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On Completely Monotonic and Related Functions

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Abstract. In this article, we study recent investigations on completely monotonic and related functions. An open problem is presented.

Let us first introduce some notations which shall be used in the article. Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{R}^+ := (0, \infty),$$

I^+ is an open interval contained in $\mathbb{R}^+$, I^o is the interior of the interval $I \subset \mathbb{R}$, $\mathbb{R}$ is the set of all real numbers, $\mathcal{R}(f)$ denotes the range of the function f and $C(I)$ is the class of all continuous functions on I.

In this review article we shall only mention the recent developments on completely monotonic and related functions. For older results on completely monotonic and absolutely monotonic functions please see, for example, [37, Chapter IV].

We now recall some definitions we shall use.

Definition 1 (See [4]). *A function f is said to be absolutely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$f^{(n)}(x) \geq 0 \quad (x \in I^o).$$

The class of all *absolutely monotonic functions* on the interval I is denoted by $AM(I)$.

Definition 2 (See [4]). *A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I^o).$$

The class of all *completely monotonic functions* on the interval I is denoted by $CM(I)$.

By Leibniz's rule for the derivative of the product function fg of order n, we can easily prove that if

$$f, g \in CM(I)(AM(I)),$$

then the product function

$$fg \in CM(I)(AM(I)).$$

Dubourdieu [5] showed

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Theorem 3. *A non-constant completely monotonic function on* $I := (a, \infty)$ *is strictly completely monotonic there - i.e.*

$$(-1)^n f^{(n)}(x) > 0 \quad (n \in \mathbb{N}_0, \quad x \in I).$$

Lorch and Szego [24] also proved this result.

Remark 4. *We notice that if the interval* (a, ∞) *in Theorem 3 is replaced by other kinds of intervals , the conclusion may not be true. We can try this example:*

$$f(x) := x^2, I := (-\infty, 0).$$

In 1989, Trimble, et al [35] introduced the following notion:

Definition 5 (See [35]). *A function* f *is said to be strongly completely monotonic on* I^+ *if, for all* $n \in \mathbb{N}_0$, $(-1)^n x^{n+1} f^{(n)}(x)$ *are nonnegative and decreasing on* I^+.

The class of such functions on the interval I^+ is denoted by $SCM(I^+)$. By the definition we know that

$$\alpha f + \beta g \in SCM(I^+)$$

if

$$f, g \in SCM(I^+) \quad \text{and} \quad \alpha, \beta \geq 0.$$

Using Leibniz's rule we can get

$$fg \in SCM(I^+)$$

if

$$f, g \in SCM(I^+).$$

Trimble, et al [35] proved

Theorem 6. $f \in SCM(\mathbb{R}^+)$ *if and only if there exists a non-negative and non-decreasing function* $\phi(t)$ *on* $[0, \infty)$ *such that*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt, \quad x \in \mathbb{R}^+.$$

It is easy to see that $SCM(I^+)$ is a nontrivial subset of $CM(I^+)$.

Definition 7 (See [2]). *A function* f *is said to be logarithmically completely monotonic on an interval* I *if* $f > 0$, $f \in C(I)$, *has derivatives of all orders on* I^o *and for* $n \in \mathbb{N}$

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \quad (x \in I^o).$$

The set of all *logarithmically completely monotonic functions* on the interval I is denoted by $LCM(I)$.
In [29] the authors proved

Theorem 8.

$$LCM(I) \subset CM(I).$$

For the interval $\mathbb{R}^+$ Horn [21] proved, in terms of logarithmically completely monotonic functions, that

$$f \in LCM(\mathbb{R}^+) \Longleftrightarrow f \not\equiv 0$$

and

$$\sqrt[n]{f} \in CM(\mathbb{R}^+) \quad (n \in \mathbb{N}).$$

A function f such that

$$\sqrt[n]{f} \in CM(\mathbb{R}^+) \quad (n \in \mathbb{N})$$

is called infinitely divisible completely monotonic (*cf.* [21]). From Horn's investigation in [21], we have

$$LCM(\mathbb{R}^+) \subset CM(\mathbb{R}^+).$$

Definition 9 (See [16]). *A function f is said to be strongly logarithmically completely monotonic on I^+ if $f > 0$ and, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1}[\ln f(x)]^{(n)}$ are nonnegative and decreasing on I^+.*

Such a function class on the interval I^+ is denoted by $SLCM(I^+)$.

It is apparent that the class $SLCM(I^+)$ is a nontrivial subclass of $LCM(I^+)$ and that if

$$f, g \in SLCM(I^+)(LCM(I)),$$

then

$$fg \in SLCM(I^+)(LCM(I)).$$

In [16] the authors proved an important relationship between $SLCM(\mathbb{R}^+)$ and $SCM(\mathbb{R}^+)$ as follows.

Theorem 10. $SLCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) = \emptyset$.

In words, a strongly logarithmically completely monotonic function on $\mathbb{R}^+$ can not be strongly completely monotonic on $\mathbb{R}^+$, or, a strongly completely monotonic function on $\mathbb{R}^+$ can not be strongly logarithmically completely monotonic on $\mathbb{R}^+$.

