, Lemma 3.10]) *Every paratopological group G has a local base at the neutral element consisting of ω-good sets.*

Lemma 3.2. *Let G be a semitopological group with identity e. Suppose that a family $\gamma \subset \mathcal{N}(e)$ satisfies the following condition:*

(a) *for every $U \in \gamma$ and $x \in U$, there exists $V \in \gamma$ such that $xV \subset U$.*

Then the set $N = \bigcap\{U \cap U^{-1} : U \in \gamma\}$ is a subgroup of G. Moreover, $UN = U$ for each $U \in \gamma$.

Proof. It is clear that $N = N^{-1}$. Let us show that N is a subgroup of G. Take $a, b \in N$ and $U \in \gamma$. It follows that $a, b \in U \cap U^{-1}$. By (a), there exists $V \in \gamma$ such that $aV \subseteq U$. Hence $ab \in aN \subseteq aV \subseteq U$. Since $b^{-1} \in U$, by (a) again we can find $W \in \gamma$ satisfying $b^{-1}W \subseteq U$, that is, $W^{-1}b \subseteq U^{-1}$. Thus, we have that $ab \in Nb \subseteq W^{-1}b \subseteq U^{-1}$ and, consequently, $ab \in U \cap U^{-1}$ for each $U \in \gamma$. Therefore, $ab \in N$. We have thus proved that N is a subgroup of G.

Next, we show that $UN = U$ for each $U \in \gamma$. For this, pick up $U \in \gamma$ and take $a \in U$. By (a), we can find $V \in \gamma$ such that $aV \subseteq U$. It follows that $aN \subseteq aV \subseteq U$. This completes the proof. □

For a Hausdorff semitopological group G with identity e, the *Hausdorff number* of G, denoted by $Hs(G)$, is the minimum cardinal number κ such that for every neighborhood U of e in G, there exists a family γ of neighborhoods of e such that $\bigcap_{V\in\gamma} VV^{-1} \subseteq U$ and $|\gamma| \leq \kappa$ (see [16]).

We know that a paratopological group G with identity e is Hausdorff if and only if $\bigcap_{V\in\mathcal{N}(e)} V^{-1}VV^{-1} = \{e\}$. This motivates the next definition.

Definition 3.3. Let G be a Hausdorff paratopological group with identity e. The bilateral Hausdorff number of G, denoted by $BHs(G)$, is the minimum cardinal number κ such that for every neighborhood $U \in \mathcal{N}(e)$, there exists a family $\gamma \subseteq \mathcal{N}(e)$ such that $\bigcap_{V\in\gamma} V^{-1}VV^{-1} \subseteq U$ and $|\gamma| \leq \kappa$.

It follows from the previous definition that $Hs(G) \leq BHs(G)$ for every Hausdorff paratopological group G. Clearly, if G is 2-oscillating, then we have the equality $Hs(G) = BHs(G)$. It will be interesting to find a Hausdorff paratopological group G such that $Hs(G) < BHs(G)$. Recall that a paratopological group G is said to be 2-*oscillating* if for every neighborhood U of the identity e in G there is a neighborhood V of e such that $V^{-1}V \subseteq UU^{-1}$. Precompact and Abelian paratopological groups are 2-oscillating (see [3]).

By [16, Proposition 2.4], every Hausdorff Lindelöf paratopological group has countable Hausdorff number. Using a similar argument, we can prove the following result.

