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Soliton Solutions of Cubic-Quintic Nonlinear Schrödinger and Variant Boussinesq Equations by the First Integral Method

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Abstract. The cubic-quintic nonlinear Schrödinger equation emerges in models of light propagation in diverse optical media, such as non-Kerr crystals, chalcogenide glasses, organic materials, colloids, dye solutions and ferroelectrics. The first integral method is an efficient method for obtaining exact solutions of some nonlinear partial differential equations. By using the extended first integral method, we construct exact solutions of a fourth-order dispersive cubic-quintic nonlinear Schrödinger equation and the variant Boussinesq system. The stability analysis for these solutions are discussed.

1. Introduction

Nonlinear Schrödinger (NLS) equation with more complex nonlinearities plays an important role in various branches of physics such as nonlinear optics [1,2], water waves [3], plasma physics, quantum mechanics, superconductivity and Bose-Einstein condensate theory. In optics, the propagation of a picosecond optical pulse in a monomode optical fiber is described by the classic NLS equation. For water waves, the NLS equation describes the evolution of the envelope of modulated nonlinear wave groups. As well as their cubic counterparts, such models are of interest by themselves, and may also have direct applications [4]. In particular, glasses and organic optical media whose dielectric response features the cubic-quintic (CQ) nonlinearity, i.e., a self-defocusing quintic correction to the self-focusing cubic Kerr effect, are known [5-7].

The cubic-quintic nonlinear Schrödinger (CQNLS) equation emerges in models of light propagation in diverse optical media, such as non-Kerr crystals [8], chalcogenide glasses [5-6], organic materials [7], colloids, dye solutions and ferroelectrics [10-15]. It has also been predicted that this complex nonlinearity can be synthesized by means of a cascading mechanism [13]. It should be noticed that, in the optics models, evolution variable z is the propagation distance. The competition of the focusing (cubic) and defocusing (quintic) nonlinear terms is the key feature of the CQNLS model, which allows for the existence of stable multidimensional structures which would be unstable in the focusing cubic nonlinear Schrödinger (NLS) equation [15-23].

Lattice models with saturable onsite nonlinear terms have been studied too. The first model of that type was introduced by Vinetskii and Kukhtarev [24-30]. Bright solitons in this model were predicted in

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1D and 2D geometries. Lattice solitons supported by saturable self-defocusing nonlinearity were created in an experiment conducted in an array of optical waveguides built in a photovoltaic medium. Dark discrete solitons were also considered experimentally and theoretically in the latter model [31-40].

Usually, the nonlinearities of the NLS equations are cubic, but there are nonlinear systems which engender cubic and quintic (CQ) nonlinearities [23-25]. The case of CQ nonlinearities opens new possibilities. For example, in nonlinear optics and fibers [23], the CQ nonlinearities can be used to describe pulse propagation in double-doped optical fibers, when the type of dopant varies along the fiber, with the value and sign of the cubic and quintic parameters that control the nonlinearities being adjusted by properly choosing the characteristics of the two dopants. In Bose-Einstein condensation [24-25], the CQ nonlinearities are used to describe the two-body and three-body interactions among atoms. The aim of this paper is to find exact soliton solutions of the cubic-quintic nonlinear Schrödinger (CQNLS) equation and the variant Boussinesq system by the first integral method.

2. Problem Formulation

For laser beam propagating in a nonlinear optical medium, a stationary state of propagation is possible, when linear diffraction is balanced by self-focusing due to a Kerr nonlinearity [1]. However, this stationary state is well known to be unstable for two (space) dimensional laser beams,leading to monotonous diffraction or catastrophic self focusing [2, 26]. It is also well known that different physical mechanisms may lead to a saturation of cubic Kerr nonlinearity thus avoiding beam collapse [3-5]. The dynamics of the slowly varying beam amplitude ψ in a PST-like medium is governed by the cubic-quintic nonlinear Schrödinger (NLS) equation

$$2ik\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+2kk_0n_2|\psi|^2\psi+2kk_0n_3|\psi|^4\psi=0, \tag{1}$$

where ψ(x, y, z) is the complex wave function, ∇^2 is the two-dimensional (2D) Laplacian, and the last two terms represent, respectively, the focusing cubic and defocusing quintic nonlinearities, and the refractive index, n, in PTS is of the form $n=n_0+n_2I+n_3I^2$, where I is the beam intensity and n_j are nonlinear coefficients with $n_2>0$, $n_3<0$, $k=\frac{w}{c}$ and $k_0=n_0k$. Normalizing this equation according to $\acute{z}=\frac{\alpha z}{z_0}$, $\acute{x}=\frac{\sqrt{\alpha}x}{r_0}$, $\acute{y}=\frac{\sqrt{\alpha}y}{r_0}$, $\acute{\psi}^2=\frac{\psi^2}{\alpha\psi_0^2}$, $z_0=-\frac{n_3}{2kk_0n_2^2}$, $\psi_0^2=-\frac{n_3}{n_2}$ and $r_0=\sqrt{-\frac{n_3}{2kk_0n_2^2}}$. Dimitrevski et al. deduced the two-dimensional NLS equation in cubic-quintic nonlinear media

$$i\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+|\psi|^2\psi-\sigma|\psi|^4\psi=0, \tag{2}$$

where σ is arbitrary constant.

