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Differences of Generalized Weighted Composition Operators from the Bloch Space into Bers-type Spaces

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Abstract. We study the boundedness and compactness of the differences of two generalized weighted composition operators acting from the Bloch space to Bers-type spaces.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, namely $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. It is well known that

$$\frac{|\sigma_a'(z)|}{1-|\sigma_a(z)|^2} = \frac{1}{1-|z|^2} \quad \text{and} \quad \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2} = 1-|\sigma_a(z)|^2.$$

For $a, z \in \mathbb{D}$, the pseudo-hyperbolic distance between z and a is given by $\rho(z,a) = \big|\sigma_a(z)\big|$. It is easy to check that $\rho(z,a)$ satisfies the following inequality:

$$\frac{1-\rho(z,a)}{1+\rho(z,a)} \le \frac{1-|z|^2}{1-|a|^2} \le \frac{1+\rho(z,a)}{1-\rho(z,a)}, \quad z,\ a \in \mathbb{D}. \tag{1}$$

An $f \in H(\mathbb{D})$ belongs to the Bloch space $\mathcal{B}$, if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

$\mathcal{B}$ is a Banach space with the above norm.

Let $\alpha \ge 0$. An $f \in H(\mathbb{D})$ belongs to the Bers-type space, denoted by H_α^∞, if

$$\|f\|_{H_\alpha^\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty.$$

Also, H_α^∞ is a Banach space with the norm $\|\cdot\|_{H_\alpha^\infty}$. When $\alpha = 0$, H_0^∞ is just the bounded analytic function space, simply denoted by H^∞.

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Throughout the paper, $S(\mathbb{D})$ denotes the set of analytic self-maps of $\mathbb{D}$. A map $\varphi \in S(\mathbb{D})$ induces a linear operator C_φ, where $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. C_φ is called the composition operator.

Let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined as following:

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \quad z \in \mathbb{D}.$$

In these five decades, there has been much work on composition operators and weighted composition operators on various Banach spaces of analytic functions. See [2, 12, 21] for a comprehensive overview of this field.

Let $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$ and n be a nonnegative integer. Let $f^{(n)}$ denote the n-th derivative of f and $f^{(0)} = f$. A linear operator $D^n_{\varphi,u}$ is defined by

$$D^n_{\varphi,u} f = u f^{(n)} \circ \varphi, \ \ f \in H(\mathbb{D}).$$

If $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u}$ is the composition operator C_φ. If $n = 0$, then $D^n_{\varphi,u}$ is just the weighted composition operator uC_φ. If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$. The operator $D^n_{\varphi,u}$ is called the generalized weighted composition operator, which includes many known operators and was introduced by Zhu in [22], and studied in [7, 8, 14–16, 18, 20, 22–24] and the references therein.

Recently, there has been an increasing interest in studying the differences of two composition operators acting on various analytic function spaces. In [13], the authors studied the differences of composition operators on the Hardy space H^2. The motivation for this is to understand the topological structure of the set of composition operators on H^2. In [9], the authors studied the topological structure of the set of composition operators on H^∞ and gave a relationship between such a problem and the boundedness and compactness of the differences of two composition operators acting from the Bloch space to H^∞. In [10], Moorhouse studied the differences of composition operators on weighted Bergman spaces. The differences of two composition operators acting on Bers-type spaces was studied in [1, 3, 17]. The differences of two generalized weighted composition operators acting on Bers-type spaces was studied in [20]. The differences of two composition operators acting on the Bloch space was studied in [4, 6, 11, 19]. In [5], the authors studied the differences of two weighted composition operators acting from the Bloch space to H^∞.

In [24], Zhu studied the boundedness and compactness of $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\alpha$. Motivated by these, in this paper, we investigate the boundedness and compactness of the differences of two generalized weighted composition operators from the Bloch space into H^∞_α. The results generalize the corresponding results in [24] on the single generalized weighted composition operator.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \approx b$.

2. Main Results and Proofs

In this section we give our main results and proofs. For this purpose, we need the following lemma, which can be proved in a standard way (see, for example, Theorem 3.11 in [2]).

