



The Generalized of Selberg's Inequalities in C^* -Module

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Abstract. We obtain a Generalized of Selberg's type inequalities in Hilbert spaces and their extensions in operators algebras, in C^* -modules and in algebras of adjointable $\mathbb{A}$ -linear maps. Some applications for improving the Bessel type inequality result are given.

1. Introduction

Let $\mathbb{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The inequality of Selberg

$$\sum_{j=1}^n \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k \rangle|} \leq \|x\|^2, \quad x, y_1, \dots, y_n \in \mathbb{H}, y_j \neq 0, \quad 0 \leq j \leq n, \quad (1)$$

is originating from analytic theory of numbers [18]. It was discovered by A. Selberg around 1949, on account of the arguments of the distribution of primes [1,4,12,13,18,19]. Since that time it has interested many mathematicians who gave it many proofs, many extensions and refinements, see [1,5,10,11,13,17]. It is useful to recall that Schwartz's inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \quad x, y \in \mathbb{H} \quad (2)$$

and Bessel's inequality

$$\sum_{j=1}^n |\langle x, y_j \rangle|^2 \leq \|x\|^2, \quad x \in \mathbb{H}, y_1, \dots, y_n \text{ are nonzero and orthogonal in } \mathbb{H}, \quad (3)$$

are special cases of Selberg's inequality. Let me cite also, on the occasion, following inequalities encountered to this subject in the literature. Thus, in chronological order of publication, we have.

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In 1958 Heilbronn's inequality [13],

$$\sum_{i=1}^n |\langle x, y_i \rangle| \leq \|x\| \left(\sum_{i,j=1}^n |\langle y_i, y_j \rangle| \right)^{\frac{1}{2}}, \quad x, y_1, \dots, y_n \in \mathbb{H}.$$

In 1971, Bombieri's inequality [1],

$$\sum_{i=1}^n |\langle x, y_i \rangle|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle y_i, y_j \rangle| \right\}, \quad x, y_1, \dots, y_n \in \mathbb{H},$$

In 1992 J.E. Pečarić's inequality [17],

$$\left| \sum_{i=1}^n c_i \langle x, y_i \rangle \right|^2 \leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle y_i, y_j \rangle| \right\}, \quad x, y_1, \dots, y_n \in \mathbb{H}.$$

Moreover, in 1998, M. Fujii and R. Nakamoto [11] obtained in a Hilbert space, the following refinement for previous inequalities,

$$|\langle y, x \rangle|^2 + \sum_{j=1}^n \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k \rangle|} \|y\|^2 \leq \|x\|^2 \|y\|^2, \quad x, y, y_1, \dots, y_n \in \mathbb{H}, \quad (4)$$

with the condition that $\langle y, y_i \rangle = 0$.

The goal of this paper is to show a generalized of selberg's inequality in Hilbert spaces and their extensions in algebras of operators, in Hilbert C^* -modules and in algebras of adjointable $\mathbb{A}$ -linear maps.

2. Preliminaries in Hilbert C^* -Modules

In this section we briefly recall the definitions and examples of Hilbert C^* -modules. For information about Hilbert C^* -module, we refer to ([8,9,16]). Our reference for C^* -algebras is ([3]). Let $\mathbb{A}$ be a C^* -algebra (not necessarily unitary) and $\mathbb{X}$ be a complex linear space.

Definition 2.1. A pre-Hilbert $\mathbb{A}$ -module is a right $\mathbb{A}$ -module $\mathbb{X}$ equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{A}$ satisfying

1. $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0$ if and only if $x = 0$ for all x in $\mathbb{X}$,
2. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all x, y, z in $\mathbb{X}$, α, β in $\mathbb{C}$,
3. $\langle x, y \rangle = \langle y, x \rangle^*$ for all x, y in $\mathbb{X}$,
4. $\langle x, ya \rangle = \langle x, y \rangle a$ for all x, y in $\mathbb{X}$, a in $\mathbb{A}$.

The map $\langle \cdot, \cdot \rangle$ is called an $\mathbb{A}$ -valued inner product of $\mathbb{X}$, and for $x \in \mathbb{X}$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ which is a norm on $\mathbb{X}$, where the latter norm denotes that in the C^* -algebra $\mathbb{A}$. This norm makes $\mathbb{X}$ into a right normed module over $\mathbb{A}$. A pre-Hilbert module $\mathbb{X}$ is called a Hilbert $\mathbb{A}$ -module if it is complete with respect to its

norm. Two typical examples of Hilbert C^* -modules are as follows:

(I) Every Hilbert space is a Hilbert C^* -module.