The following result (see [16]) also reveals a relationship between $SLCM(I^+)$ and $SCM(I^+)$.

Theorem 11. *Suppose that*

$$f \in C(I^+), \quad f > 0 \quad \text{and} \quad f' \in SCM(I^+).$$

If

$$xf'(x) \geq f(x) \quad (x \in I^+),$$

then

$$\frac{1}{f} \in SLCM(I^+).$$

Remark 12. *The following condition:*

$$xf'(x) \geq f(x) \quad (x \in I^+)$$

in Theorem 11 cannot be dropped. See the counterexample contained in [16, Remark 2]

If in Definition 2 the set $\mathbb{N}_0$ is replaced with the set $\mathbb{N}$, the function will be called *almost completely monotonic function* [34]. If in Definition 5 the set $\mathbb{N}_0$ is replaced with the set $\mathbb{N}$, the function will be called *almost strongly completely monotonic function* [16]. The class of all almost completely monotonic functions on the interval I and the class of all almost strongly completely monotonic functions on the interval I^+ are denoted by $ACM(I)$ and by $ASCM(I^+)$, respectively. These two terminologies are useful to simplify the statements of the results.

The following result is the Lemma 2 of [34], which shows a property of $ACM(I)$ and $ASCM(I^+)$.

Theorem 13.

1. $-f \in ACM(I)$ *if and only if* $f \in C(I)$ *and* $f' \in CM(I^o)$.
2. $-f \in ASCM(I^+)$ *implies* $f' \in SCM(I^+)$.

Remark 14. *Please note that the converse of Theorem 13(2) is not true. See the counterexample give in [34, Remark 2].*

The result below can be derived directly from the definitions.

Theorem 15.

1. $f \in LCM(I)$ *if and only if* $f > 0$ *and* $\ln f \in ACM(I)$.
2. $f \in SLCM(I^+)$ *if and only if* $f > 0$ *and* $\ln f \in ASCM(I^+)$.

The following two results (see [16]) show relationships between $SLCM(I^+)$ and $ASCM(I^+)$.

Theorem 16. $SLCM(I^+) \subset ASCM(I^+)$.

That is, a strongly logarithmically completely monotonic function on the interval I^+ must be almost strongly completely monotonic on I^+.

Theorem 17. *Suppose that*

$$f \in C(I^+), f > 0 \quad \text{and} \quad -f \in ASCM(I^+).$$

Then

$$\frac{1}{f} \in SLCM(I^+).$$

In [16] authors also showed

Theorem 18. *Suppose that*

$$f \in C(I), \quad f > 0 \quad \text{and} \quad f' \in CM(I^o).$$

Then

$$\frac{1}{f} \in LCM(I).$$

In [6] the following result was obtained, which shows a relationship between $ACM(I)$ and $LCM(I)$.

Theorem 19. *Suppose that*

$$f > 0 \quad \text{and} \quad -f \in ACM(I),$$

then

$$\frac{1}{f} \in LCM(I).$$

For the interpolation of sequences by completely monotonic functions, the author [10] established

Theorem 20. *Suppose that the sequence $\{\mu_n\}_0^\infty$ is completely monotonic, then for any $\varepsilon \in (0,1)$, there exists a continuous interpolating function $f(x)$ on the interval $[0,\infty)$ such that $f|_{[0,\varepsilon]}$ and $f|_{[\varepsilon,\infty)}$ are both completely monotonic and*

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

Remark 21. *From the result of widder (see Theorem 14b of [37, Chapter IV]) we know that if the sequence $\{\mu_n\}_{n=0}^\infty$ is not minimal completely monotonic, then there does not exist a function $f \in CM[0,\infty)$ such that*

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

For the notion of a completely monotonic sequence see [20]. For the notion of a minimal completely monotonic sequence see [38].

As a corollary of Theorem 20, the following result was also given in [10].

Theorem 22. *Suppose that the sequence $\{\mu_n\}_{n=0}^\infty$ is completely monotonic. Then there exists a completely monotonic interpolating function $g(x)$ on the interval $[1,\infty)$ such that*

$$g(n) = \mu_n, \quad n \in \mathbb{N}.$$

Regarding the compositions of completely monotonic and related functions, the following two results were given in [37, Chapter IV].

Theorem 23. *Suppose that*

$$f \in AM(I_1), \quad g \in AM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in AM(I)$.

Theorem 24. *Suppose that*

$$f \in AM(I_1), \quad g \in CM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in CM(I)$.

Remark 25. *We observe that* $f \circ g$ *may neither belong to* $CM(I)$ *nor belong to* $AM(I)$ *when*

$$f \in CM(I_1), \quad g \in AM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

See the counterexample given in [6, Remark 1.5].

The result below (see [25, Theorem 5]) is a converse of Theorem 24.

Theorem 26. *Let* f *be defined on* $[0, \infty)$. *If, for each* $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$, *then* $f \in AM(\mathbb{R}^+)$.

The following result was given in [26].