Lemma 1. *Let $\alpha > 0$, $\varphi, \psi \in S(\mathbb{D})$, $u, v \in H(\mathbb{D})$ and n be a positive integer. Then $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is compact if and only if $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|(D^n_{\varphi,u} - D^n_{\psi,v}) f_k\|_{H^\infty_\alpha} \to 0$ as $k \to \infty$.*

Lemma 2. [21] *For every positive integer n, $f \in \mathcal{B}$ if and only if $\sup_{z\in\mathbb{D}}(1 - |z|^2)^n |f^{(n)}(z)| < \infty$. Moreover, the following asymptotic relationship holds*

$$\|f\|_{\mathcal{B}} \approx \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^n |f^{(n)}(z)|.$$

Lemma 3. *For every positive integer n, if $f \in \mathcal{B}$, then*

$$|(1-|z|^2)^n f^{(n)}(z) - (1-|w|^2)^n f^{(n)}(w)| \le C\|f\|_{\mathcal{B}}\rho(z,w), \quad z,w \in \mathbb{D}.$$

Proof. Let $f \in \mathcal{B}$. By Lemma 2, we see that $f^{(n)} \in H_n^\infty$, moreover, $\|f^{(n)}\|_{H_n^\infty} \le C\|f\|_{\mathcal{B}}$. By Lemma 3.2 in [3] or Lemma 2.3 in [11], we have

$$|(1-|z|^2)^n f^{(n)}(z) - (1-|w|^2)^n f^{(n)}(w)| \le C\|f^{(n)}\|_{H_n^\infty}\rho(z,w) \le C\|f\|_{\mathcal{B}}\rho(z,w), \quad z,w \in \mathbb{D}.$$

This completes the proof of this Lemma.

Remark 1. From the remark 3.3 of [3], for every $f \in \mathcal{B}$, we have

$$|(1-|z|^2)^n f^{(n)}(z) - (1-|w|^2)^n f^{(n)}(w)| \le C\rho(z,w)\sup_{z\in\mathbb{D}_r}(1-|z|^2)^n|f^{(n)}(z)|, \quad z,w \in \mathbb{D}_r, \tag{2}$$

where $\mathbb{D}_r = \{z \in \mathbb{D} : |z| \le r < 1\}$.

For the simplicity of this paper, we denote

$$M_{u,\varphi}(z) = \frac{(1-|z|^2)^\alpha u(z)}{(1-|\varphi(z)|^2)^n}, \qquad M_{v,\psi}(z) = \frac{(1-|z|^2)^\alpha v(z)}{(1-|\psi(z)|^2)^n}. \tag{3}$$

Theorem 1. *Let $\alpha > 0$, $\varphi, \psi \in S(\mathbb{D})$, $u, v \in H(\mathbb{D})$ and n be a positive integer. Then the following statements are equivalent.*

(*a*) $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ *is bounded;*

(*b*)

$$\sup_{z\in\mathbb{D}}\left|M_{u,\varphi}(z)\right|\rho(\varphi(z),\psi(z)) < \infty \tag{4}$$

and

$$\sup_{z\in\mathbb{D}}\left|M_{u,\varphi}(z) - M_{v,\psi}(z)\right| < \infty; \tag{5}$$

(*c*) (5) *holds and*

$$\sup_{z\in\mathbb{D}}\left|M_{v,\psi}(z)\right|\rho(\varphi(z),\psi(z)) < \infty. \tag{6}$$

Proof. $(b) \Rightarrow (c)$. Assume that (4) and (5) hold. It is clear that

$$\left|M_{v,\psi}(z)\right| \le \left|M_{u,\varphi}(z)\right| + \left|M_{u,\varphi}(z) - M_{v,\psi}(z)\right|.$$

Multiplying the last inequality by $\rho(\varphi(z),\psi(z))$ and notice that $\rho(\varphi(z),\psi(z)) \le 1$, we get

$$\begin{aligned}\left|M_{v,\psi}(z)\right|\rho(\varphi(z),\psi(z)) &\le \left|M_{u,\varphi}(z)\right|\rho(\varphi(z),\psi(z)) + \left|M_{v,\psi}(z) - M_{u,\varphi}(z)\right|\rho(\varphi(z),\psi(z)) \\ &\le \left|M_{u,\varphi}(z)\right|\rho(\varphi(z),\psi(z)) + \left|M_{u,\varphi}(z) - M_{v,\psi}(z)\right|,\end{aligned}$$

which together with the assumptions implies the desired result.