(II) Every C^* -algebra $\mathbb{A}$ is a Hilbert $\mathbb{A}$ -module via $\langle a, b \rangle = a^*b$ ($a, b \in \mathbb{A}$).

Notice that the inner product structure of a C^* -algebra is essentially more complicated than complex numbers.

One may define an $\mathbb{A}$ -valued norm $|\cdot|$ by $|x| = \langle x, x \rangle^{\frac{1}{2}}$. Clearly, $\|x\| = \||x|\|$ for each $x \in \mathbb{X}$. It is known that $|\cdot|$ does not satisfy the triangle inequality in general.

We recall also the definition and the following result, they will be used in the later.

Definition 2.2. An adjointable module map $t : E \rightarrow F$ has a polar decomposition if there is a partial isometry $u : E \rightarrow F$ such that $T = u|T|$, $t = u|t|$ and $\text{Ker}(u) = \text{Ker}(t)$, $\text{Ran}(u) = \overline{\text{Ran}(t)}$, $\text{ker}(u^* = \text{ker}(t^*))$ and $\text{Ran}(u^*) = \text{Ran}(|t|)$.

In general bounded adjointable $\mathbb{A}$ -module maps between Hilbert $\mathbb{A}$ -modules do not have polar composition, but M. Joita [14] has given a necessary and sufficient condition for bounded adjointable module maps to admit polar decomposition.

Theorem 2.3. A bounded adjointable operator t has polar decomposition if and only if $\text{Ran}(t)$ and $\text{Ran}(|t|)$ are orthogonal direct summands.

The following lemma is useful to prove this Selberg's inequality.

Lemma 2.4. (see [6]) Let $\mathbb{A}$ be a C^* -algebra $a, b, c \in \mathbb{A}$. Then

$$a^*cb + b^*c^*a \leq \|c\|(|a|^2 + |b|^2). \quad (5)$$

3. Generalized of the Selberg's Inequality for Hilbert Space

We start our work by presenting a the generalized of the Selberg's inequality for Hilbert space.

Theorem 3.1. Let $\mathbb{H}$ be an Hilbert space and y_{ij} be a non zero vectors in $\mathbb{H}$, such that y_{ij} is orthogonal to y_{kj} for all $i \neq k$ in $\{1, \dots, m\}$ and $j \in \{1, \dots, n_i\}$. If $x \in \mathbb{H}$ then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} |\langle y_{ij}, y_{ik} \rangle|} \leq \|x\|^2. \quad (6)$$

Proof. Let α_{ij} , $1 \leq i \leq m$, $1 \leq j \leq n_i$ be scalars elements in $\mathbb{C}$. We can write

$$0 \leq \left\| x - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\|^2$$

and

$$\begin{aligned}
 & \left\| x - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\|^2 \\
 &= \left\langle x - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij}, x - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\rangle \\
 &= \langle x, x \rangle - \left\langle x, \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\rangle - \left\langle \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij}, x \right\rangle + \left\langle \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij}, \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\rangle \\
 &= \|x\|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} \langle x, y_{ij} \rangle - \sum_{i=1}^m \sum_{j=1}^{n_i} \overline{\alpha_{ij}} \langle y_{ij}, x \rangle + \sum_{i=1}^m \sum_{j,l=1}^{n_i} \overline{\alpha_{ij}} \alpha_{il} \langle y_{ij}, y_{il} \rangle
 \end{aligned}$$

We choose

$$\alpha_{ij} = \frac{\langle y_{ij}, x \rangle}{\sum_{k=1}^{n_i} |\langle y_{ij}, y_{ik} \rangle|},$$

then we get

$$\begin{aligned}
 \left\| x - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} \right\|^2 &= \|x\|^2 - 2 \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle x, y_{ij} \rangle|^2}{\sum_{k=1}^{n_i} |\langle y_{ij}, y_{ik} \rangle|} + \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle x, y_{ij} \rangle|^2}{\sum_{k=1}^{n_i} |\langle y_{ij}, y_{ik} \rangle|} \\
 &= \|x\|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle x, y_{ij} \rangle|^2}{\sum_{k=1}^{n_i} |\langle y_{ij}, y_{ik} \rangle|}.
 \end{aligned}$$

Then

$$\|x\|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle x, y_{ij} \rangle|^2}{\sum_{k=1}^{n_i} |\langle y_{ij}, y_{ik} \rangle|} \geq 0, \quad (7)$$

which ends the proof. $\square$

The following Selbergs inequality in [10] can be obtained by taking $m = 1$ in Theorem 3.1.