Proposition 3.4. *Every Hausdorff Lindelöf paratopological group satisfies the inequality $BHs(G) \leq \omega$.*

Proof. Take $U \in \mathcal{N}(e)$. Since G is a Hausdorff paratopological group, for each $x \in G \setminus U$ there exists $V_x \in \mathcal{N}(e)$ such that $V_x x V_x^2 \cap V_x = \emptyset$ or, equivalently, $xV_x \cap V_x^{-1}V_xV_x^{-1} = \emptyset$. The set $G \setminus U$ is closed in G and the family $\{xV_x : x \in G \setminus U\}$ is an open cover of $G \setminus U$, so there exists a countable subset $S \subseteq G \setminus U$ such that the family $\{xV_x : x \in S\}$ covers $G \setminus U$. It follows that $\bigcap_{x\in S} V_x^{-1}V_xV_x^{-1} \subseteq U$. Therefore, $BHs(G) \leq \omega$. □

Theorem 3.5. *If G is a Hausdorff Lindelöf paratopological group, then G is ω-admissible.*

Proof. Take a sequence $\{U_n : n \in \omega\} \subseteq \mathcal{N}(e)$. By Lemma 3.1, for each $n \in \omega$ there exists $U_n^* \in \mathcal{N}^*(e)$ such that $U_n^* \subseteq U_n$. By induction, we will construct a sequence $\{\gamma_n : n \in \omega\}$ such that for every $n \in \omega$:

(i) $\gamma_n \subseteq \mathcal{N}^*(e)$ and $|\gamma_n| \leq \omega$;
(ii) $\gamma_n \subseteq \gamma_{n+1}$;
(iii) γ_n is closed under finite intersections;
(iv) for every $U \in \gamma_n$ and $x \in U$, there exists $V \in \gamma_{n+1}$ such that $xV \subseteq U$;
(v) $\bigcap_{V\in\gamma_{n+1}} V^{-1}VV^{-1} \subseteq \bigcap \gamma_n$.

Let γ_0 be the minimal family containing $\{U_n^* : n \in \omega\}$ and closed under finite intersections. Suppose that we have defined γ_n. As $\gamma_n \subseteq \mathcal{N}^*(e)$, there exists a countable family $\lambda_{n,1} \subseteq \mathcal{N}^*(e)$ such that for each $U \in \gamma_n$ and $x \in U$, there exists $V \in \lambda_{n,1}$ satisfying $xV \subseteq U$. Since G is a Hausdorff Lindeöf space, Proposition 3.4 implies that we can find a countable family $\lambda_{n,2} \subseteq \mathcal{N}^*(e)$ such that for every $U \in \gamma_n$, we have $\bigcap_{V\in\lambda_{n,2}} V^{-1}VV^{-1} \subseteq U$. Let γ_{n+1} be the minimal family containing $\gamma_n \cup \left(\bigcup_{i=1}^2 \lambda_{n,i}\right)$ and closed under finite intersections. Clearly, γ_{n+1} satisfies (i)–(v). This finishes our construction.

Put $\gamma = \bigcup_{n\in\omega} \gamma_n$. By construction, γ satisfies condition (a) in Lemma 3.2. Thus, $H = \bigcap\{U \cap U^{-1} : U \in \gamma\}$ is a subgroup of G. By item (v), $H = \bigcap\{UU^{-1} : U \in \gamma\}$. It follows that $H = \bigcap\{U : U \in \gamma\} = \bigcap\{\overline{U} : U \in \gamma\} = \bigcap\{U^{-1}UU^{-1} : U \in \gamma\}$.

Let p be the quotient function from G onto G/H, the quotient space given by the left cosets. Let us show that G/H is a Hausdorff space. Take $x, y \in G$ such that $p(x) \neq p(y)$. So $x^{-1}y \notin H$. Since $H = \bigcap\{UU^{-1} : U \in \gamma\}$, we can find $U \in \gamma$ with $x^{-1}y \notin UU^{-1}$. It follows that $xU \cap yU = \emptyset$. By Lemma 3.2, $xUH \cap yUH = \emptyset$. Hence $p(xU) \cap p(yU) = \emptyset$. We have thus proved that G/H is Hausdorff. Corollary 2.5, implies that $X = (G/H)_{sr}$ is a Tychonoff space. By [5, Theorem 5.1.2], X is paracompact.