$(c) \Rightarrow (a)$. Suppose that (5) and (6) hold. For any $f \in \mathcal{B}$, by Lemmas 2 and 3, we have

$$
\begin{aligned}
\|(D^n_{\varphi,u} - D^n_{\psi,v})f\|_{H^\infty_\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(D^n_{\varphi,u} - D^n_{\psi,v})f(z)| \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f^{(n)}(\varphi(z))u(z) - f^{(n)}(\psi(z))v(z)| \\
&= \sup_{z\in\mathbb{D}}|M_{u,\varphi}(z)f^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n - M_{v,\psi}(z)f^{(n)}(\psi(z))(1-|\psi(z)|^2)^n| \\
&\le \sup_{z\in\mathbb{D}}|M_{u,\varphi}(z) - M_{v,\psi}(z)||f^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n| \\
&\quad + \sup_{z\in\mathbb{D}}|M_{v,\psi}(z)||f^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n - f^{(n)}(\psi(z))(1-|\psi(z)|^2)^n| \\
&\le C\|f\|_{\mathcal{B}}\sup_{z\in\mathbb{D}}|M_{u,\varphi}(z) - M_{v,\psi}(z)| + C\|f\|_{\mathcal{B}}\sup_{z\in\mathbb{D}}|M_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \\
&< \infty.
\end{aligned}
$$

Therefore $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is bounded.

$(a) \Rightarrow (b)$. Suppose that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is bounded. For any point $w \in \mathbb{D}$ such that $|\varphi(w)| \ge 1/2$, set

$$
g_w(z) = \frac{1}{n!\overline{\varphi(w)}^n} \cdot \frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z} \tag{7}
$$

and let f_w be an analytic function with $f_w(0) = 0, f'_w(0) = 0, \cdots, f^{(n-1)}_w(0) = 0$ and

$$
f^{(n)}_w(z) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{1+n}}\sigma_{\varphi(w)}(z).
$$

It is easy to check that $f_w, g_w \in \mathcal{B}$. Moreover,

$$
\begin{aligned}
g^{(n)}_w(\varphi(w)) &= \frac{1}{(1-|\varphi(w)|^2)^n}, \qquad g^{(n)}_w(\psi(w)) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}\psi(w))^{1+n}}, \\
f^{(n)}_w(\varphi(w)) &= 0, \qquad f^{(n)}_w(\psi(w)) = \frac{(1-|\varphi(w)|^2)\sigma_{\varphi(w)}(\psi(w))}{(1-\overline{\varphi(w)}\psi(w))^{1+n}}.
\end{aligned}
$$

Since $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is bounded, we obtain

$$
\begin{aligned}
\infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})f_w\|_{H^\infty_\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f^{(n)}_w(\varphi(z))u(z) - f^{(n)}_w(\psi(z))v(z)| \\
&\ge (1-|w|^2)^\alpha|f^{(n)}_w(\varphi(w))u(w) - f^{(n)}_w(\psi(w))v(w)| \\
&= (1-|w|^2)^\alpha\frac{|v(w)|(1-|\varphi(w)|^2)\rho(\varphi(w),\psi(w))}{|1-\overline{\varphi(w)}\psi(w)|^{1+n}} \\
&= \left|M_{v,\psi}(w)\frac{(1-|\psi(w)|^2)^n(1-|\varphi(w)|^2)\rho(\varphi(w),\psi(w))}{(1-\overline{\varphi(w)}\psi(w))^{1+n}}\right|
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
\infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})g_w\|_{H^\infty_\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g^{(n)}_w(\varphi(z))u(z) - g^{(n)}_w(\psi(z))v(z)| \\
&\ge (1-|w|^2)^\alpha|g^{(n)}_w(\varphi(w))u(w) - g^{(n)}_w(\psi(w))v(w)| \\
&= \left|M_{u,\varphi}(w) - M_{v,\psi}(w)\frac{(1-|\psi(w)|^2)^n(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{1+n}}\right|.
\end{aligned} \tag{9}
$$

Multiplying (9) by $\rho(\varphi(w),\psi(w))$ and from (8) we obtain

$$\sup_{|\varphi(w)|>\frac{1}{2}} \left|M_{u,\varphi}(w)\right|\rho(\varphi(w),\psi(w)) < \infty. \tag{10}$$