Theorem 3.2 (See [10]). Let $\mathbb{H}$ be a Hilbert space and $y_1 \dots y_n$ non zero vectors in $\mathbb{H}$. If $x \in \mathbb{H}$, then

$$\sum_{j=1}^n \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k \rangle|} \leq |x|^2. \quad (8)$$

The following refinement of Selbergs inequality in [11] can be obtained by taking $m = 2$ and $n_2 = 1$ in Theorem 3.1.

Theorem 3.3 (See [11]). Let $\mathbb{H}$ be a Hilbert space, y and $y_1 \dots y_n$ non zero vectors in $\mathbb{H}$ such that $\langle y, y_j \rangle = 0$ for $j = 1 \dots n$. If $x \in \mathbb{H}$ then

$$|\langle y, x \rangle|^2 + \sum_{j=1}^n \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k \rangle|} \|y\|^2 \leq |x|^2 \|y\|^2. \quad (9)$$

The following theorems give the generalized Selberg's inequality in an algebra of operators in Hilbert space.

Theorem 3.4. Let $\mathbb{H}$ be a Hilbert space and $T = U|T|$ be the polar decomposition of an operator T on $\mathbb{H}$, $\{y_{ij}; i = 1, 2, \dots, m \text{ and } j = 1 \dots n_i\} \not\subset \ker(T^*)$ and $\alpha \in [0, 1]$.

If $(|T|^{1-\alpha}U^*y_{ij}, |T|^{1-\alpha}U^*y_{kj}) = 0$ for all j and $k \neq i$, then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle Tx, y_{ij} \rangle|^2}{\sum_{k=1}^{n_i} |\langle |T^*|^{2(1-\alpha)} y_{ik}, y_{ij} \rangle|} \leq \| |T|^\alpha x \|^2 \quad (10)$$

holds for all $x \in \mathbb{H}$.

Proof. We replace, in Theorem 3.1 respectively x and y_{ij} by $|T|^\alpha x$ and $|T|^{1-\alpha}U^*y_{ij}$, for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1 \dots n_i\}$. Then we have the result. $\square$

Theorem 3.5. Let $\mathbb{H}$ be a Hilbert space and $T = U|T|$ be the polar decomposition of an operator T on $\mathbb{H}$, $\{y_{ij}; i = 1, 2, \dots, m \text{ and } j = 1 \dots n_i\} \not\subset \ker(T^*)$ and $\alpha, \alpha_1, \dots, \alpha_m$, such that $\alpha + \alpha_i \geq 1$, for all $i = 1, \dots, m$. If $\langle |T|^{\alpha_i}U^*y_{ij}, |T|^{\alpha_k}U^*y_{kj} \rangle = 0$ for all j and $k \neq i$, then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle |T|^{\alpha_i+\alpha-1}x, y_{ij} \rangle|^2}{\sum_{k=1}^{n_i} |\langle |T^*|^{2\alpha_i} y_{ik}, y_{ij} \rangle|} \leq \| |T|^\alpha x \|^2 \quad (11)$$

holds for all $x \in \mathbb{H}$.

Proof. We replace x and y_{ij} by $|T|^\alpha x$ and $|T|^{\alpha_i}U^*y_{ij}$ in Theorem 3.1 respectively. Then we have the result. $\square$

This two previous theorems are the generalized of the following result obtained in [11]:

Theorem 3.6 (See [11]). Let $\mathbb{H}$ be a Hilbert space and $T = U|T|$ be the polar decomposition of an operator T on $\mathbb{H}$, $\{y_j; j = 1, 2, \dots, n\} \not\subset \ker(T^*)$ and $\alpha \in [0, 1]$. If $\langle U|T|^{1-\alpha}y, y_j \rangle = 0$ for all j and $i \neq k$, then

$$|\langle |T|^\alpha x, y \rangle|^2 + \sum_{j=1}^n \frac{|\langle Tx, y_j \rangle|^2}{\sum_{k=1}^n |\langle |T^*|^{2(1-\alpha)} y_j, y_k \rangle|} \|y\|^2 \leq \| |T|^\alpha x \|^2 \|y\|^2$$

holds for all $x \in \mathbb{H}$.