Let us show that X has G_δ-diagonal. Put $\mathcal{U}(U) = \bigcup_{x\in G} Int\overline{p(xU)} \times Int\overline{p(xU)}$, for every $U \in \gamma$. Clearly, $\mathcal{U}(U)$ is open in $X \times X$ and contains the diagonal Δ_X. Take $a, b \in G$ such that $p(a) \neq p(b)$. It follows that $b^{-1}a \notin H = \bigcap\{U^{-1}UU^{-1} : U \in \gamma\}$. Therefore, there exists $U \in \gamma$ such that $b^{-1}a \notin U^{-1}UU^{-1}$. Take $V \in \gamma$ with $V^2 \subseteq U$. We claim that $(p(a), p(b)) \notin \mathcal{U}(V)$. Suppose the contrary. Then, we can find $x \in G$ such that $p(a), p(b) \in Int\overline{p(xV)}$. It follows from Lemmas 2.1 and 3.2 that $Int\overline{p(xV)} = Int\,p(\overline{xV})$. Therefore, $p(a) \in Int\,p(\overline{xV}) \subseteq p(xVV^{-1})$ and $p(b) \in Int\,p(x\overline{V}) \subseteq p(xV^{-1}V)$. Hence $a \in xVV^{-1}H \subseteq xVV^{-2}$ and $b \in xV^{-1}VH \subseteq xV^{-1}V^2$. It follows that $b^{-1}a \in (V^{-2}Vx^{-1})(xVV^{-2}) = V^{-2}V^2V^{-2} \subseteq U^{-1}UU^{-1}$. This contradicts the choice of U. We have thus proved that $\bigcap_{U\in\gamma} \mathcal{U}(U) = \Delta_X$.

Finally, since every Hausdorff paracompact space with a G_δ-diagonal is submetrizable (see [6, Corollary 2.9]), the space X is submetrizable. The topology on $X = (G/H)_{sr}$ is weaker than the topology on G/H and, consequently, G/H is submetrizable. This completes the proof. □

Corollary 3.6. ([9],[12]) *Every Hausdorff Lindelöf paratopological group with countable pseudocharacter is submetrizable.*

According to [17] (also [18]), given a semitopological group G, the T_i-*reflection* of G for $i \in \{0, 1, 2, 3\}$ is defined as a pair $(H, \varphi_{G,i})$ where H is a semitopological group satisfying the T_i separation axiom and $\varphi_{G,i}$ is a continuous homomorphism of G onto H with the following property: for every continuous mapping $f\colon G \to X$ to a T_i-space X, there exists a continuous mapping $h\colon H \to X$ such that $f = h \circ \varphi_{G,i}$.

$$\begin{tikzcd} G \arrow[r, "\varphi_{G,i}"] \arrow[d, "f"'] & H \arrow[dl, "h"] \\ X & \end{tikzcd}$$

Similarly, a *regular* (*Tychonoff*) *reflection* of a semitopological group G is defined. As is customary, by 'regular' we mean 'T_1 & T_3'.

Abusing of terminology, we will usually refer to $T_0(G)$, $T_1(G)$, $T_2(G)$, $Reg(G)$ and $Tych(G)$ as the T_0-, T_1-, Hausdorff, regular and Tychonoff reflection, respectively, of the group G.

Problem 3.7. *Let H be a subgroup of a regular Lindelöf paratopological group G such that the space G/H is Hausdorff (regular) and it has countable pseudocharacter. Is G/H submetrizable?*

Theorem 3.8. *Let G be an ω-admissible paratopological group. Then*

i) $T_i(G)$ is ω-admissible for each $i = 0, 1, 2$;

ii) $T_3(G)$ is ω-admissible;

iii) $Reg(G)$ is ω-admissible;

iv) $Tych(G)$ is ω-admissible.