If $|\varphi(w)| \le \frac{1}{2}$, taking $h_w(z) = \frac{(z-\psi(w))^{n+1}}{(n+1)!} \in \mathcal{B}$ and using the boundedness of $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$, we obtain

$$\begin{aligned} \infty > \|(D^n_{\varphi,u} - D^n_{\psi,v})h_w\|_{H^\infty_\alpha} &\ge (1-|w|^2)^\alpha |h^{(n)}_w(\varphi(w))u(w) - h^{(n)}_w(\psi(w))v(w)| \\ &= (1-|w|^2)^\alpha \left|u(w)(\varphi(w)-\psi(w))\right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{|\varphi(w)|\le\frac{1}{2}} \left|M_{u,\varphi}(w)\right|\rho(\varphi(w),\psi(w)) &= \sup_{|\varphi(w)|\le\frac{1}{2}} \frac{(1-|w|^2)^\alpha\left|u(w)(\varphi(w)-\psi(w))\right|}{(1-|\varphi(w)|^2)^n\left|1-\overline{\varphi(w)}\psi(w)\right|} \\ &\le (1-|w|^2)^\alpha |u(w)(\varphi(w)-\psi(w))\Big| < \infty. \end{aligned} \tag{11}$$

From (10) and (11), we get (4).

Exchanging φ and ψ in functions $f_w(z)$, $g_w(z)$ and $h_w(z)$, similarly to the above proof we can obtain (6).

Next, we prove that (5) holds. For $|\varphi(w)| > \frac{1}{2}$, by (9), we also have

$$\begin{aligned} \infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})g_w\|_{H^\infty_\alpha} \\ &\ge \left|M_{u,\varphi}(w) - M_{v,\psi}(w)\frac{(1-|\psi(w)|^2)^n(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{1+n}}\right| \\ &= \left|M_{u,\varphi}(w) - M_{v,\psi}(w) + M_{v,\psi}(w)(1-\frac{(1-|\psi(w)|^2)^n(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{1+n}})\right| \\ &\ge \left|M_{u,\varphi}(w) - M_{v,\psi}(w)\right| - \left|M_{v,\psi}(w)\right| \cdot \left|g^{(n)}_w(\varphi(w))(1-|\varphi(w)|^2)^n - g^{(n)}_w(\psi(w))(1-|\psi(w)|^2)^n\right|. \end{aligned} \tag{12}$$

From Lemma 3 and (6), we see that

$$\begin{aligned} &\left|M_{v,\psi}(w)\right| \cdot \left|g^{(n)}_w(\varphi(w))(1-|\varphi(w)|^2)^n - g^{(n)}_w(\psi(w))(1-|\psi(w)|^2)^n\right| \\ \le\ &\|g_w\|_{\mathcal{B}}|M_{v,\psi}(w)|\rho(\varphi(w),\psi(w)) < \infty, \end{aligned}$$

which with (12) imply that $|M_{u,\varphi}(w) - M_{v,\psi}(w)| < \infty$ holds for all $w \in \mathbb{D}$ with $|\varphi(w)| > \frac{1}{2}$.

When $|\varphi(w)| \le \frac{1}{2}$ and $|\psi(w)| > \frac{3}{4}$, then using (1) we get $\rho(\varphi(w),\psi(w)) > \frac{5}{27}$. From (4) and (6), we get $|M_{u,\varphi}(w) - M_{v,\psi}(w)| < \infty$.

When $|\varphi(w)| \le \frac{1}{2}$ and $|\psi(w)| \le \frac{3}{4}$, since $\frac{z^n}{n!} \in \mathcal{B}$, we get

$$\begin{aligned} \infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})(\frac{z^n}{n!})\|_{H^\infty_\alpha} = (1-|w|^2)^\alpha\left|u(w)-v(w)\right| \\ &= \left|(M_{u,\varphi}(w) - M_{v,\psi}(w))(1-|\varphi(w)|^2)^n + M_{v,\psi}(w)[(1-|\varphi(w)|^2)^n - (1-|\psi(w)|^2)^n]\right| \\ &\ge \left|M_{u,\varphi}(w) - M_{v,\psi}(w)\right|(1-|\varphi(w)|^2)^n - \left|M_{v,\psi}(w)[(1-|\varphi(w)|^2)^n - (1-|\psi(w)|^2)^n]\right|. \end{aligned} \tag{13}$$