Theorem 3.7 (See [11]). Let $\mathbb{H}$ be a Hilbert space. Suppose that $\{y_j; j = 1, 2, \dots, n\} \not\subset \ker(T^*)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1 \geq \alpha$. If $(|T^*|^{\beta+1-\alpha}y, y_j) = 0$ for all j , then

$$|\langle |T|^{\alpha+\beta-1}x, y \rangle|^2 + \sum_{j=1}^n \frac{|\langle Tx, y_j \rangle|^2 \| |T^*|^\beta y \|^2}{\sum_{k=1}^n |\langle |T^*|^{2(1-\alpha)} y_j, y_k \rangle|} \leq \| |T|^\alpha x \|^2 \| |T^*|^\beta y \|^2$$

holds for all $x \in \mathbb{H}$. In particular, if $\langle |T^*|^{2(1-\alpha)}y, y_j \rangle = 0$ for $\alpha \in [0, 1]$, then

$$|\langle Tx, y \rangle|^2 + \sum_{j=1}^n \frac{|\langle Tx, y_j \rangle|^2 \| |T^*|^{1-\alpha}y \|^2}{\sum_{k=1}^n |\langle |T^*|^{2(1-\alpha)} y_j, y_k \rangle|} \leq \| |T|^\alpha x \|^2 \| |T^*|^{1-\alpha}y \|^2$$

holds for all $x \in \mathbb{H}$.

4. Generalized of Selberg's Inequality in C^* -Module

The following theorem gives an extension of generalized Selberg's inequality in a Hilbert $\mathbb{A}$ -module.

Theorem 4.1. Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$ -module and y_{ij} be a non zero vectors in $\mathbb{X}$, such that y_{ij} is orthogonal to y_{kj} for all $i \neq k$ in $\{1, \dots, m\}$ and $j \in \{1, \dots, n_i\}$. If $x \in \mathbb{X}$ then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|} \leq |x|^2. \quad (12)$$

Proof. For any $\alpha_{ij} \in \mathbb{A}$, we have

$$\begin{aligned} 0 &\leq \left| x - \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij} \right|^2 = \left\langle x - \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij}, x - \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij} \right\rangle \\ &= \langle x, x \rangle - \left\langle \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij}, x \right\rangle - \left\langle x, \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij} \right\rangle + \left\langle \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij}, \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij} \right\rangle \\ &= |x|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^* \langle y_{ij}, x \rangle - \sum_{i=1}^m \sum_{j=1}^{n_i} \langle x, y_{ij} \rangle \alpha_{ij} + \sum_{i=1}^m \sum_{j,l=1}^{n_i} \alpha_{ij}^* \langle y_{ij}, y_{il} \rangle \alpha_{il} \\ &= |x|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^* \langle y_{ij}, x \rangle - \sum_{i=1}^m \sum_{j=1}^{n_i} \langle x, y_{ij} \rangle \alpha_{ij} + \frac{1}{2} \sum_{i=1}^m \sum_{j,l=1}^{n_i} (\alpha_{ij}^* \langle y_{ij}, y_{il} \rangle \alpha_{il} \\ &\quad + \alpha_{il}^* \langle y_{il}, y_{ij} \rangle \alpha_{ij}). \end{aligned}$$

By Lemma (2.4) we get

$$\alpha_{ij}^* \langle y_{ij}, y_{il} \rangle \alpha_{il} + \alpha_{il}^* \langle y_{il}, y_{ij} \rangle \alpha_{ij} \leq |\alpha_{ij}|^2 \|\langle y_{ij}, y_{il} \rangle\| + |\alpha_{il}|^2 \|\langle y_{ij}, y_{il} \rangle\|$$

then

$$\begin{aligned} \left| x - \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij} \right|^2 &\leq |x|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \alpha_{ij}^* \langle y_{ij}, x \rangle \\ &\quad - \sum_{i=1}^m \sum_{j=1}^{n_i} \langle x, y_{ij} \rangle \alpha_{ij} + \frac{1}{2} \sum_{i=1}^m \sum_{j,l=1}^{n_i} |\alpha_{ij}|^2 \|\langle y_{ij}, y_{il} \rangle\| + |\alpha_{il}|^2 \|\langle y_{il}, y_{ij} \rangle\|. \end{aligned} \quad (13)$$