Proof. i) Fix $i \in \{0, 1, 2\}$. Let $\{U_n : n \in \omega\}$ be sequence of open neighborhoods of the identity in $T_i(G)$. According to [17] and [18], $T_i(G) \cong G/N$, where N is a normal subgroup of G. Consider $\varphi_{G,i}\colon G \to T_i(G)$. For each $n \in \omega$, there exists an open neighborhood V_n of the identity e in G such that $\varphi_{G,i}(V_n) \subseteq U_n$. Since G is ω-admissible, there exists a subgroup H of G such that $H \subseteq \bigcap_{n\in\omega} V_n$ and the left quotient space G/H is submetrizable. Then $M = \varphi_{G,i}(H) \subseteq \bigcap_{n\in\omega} \varphi_{G,i}(V_n) \subseteq \bigcap_{n\in\omega} U_n$. Let us show that the left quotient space $T_i(G)/M$ is submetrizable. Consider the quotient functions $p\colon G \to G/H$ and $q\colon T_i(G) \to T_i(G)/M$. Since G/H is submetrizable, there exists a bijective continuous function from G/H onto a metrizable space X. Since X is a T_i space, we can find a continuous function $h\colon T_i(G) \to X$ such that $h \circ \varphi_{G,i} = f \circ p$. We claim that there exists a function $g\colon T_i(G)/M \to X$ such that $g \circ q \circ \varphi_{G,i} = f \circ p$.

Indeed, take $y \in T_i(G)/M$. There exists $x \in G$ such that $q(\varphi_{G,i}(x)) = y$. We have that $f(p(\varphi_{G,i}^{-1}(q^{-1}(y)))) = f(p(\varphi_{G,i}^{-1}(\varphi_{G,i}(xH)))) = f(p(xHN)) = f(p(xNH)) = f(p(xN)) = h(\varphi_{G,i}(xN)) = \{h(\varphi_{G,i}(x))\}$. This proves our claim.

Since $q \circ \varphi_{G,i}$ is open and $f \circ p$ is continuous, the function $g\colon T_i(G)/M \to X$ is continuous.

Let us show that g is injective. Take $x, y \in G$ such that $q(\varphi_{G,i}(x)) \neq q(\varphi_{G,i}(y))$. This implies that $y^{-1}x \notin HN$. Hence $y^{-1}x \notin H$. So $p(x) \neq p(y)$. Since f is injective, we have that $f(p(x)) \neq f(p(y))$. It follows that g is injective.

We have thus proved that $T_i(G)/M$ is submetrizable.

ii) Since G is a paratopological group, $T_3(G) = G_{sr}$. Let $\{Int\overline{U_n} : n \in \omega\}$ be a sequence of open neighborhoods of the identity in G_{sr}. Since G is ω-admissible, there exists a subgroup $H \subset \bigcap_{n\in\omega} Int\overline{U_n}$ such that G/H is submetrizable, that is, there is a condensation p from G/H onto a metrizable space M. Consider the quotient maps π_1 and π_2 from G onto G/H and from G_{sr} onto G_{sr}/H, respectively. Notice that, as functions, the equality $\pi_1 = \pi_2$ holds and, consequently, we have $p \circ \pi_1 = p \circ \pi_2$. Since $p \circ \pi_1$ is a continuous function from G onto a metrizable space, $p \circ \pi_2$ is also continuous. The definition of quotient topology implies that p is continuous as a function from G_{sr}/H onto M. This proves ii).

iii) By [17, Proposition 3.7], $Reg(G) = T_0(T_3(G))$. It remains to apply i)–ii).

iv) The paratopological group $Reg(G)$ is regular. By Corollary 2.3, the space $Reg(G)$ is Tychonoff. So $Tych(G) = Reg(G)$. □

Following [13], we say that a semitopological group G has *countable symmetry number* if for every open neighborhood U of the identity e in G, there exists a countable family γ of open neighborhoods of e in G such that $\bigcap_{V\in\gamma} V^{-1} \subseteq U$.

The following result may be of interest in itself.