Since the derivative of the function $g(x) = (1-x^2)^n$ is bounded on $[0,1]$, we have

$$\left|(1-|\varphi(w)|^2)^n - (1-|\psi(w)|^2)^n\right| \le C\left||\varphi(w)| - |\psi(w)|\right| \le \rho(\varphi(w),\psi(w)),$$

and hence

$$\left|M_{v,\psi}(w)[(1-|\varphi(w)|^2)^n - (1-|\psi(w)|^2)^n]\right| \le |M_{v,\psi}(w)|\rho(\varphi(w),\psi(w)),$$

which together with (6) and (13) implies $|M_{u,\varphi}(w) - M_{v,\psi}(w)| < \infty$, when $|\varphi(w)| \le \frac{1}{2}$ and $|\psi(w)| \le \frac{3}{4}$. Therefore we conclude that $\sup_{w\in\mathbb{D}} |M_{u,\varphi}(w) - M_{v,\psi}(w)| < \infty$. The proof is complete. □

When $u = v$, we get some characterizations of the boundedness of $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{B} \to H^\infty_\alpha$.

Corollary 1. *Let $\alpha > 0$, $\varphi, \psi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$ and n be a positive integer. Then the following statements are equivalent.*

(*a*) $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{B} \to H^\infty_\alpha$ *is bounded;*

(*b*) $\sup_{z\in\mathbb{D}} \left|M_{u,\varphi}(z)\right|\rho(\varphi(z), \psi(z)) < \infty,\ \sup_{z\in\mathbb{D}} \left|M_{u,\psi}(z)\right|\rho(\varphi(z), \psi(z)) < \infty;$

(*c*) $\sup_{z\in\mathbb{D}} \left|M_{u,\varphi}(z)\right|\rho(\varphi(z), \psi(z)) < \infty,\ \sup_{z\in\mathbb{D}} \left|M_{u,\varphi}(z) - M_{u,\psi}(z)\right| < \infty;$

(*d*) $\sup_{z\in\mathbb{D}} \left|M_{u,\psi}(z)\right|\rho(\varphi(z), \psi(z)) < \infty,\ \sup_{z\in\mathbb{D}} \left|M_{u,\varphi}(z) - M_{u,\psi}(z)\right| < \infty.$

Proof. Set $u = v$ in Theorem 1, we see that (a), (c) and (d) are equivalent. We only need to prove $(b) \Rightarrow (c)$. Now assume that (b) holds. It is easy to see that

$$\sup_{\rho(\varphi(z),\psi(z))>\frac{1}{2}} \left|M_{u,\varphi}(z) - M_{u,\psi}(z)\right| < \infty.$$

If $\rho(\varphi(z), \psi(z)) \le \frac{1}{2}$, then from (1), we have

$$\begin{aligned}\left|M_{u,\varphi}(z) - M_{u,\psi}(z)\right| &= \frac{(1-|z|^2)^\alpha|u(z)|}{(1-|\varphi(z)|^2)^n}\left|1-\left(\frac{1-|\varphi(z)|^2}{1-|\psi(z)|^2}\right)^n\right| \le \frac{(1-|z|^2)^\alpha|u(z)|}{(1-|\varphi(z)|^2)^n}\left|1-\left(\frac{1-\rho(\varphi(z),\psi(z))}{1+\rho(\varphi(z),\psi(z))}\right)^n\right| \\ &\le \left|M_{u,\varphi}(z)\right|\rho(\varphi(z), \psi(z)) < \infty.\end{aligned}$$

Therefore, we obtain $\sup_{z\in\mathbb{D}} \left|M_{u,\varphi}(z) - M_{u,\psi}(z)\right| < \infty$. The proof is complete.

Theorem 2. *Let $\alpha > 0$, $\varphi, \psi \in S(\mathbb{D})$, $u, v \in H(\mathbb{D})$ and n be a positive integer. Then $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is compact if and only if $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is bounded and the following equalities hold.*

$$\lim_{|\varphi(z)|\to1} \left|M_{u,\varphi}(z)\right|\rho(\varphi(z), \psi(z)) = 0; \tag{14}$$

$$\lim_{|\psi(z)|\to1} \left|M_{v,\psi}(z)\right|\rho(\varphi(z), \psi(z)) = 0; \tag{15}$$

$$\lim_{|\varphi(z)|\to1,|\psi(z)|\to1} |M_{u,\varphi}(z) - M_{v,\psi}(z)| = 0. \tag{16}$$