We choose

$$\alpha_{ij} = \frac{\langle y_{ij}, x \rangle}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|},$$

then we obtain

$$\begin{aligned}
 \left| x - \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij} \alpha_{ij} \right|^2 &\leq |x|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|} - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|} \\
 &+ \frac{1}{2} \sum_{i=1}^m \sum_{j,l=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2 \|\langle y_{ij}, y_{il} \rangle\|}{(\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|)^2} + \frac{1}{2} \sum_{i=1}^m \sum_{j,l=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2 \|\langle y_{ij}, y_{il} \rangle\|}{(\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|)^2} \\
 &= |x|^2 - 2 \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|} + \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|} \\
 &= |x|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|}.
 \end{aligned}$$

Then

$$|x|^2 - \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\|} \geq 0, \quad (14)$$

which completes the proof. $\square$

The previous theorem is a generalized of the following Selberg's and refinement of selberg's inequalities :

Theorem 4.2 (See [2]). Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$ -module, $y_1 \dots y_n$ non zero vectors in $\mathbb{X}$. If $x \in \mathbb{X}$ then

$$\sum_{j=1}^n \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k \rangle|} \leq |x|^2. \quad (15)$$

Theorem 4.3 (See [2]). Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$ -module, y and $y_1 \dots y_n$ non zero vectors in $\mathbb{X}$ such that $\langle y, y_j \rangle = 0$ for $j = 1 \dots n$. If $x \in \mathbb{X}$ then

$$|\langle y, x \rangle|^2 + \sum_{j=1}^n \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k \rangle|} \|y\|^2 \leq |x|^2 \|y\|^2. \quad (16)$$

By Theorem 4.1 we can obtained a generalized of Bombieri's inequality in C^* -module ,

Theorem 4.4. Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$ -module and y_{ij} be a non zero vectors in $\mathbb{X}$, such that y_{ij} is orthogonal to y_{kj} for all $i \neq k$ in $\{1, \dots, m\}$ and $j \in \{1, \dots, n_i\}$. If $x \in \mathbb{X}$ then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} |\langle y_{ij}, x \rangle|^2 \leq |x|^2 \max_{1 \leq i \leq m, 1 \leq j \leq n_i} \sum_{k=1}^{n_i} \|\langle y_{ij}, y_{ik} \rangle\| \quad (17)$$

The following theorems extend previous results to algebra of operators in Hilbert C^* -module.

Theorem 4.5. Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$ -module and $t = u|t|$ be the polar decomposition of adjointable module map t on $\mathbb{X}$, $\{y_{ij}; i = 1, 2, \dots, m \text{ and } j = 1 \dots n_i\} \not\subset \ker(t^*)$ and $\alpha \in [0, 1]$.

If $\langle |t|^{1-\alpha} u^* y_{ij}, |t|^{1-\alpha} u^* y_{kj} \rangle = 0$ for all j and $k \neq i$, then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, tx \rangle|^2}{\sum_{k=1}^{n_i} \|\langle |t^*|^{2(1-\alpha)} y_{ik}, y_{ij} \rangle\|} \leq \| |t|^\alpha x \|^2 \quad (18)$$

holds for all $x \in \mathbb{X}$.

Proof. We replace x and y_{ij} by $|t|^\alpha x$ and $|t|^{1-\alpha} u^* y_{ij}$ in Theorem 4.1 respectively. Then we have the result. $\square$

Theorem 4.6. Let $\mathbb{X}$ be a Hilbert $\mathbb{A}$ -module and $t = u|t|$ be the polar decomposition adjointable module map t on $\mathbb{X}$, $\{y_{ij}; i = 1, 2, \dots, m \text{ and } j = 1 \dots n_i\} \not\subset \ker(t^*)$ and $\alpha, \alpha_1, \dots, \alpha_m$, such that $\alpha + \alpha_i \geq 1$, for all $i = 1, \dots, m$. If $\langle |t|^{\alpha_i} u^* y_{ij}, |t|^{\alpha_k} u^* y_{kj} \rangle = 0$ for all j and $k \neq i$, then

$$\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{|\langle y_{ij}, t |t|^{\alpha_i + \alpha - 1} x \rangle|^2}{\sum_{k=1}^{n_i} \|\langle |t^*|^{2\alpha_i} y_{ik}, y_{ij} \rangle\|} \leq \| |t|^\alpha x \|^2 \quad (19)$$

holds for all $x \in \mathbb{X}$.

Proof. We replace x and y_{ij} by $|t|^\alpha x$ and $|t|^{\alpha_i} u^* y_{ij}$ in Theorem 4.1 respectively. Then we have the result. $\square$

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