Theorem 27. *Suppose that*

$$f \in CM(I_1), \quad g \in C(I), \quad g' \in CM(I^o) \quad \text{and} \quad \mathcal{R}(g) \subset I_1,$$

then $f \circ g \in CM(I)$.

In [25] the authors gave an interesting result related to Theorem 27 as follows.

Theorem 28. *For each function* $f \in CM(I)$, *where* $I := [0, \infty)$, *there exists a function* g *on* I *such that*

$$g(0) = 0, \quad f \circ g \in CM(I) \quad \text{and} \quad g' \notin CM(\mathbb{R}^+).$$

This result shows that the condition:

$$g' \in CM(I^o)$$

in Theorem 27 is not a necessary condition.

In [18], the following results were shown.

Theorem 29. *Suppose that*

$$f \in ACM(I_1), \quad g \in C(I), \quad g' \in CM(I^o) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in ACM(I)$.

Corollary 30. *Suppose that*

$$f \in ACM(I_1), \quad -g \in ACM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in ACM(I)$.

Theorem 31. *Suppose that*

$$f \in LCM(I_1), \quad g \in C(I), \quad g' \in CM(I^o) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in LCM(I)$.

Theorem 32. *Suppose that*

$$f \in SLCM(I_1^+), \ \ g' \in SCM(I^+) \quad and \quad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geq g(x) \quad (x \in I^+),$$

then $f \circ g \in SLCM(I^+)$.

Remark 33. *The condition:*

$$2xg'(x) \geq g(x) \quad (x \in I^+)$$

in Theorem 32 cannot be waived. See the counterexample given in [18, Remark 3].

Theorem 34. *Suppose that*

$$f \in LCM(I_1), \quad -g \in ACM(I) \quad and \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in LCM(I)$.

Theorem 35. *Let* I_1 *and* I *be open intervals. Also let* f *and* g *be defined on* I_1 *and* I*, respectively. If*

$$f' \in LCM(I_1), \ \ g' \in LCM(I) \quad and \quad \mathcal{R}(g) \subset I_1,$$

then $(f \circ g)' \in LCM(I)$.

In [34], the authors established the results below.

Theorem 36. *Suppose that*

$$f \in AM(I), g \in ASCM(I^+) \quad and \quad \mathcal{R}(g) \subset I.$$

Then $f \circ g \in ASCM(I^+)$.

Theorem 37. *Let* I_1 *be an open interval and* f *defined on* I_1.

1. *If* $f' \in AM(I_1), g \in ACM(I)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in ACM(I)$.
2. *If* $f' \in AM(I_1), g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $-(f \circ g)' \in ASCM(I^+)$.

Theorem 38. *Suppose that*

$$f \in ASCM(I_1^+), \ \ g' \in SCM(I^+) \quad and \quad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geq g(x) \quad (x \in I^+),$$

then $f \circ g \in ASCM(I^+)$.

Remark 39. *The condition:*

$$2xg'(x) \geq g(x) \quad (x \in I^+)$$

in Theorem 38 can not be waived even if the hypothesis $f \in ASCM(I_1^+)$ *is replaced by a stronger condition:* $f \in SCM(I_1^+)$*. See the counterexample given in [34, Remark 1].*

We shall also mention Lemma 3 of [34] here.

Theorem 40. *Suppose that*

$$f \in AM(I_1), g \in ACM(I) \quad and \quad \mathcal{R}(g) \subset I_1.$$

Then

$$f \circ g \in CM(I).$$

Remark 41. *Theorem 8 can be regarded as a corollary of Theorem 40. In fact, let*

$$f \in LCM(I),$$

then

$$\ln f \in ACM(I).$$

Since

$$e^x \in AM(\mathbb{R}),$$

by Theorem 40, we obtain

$$e^{\ln f} = f \in CM(I).$$

In [6] the following results were established.

Theorem 42. *Suppose that*

$$f \in ACM(I_1), \quad -g \in ASCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then

$$f \circ g \in ASCM(I^+).$$

Theorem 43. *Suppose that*

$$f \in LCM(I_1), \quad -g \in ASCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then

$$f \circ g \in SLCM(I^+).$$

Theorem 44. *Let I_1 and I be open intervals, and let f and g be defined on I_1 and I respectively. If*

$$f' \in CM(I_1), \;\; g' \in CM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then

$$(f \circ g)' \in CM(I).$$

Theorem 45. *Let f and g be defined on I_1^+ and I^+ respectively. Suppose that*

$$f' \geq 0, \;\; f' \in ASCM(I_1^+), \;\; g' \in SCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geq g(x) \quad (x \in I^+),$$

then

$$(f \circ g)' \in ASCM(I^+).$$

Open Problem

Can the condition:

$$2xg'(x) \geq g(x) \quad (x \in I^+)$$

in Theorem 45 be waived?

There are also a lot of investigations on specific completely monotonic or related functions and their applications. For several recent works, see (for example) [1, 3, 7–9, 11–15, 17, 19, 22, 23, 27, 28, 30–33, 36, 39, 40].

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