Proof. Necessity. Assume that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is compact. Then it is clear that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{B} \to H^\infty_\alpha$ is bounded. Let $\{z_k\}$ be a sequence of points in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define

$$f_k(z) = f_{z_k}(z) \ \text{ and } \ g_k(z) = g_{z_k}(z)$$

as in the proof of Theorem 1. From (8) and (9), we see that

$$\|(D^n_{\varphi,u} - D^n_{\psi,v})f_k\|_{H^\infty_\alpha} \ge \left|M_{v,\psi}(z_k)\frac{\rho(\varphi(z_k), \psi(z_k))(1-|\psi(z_k)|^2)^n(1-|\varphi(z_k)|^2)}{(1-\overline{\varphi(z_k)}\psi(z_k))^{n+1}}\right|, \tag{17}$$

$$\|(D^n_{\varphi,u} - D^n_{\psi,v})g_k\|_{H^\infty_\alpha} \ge \left|M_{u,\varphi}(z_k) - M_{v,\psi}(z_k)\frac{(1-|\varphi(z_k)|^2)(1-|\psi(z_k)|^2)^n}{(1-\overline{\varphi(z_k)}\psi(z_k))^{n+1}}\right|,$$

and hence

$$\begin{aligned}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_k\|_{H^\infty_\alpha} &\geq \rho(\varphi(z_k),\psi(z_k))\|(D^n_{\varphi,u}-D^n_{\psi,v})g_k\|_{H^\infty_\alpha}\\ &\geq \left|M_{u,\varphi}(z_k)\rho(\varphi(z_k),\psi(z_k))-\frac{M_{v,\psi}(z_k)\rho(\varphi(z_k),\psi(z_k))(1-|\psi(z_k)|^2)^n}{(1-\overline{\varphi(z_k)}\psi(z_k))^{n+1}(1-|\varphi(z_k)|^2)^{-1}}\right|. \qquad (18)\end{aligned}$$

Since $D^n_{\varphi,u}-D^n_{\psi,v}$ is compact, by Lemma 1, we have $\|(D^n_{\varphi,u}-D^n_{\psi,v})f_k\|_{H^\infty_\alpha}\to 0$ and $\|(D^n_{\varphi,u}-D^n_{\psi,v})g_k\|_{H^\infty_\alpha}\to 0$ as $k\to\infty$. From (17) and (18), we conclude that (14) holds. Exchanging φ and ψ in $f_k(z)$ and $g_k(z)$ and similar to the above proof, we can prove that (15) holds.

Next we prove that (16) holds. From (12), we have

$$\|(D^n_{\varphi,u}-D^n_{\psi,v})g_k\|_{H^\infty_\alpha}\geq \left|M_{u,\varphi}(z_k)-M_{v,\psi}(z_k)\right|-\left|M_{v,\psi}(z_k)\left[g_k^{(n)}(\varphi(z_k))(1-|\varphi(z_k)|^2)^n-g_k^{(n)}(\psi(z_k))(1-|\psi(z_k)|^2)^n\right]\right|.$$

From Lemma 3 and (15), we have

$$\begin{aligned}&\left|M_{v,\psi}(z_k)\left[g_k^{(n)}(\varphi(z_k))(1-|\varphi(z_k)|^2)^n-g_k^{(n)}(\psi(z_k))(1-|\psi(z_k)|^2)^n\right]\right|\\ \leq\ & \|g_k\|_{\mathcal{B}}|M_{v,\psi}(z_k)|\rho(\varphi(z_k),\psi(z_k))\to 0\end{aligned}$$

as $|\psi(z_k)|\to 1$. Since $\|(D^n_{\varphi,u}-D^n_{\psi,v})g_k\|_{H^\infty_\alpha}\to 0$ as $|\varphi(z_k)|\to 1$, we get $\left|M_{u,\varphi}(z_k)-M_{v,\psi}(z_k)\right|\to 0$ as $|\varphi(z_k)|\to 1$ and $|\psi(z_k)|\to 1$. This implies that (16) holds.