Proposition 3.9. *Every ω-admissible paratopological group G has countable symmetry number.*

Proof. Let U be an open neighborhood of the identity e in G. Since G is ω-admissible, there exists $H \leq G$ such that $H \subseteq U$ and G/H is submetrizable. We have that G/H has countable pseudocharacter. This implies that we can find a countable family γ of open neighborhoods of e in G such that $\bigcap_{V\in\gamma} p(V) = \{H\}$, where p is the quotient function from G onto G/H. It follows that $\bigcap_{V\in\gamma} VH = H$. Therefore, $\bigcap_{V\in\gamma} V^{-1} \subseteq H \subseteq U$. This completes the proof. □

Proposition 3.9 permits us to construct an example of an ω-narrow paratopological group which is not ω-admissible. In addition, the next example answers in the negative [11, Problem 3].

Example 3.10. There exists an Abelian Tychonoff ω-narrow paratopological group H which is not ω-admissible. In fact, H has uncountable symmetry number.

Proof. Let $\mathbb{Z}$ be the discrete group of integers and κ an uncountable cardinal. For a finite set $A \subseteq \kappa$, we define a set $U_A \subseteq \mathbb{Z}^\kappa$ by

$$U_A = \{x \in \mathbb{Z}^\kappa : x(\alpha) = 0 \text{ if } \alpha \in A \text{ and } x(\alpha) \geq 0 \text{ if } \alpha \in \kappa \setminus A\}.$$

The family $\mathcal{U} = \{U_A : A \subseteq \kappa, |A| < \omega\}$ is a local base at the neutral element of $\mathbb{Z}^\kappa$ for a topology τ such that $G = (\mathbb{Z}^\kappa, \tau)$ is a completely regular paratopological group (see [16, Example 2.9]). Define the subset H of $\mathbb{Z}^\kappa$ as follows: $x \in H$ if there exists a positive integer n_x such that $|x(\alpha)| < n_x$ for each $\alpha \in \kappa$. Clearly, H is a subgroup of G. Let us show that $(H, \tau|_H)$ is ω-narrow. Take a finite subset A of κ and put $V = H \cap U_A$. For each $r \in \mathbb{Z}$ Consider the subset

$$D_r = \{x \in \mathbb{Z}^\kappa : x(\alpha) = r \text{ if } \alpha \notin A\}.$$

It is easy to see that $D_r \subseteq H$. Since A is finite and $\mathbb{Z}$ is countable, the subset D_r is countable. Put $D = \bigcup_{r\in\mathbb{Z}} D_r$. Clearly, $D \subseteq H$ and D is countable.

Take $x \in H$. Then, there exists a positive integer n such that $|x(\alpha)| < n$ for each $\alpha \in \kappa$. Choose $d \in D_{-n}$ such that $d(\alpha) = x(\alpha)$ for every $\alpha \in A$. Consider $v \in V$ such that $v(\alpha) = x(\alpha) + n$ if $\alpha \in \kappa \setminus A$. Of course, $v(\alpha) = 0$ if $\alpha \in A$. We claim that $d + v = x$. Indeed, $d(\alpha) + v(\alpha) = -n + x(\alpha) + n = x(\alpha)$ if $\alpha \in \kappa \setminus A$. On the other hand, $d(\alpha) + v(\alpha) = x(\alpha) + 0 = x(\alpha)$ if $\alpha \in A$. We have thus proved that $D + V = H$.

We will prove that H has uncountable symmetry number. Put $U = H \cap U_\emptyset$. Let $\{A_n : n \in \omega\}$ be a sequence of finite subset of κ and put $U_n = H \cap U_{A_n}$ for each $n \in \omega$. The set $A = \bigcup_{n\in\omega} A_n$ is a countable subset of κ. Since κ is uncountable, we can choose $k \in \kappa \setminus A$. Take $x \in H$ satisfying $h(\alpha) = 0$ if $\alpha \neq k$, and $x(k) = -1$. It is easy to see that $x \in \bigcap_{n\in\omega} U_n^{-1}$, but $x \notin U = H \cap U_\emptyset$. This shows that H has uncountable symmetry number.

Since H has uncountable symmetry number, Proposition 3.9 implies that H is not ω-admissible. □

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