Sufficiency. Since assume that $D^n_{\varphi,u}-D^n_{\psi,v}:\mathcal{B}\to H^\infty_\alpha$ is bounded, we see that (4), (5) and (6) hold. By the assumption that (14), (15) and (16) hold, then for any $\varepsilon>0$, there exists an $r\in(0,1)$ such that

$$\left|M_{u,\varphi}(z)\right|\rho(\varphi(z),\psi(z))<\varepsilon \ \text{ when } |\varphi(z)|>r; \qquad (19)$$

$$\left|M_{v,\psi}(z)\right|\rho(\varphi(z),\psi(z))<\varepsilon \ \text{ when } |\psi(z)|>r; \qquad (20)$$

$$|M_{u,\varphi}(z)-M_{v,\psi}(z)|<\varepsilon \ \text{ when } |\varphi(z)|>r \ \text{ and } \ |\psi(z)|>r. \qquad (21)$$

Let $(l_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}$ such that $\|l_k\|_{\mathcal{B}}\leq 1$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. It is easy to see that

$$\begin{aligned}\|(D^n_{\varphi,u}-D^n_{\psi,v})l_k\|_{H^\infty_\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(D^n_{\varphi,u}-D^n_{\psi,v})l_k(z)|\\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|l_k^{(n)}(\varphi(z))u(z)-l_k^{(n)}(\psi(z))v(z)|\\ &= \sup_{z\in\mathbb{D}}\left|M_{u,\varphi}(z)l_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n-M_{v,\psi}(z)l_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^n\right|\\ &= \sup_{z\in\mathbb{D}}\left|F_k(z)+G_k(z)\right|, \qquad (22)\end{aligned}$$

where

$$\begin{aligned}F_k(z) &= (M_{u,\varphi}(z)-M_{v,\psi}(z))l_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n,\\ G_k(z) &= M_{v,\psi}(z)\left[l_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n-l_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^n\right].\end{aligned}$$

In order to prove that $D^n_{\varphi,u}-D^n_{\psi,v}$ is compact, we only need to prove that $\|(D^n_{\varphi,u}-D^n_{\psi,v})l_k\|_{H^\infty_\alpha}\to 0$ as $k\to\infty$ by Lemma 1.

(i) When $|\varphi(z)|\leq r$ and $|\psi(z)|\leq r$, by (5), we have

$$|F_k(z)|\leq \sup_{|\varphi(z)|\leq r}|l_k^{(n)}(\varphi(z))|=\sup_{w\in\mathbb{D}_r}|l_k^{(n)}(w)|,$$

where $\mathbb{D}_r = \{z \in \mathbb{D} : |z| \leq r < 1\}$. From Remark 1 and (6), we get

$$|G_k(z)| \leq C|M_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \sup_{z\in\mathbb{D}_r}(1-|z|^2)^n|l_k^{(n)}(z)| \leq C\sup_{z\in\mathbb{D}_r}(1-|z|^2)^n|l_k^{(n)}(z)| \leq \sup_{z\in\mathbb{D}_r}|l_k^{(n)}(z)|. \tag{23}$$

(ii) When $|\varphi(z)| \leq r$ and $|\psi(z)| > r$, similar to the case case (i), we obtain $|F_k(z)| \leq \sup_{|w|\leq r}|l_k^{(n)}(w)|$. Using Lemma 3 and (20), we obtain

$$|G_k(z)| \leq C\|l_k\|_{\mathcal{B}}|M_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \leq \varepsilon.$$

(iii) When $|\varphi(z)| > r$ and $|\psi(z)| > r$, by Lemma 3 and (21), we have

$$|F_k(z)| \leq C|M_{u,\varphi}(z) - M_{v,\psi}(z)|\|l_k\|_{\mathcal{B}} \leq \varepsilon.$$

Also, similar to the case (ii), we get $|G_k(z)| \leq \varepsilon$.

(iv) When $|\varphi(z)| > r$ and $|\psi(z)| \leq r$, we reset

$$M_{u,\varphi}(z)l_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n - M_{v,\psi}(z)l_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^n = H_k(z) + Q_k(z), \tag{24}$$

where

$$\begin{aligned} H_k(z) &= -(M_{v,\psi}(z) - M_{u,\varphi}(z))l_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^n, \\ Q_k(z) &= -M_{u,\varphi}(z)\Big[l_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^n - l_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^n\Big]. \end{aligned}$$

Using (5) again, we have

$$|H_k(z)| \leq \sup_{|\psi(z)|\leq r}|l_k^{(n)}(\psi(z))| = \sup_{w\in\mathbb{D}_r}|l_k^{(n)}(w)|.$$

Applying Lemma 3 and (19), we obtain

$$|Q_k(z)| \leq C\|l_k\|_{\mathcal{B}}|M_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) \leq \varepsilon.$$

Therefore, from (22), (24) and the discussion of the above cases, we obtain

$$\|(D^n_{\varphi,u} - D^n_{\psi,v})l_k\|_{H^\infty_\alpha} \leq \varepsilon + \sup_{|w|\leq r}|l_k^{(n)}(w)|. \tag{25}$$

In view of the fact that $\{w \in \mathbb{D} : |w| \leq r\}$ is compact, we see that $\|(D^n_{\varphi,u} - D^n_{\psi,v})l_k\|_{H^\infty_\alpha} \to 0$ as $k \to \infty$ from (25). The proof is complete.

Corollary 2. *Let $\alpha > 0$, $\varphi, \psi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$ and n be a positive integer. Then $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{B} \to H^\infty_\alpha$ is compact if and only if $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{B} \to H^\infty_\alpha$ is bounded,*

$$\lim_{|\varphi(z)|\to 1}\big|M_{u,\varphi}(z)\big|\rho(\varphi(z),\psi(z)) = 0 \ \textit{and} \ \lim_{|\psi(z)|\to 1}\big|M_{u,\psi}(z)\big|\rho(\varphi(z),\psi(z)) = 0.$$

Proof. Necessity. This implication is obvious from Theorem 2 by taking $u = v$.

Sufficiency. Assume that $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{B} \to H^\infty_\alpha$ is bounded,

$$\lim_{|\varphi(z)|\to 1}\big|M_{u,\varphi}(z)\big|\rho(\varphi(z),\psi(z)) = 0 \text{ and } \lim_{|\psi(z)|\to 1}\big|M_{u,\psi}(z)\big|\rho(\varphi(z),\psi(z)) = 0.$$

By Theorem 2, to prove that $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{B} \to H^\infty_\alpha$ is compact, we only need to prove that

$$\lim_{|\varphi(z)|\to 1, |\psi(z)|\to 1}\big|M_{u,\varphi}(z) - M_{u,\psi}(z)\big| = 0. \tag{26}$$

Suppose (26) does not hold, then there exist $\varepsilon_0 > 0$ and a sequence $\{z_k\}$ in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ and $|\psi(z_k)| \to 1$ as $k \to \infty$, but

$$\left|M_{u,\varphi}(z_k) - M_{u,\psi}(z_k)\right| \geq \varepsilon_0. \tag{27}$$

First, we claim that $\rho(\varphi(z_k), \psi(z_k)) \to 0$ as $k \to \infty$. In fact, if it is not the case, then there exists a subsequence $\{z_{n_k}\}$ in $\{z_k\}$ such that $\rho(\varphi(z_{n_k}), \psi(z_{n_k})) \to b > 0$. On the other hand, we have

$$\lim_{k\to\infty} \left|M_{u,\varphi}(z_{n_k})\right| \rho(\varphi(z_{n_k}), \psi(z_{n_k})) = 0, \ \lim_{k\to\infty} \left|M_{u,\psi}(z_{n_k})\right| \rho(\varphi(z_{n_k}), \psi(z_{n_k})) = 0.$$

Hence

$$\lim_{k\to\infty} \left|M_{u,\varphi}(z_{n_k})\right| = 0, \ \lim_{k\to\infty} \left|M_{u,\psi}(z_{n_k})\right| = 0,$$

which contradict with (27).

So we assume $\rho(\varphi(z_k), \psi(z_k)) \leq \frac{1}{2}$ for large enough k. Similarly to the proof of Corollary 1, we have

$$\left|M_{u,\varphi}(z_k) - M_{u,\psi}(z_k)\right| \leq \left|M_{u,\varphi}(z_k)\right| \rho(\varphi(z_k), \psi(z_k)) \to 0,$$

which contradict with (27) again. Therefore, (26) holds. The proof is complete.

From Theorems 1 and 2 with $v(z) = 0$, we obtain the characterization of the boundedness and compactness of $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\alpha$ (see [24]).

Corollary 3. *Let $\alpha > 0$, $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$ and n be a positive integer. Then the following statements hold.*

(a) The operator $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\alpha$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)^\alpha |u(z)|}{(1 - |\varphi(z)|^2)^n} < \infty.$$

(b) The operator $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\alpha$ is compact if and only if $D^n_{\varphi,u} : \mathcal{B} \to H^\infty_\alpha$ is bounded and

$$\lim_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)^\alpha |u(z)|}{(1 - |\varphi(z)|^2)^n} = 0.$$

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