3. The Extended First Integral Method

Consider the nonlinear partial differential equation in the form

$$F(u,u_x,u_t,u_{xx},.........)=0, \tag{3}$$

where $u(x,t)=f(\xi)$ is the solution of nonlinear partial differential equation (3). The nonlinear partial differential equation (3) is transformed to nonlinear ordinary differential equation as

$$G(f(\xi),\frac{\partial f(\xi)}{\partial\xi},\frac{\partial^2 f(\xi)}{\partial\xi^2},........)=0. \tag{4}$$

Next, we introduce a new independent variable

$$X(\xi)=f(\xi),\qquad Y=\frac{\partial f(\xi)}{\partial\xi}, \tag{5}$$

which leads a system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \qquad \frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)). \tag{6}$$

By the qualitative theory of ordinary differential equations [41], if we can find the integrals to equation (6) under the same conditions, then the general solutions to equation (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integral, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first to equation (6) which reduces equation (4) to a first order integralable ordinary differential equation. An exact solution to equation (3) is then obtained by solving this equation. Now, let us recall the division theorem:

Division Theorem. Suppose that P(w,z) and Q(w,z) are polynomials in C[w,z]; and P(w,z) is irreducible in C[w,z]. If Q(w,z) vanishes at all zero points of P(w,z), then there exists a polynomial G(w,z) in C[w,z] such that $Q(w,z) = P(w,z)G(w,z)$.

4. Application of the Methods

4.1. *The Cubic-Quintic nonlinear Schrödinger equation*

In this section, we use the transformation equation (4) into equation (2), using equation (5) we get

$$\dot{X}(\xi) = Y(\xi)\,, \quad \dot{Y}(\xi) = \frac{1}{2}(\gamma + \alpha^2 + \beta^2)X(\xi) - \frac{1}{2}X^3(\xi) + \frac{1}{2}\sigma X^5(\xi). \tag{7}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of equation (7) and

$$Q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{8}$$

where $a_i(X)$ $(i = 0, 1, ..., m)$, are polynomials of X and $a_m(X) \neq 0$. Equation (8) is called the first integral to equation (7). Due to the Division Theorem, there exists a polynomial g(X) + h(X)Y, in the complex domain C[X,Y] such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)\, Y^i. \tag{9}$$

In this example, we take two different cases, assuming that m=1 and m= 2 in equation (8).

Case A: Suppose that m=1, by comparing with the coefficients of Y^i $(i = 0, 1, 2)$ on both sides of equation (9), we have

$$\dot{a}_1(X) = a_1(X)h(X), \tag{10}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{11}$$

$$a_1(X)\left(\frac{1}{2}(\gamma + \alpha^2 + \beta^2)X - \frac{1}{2}X^3 + \frac{1}{2}\sigma X^5\right) = g(X)a_0(X). \tag{12}$$

Since $a_i(X)$ (i=0,1) are polynomials, then from equation (10), we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of g(X) and $a_0(X)$, we conclude that $deg\,(g(X))$=2 only. Suppose that $g(X) = A_1X^2 + B_1X + A_0$, then we find $a_0(X)$,

$$a_0(X) = \frac{1}{3}A_1X^3 + \frac{1}{2}B_1X^2 + A_0X + B_0, \tag{13}$$

where B_0 is arbitrary integration constant.

Substituting $a_0(X)$ and g(X) into equation (12) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0 = B_1 = 0,\ A_0 = \frac{\sqrt{\alpha^2+\beta^2+\gamma}}{\sqrt{2}},\ A_1 = -\frac{3}{4\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}},\ \sigma = \frac{3}{16(\alpha^2+\beta^2+\gamma)} \tag{14}$$

and

$$B_0 = B_1 = 0,\ A_0 = -\frac{\sqrt{\alpha^2+\beta^2+\gamma}}{\sqrt{2}},\ A_1 = \frac{3}{4\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}},\ \sigma = \frac{3}{16(\alpha^2+\beta^2+\gamma)}, \tag{15}$$

where B_0, γ, β and α are arbitrary constants. Using the conditions (14) in (8), we obtain

$$I)\quad Y(\xi) = -\frac{X\left(-X^2 + 4(\alpha^2+\beta^2+\gamma)\right)}{4\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}}, \tag{16}$$

combining equation (16) with equation (7), we obtain

$$X(\xi) = \pm\frac{2\sqrt{-\alpha^2-\beta^2-\gamma}\,\xi_0}{\sqrt{e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}\,\xi\right)+\xi_0}}} \tag{17}$$

and the exact solution to Cubic-Quintic nonlinear Schr*ö*dinger equation can be written as

$$\psi(x,y,z) = \pm e^{i(\alpha x+\beta y+\gamma z)}\frac{2\sqrt{-\alpha^2-\beta^2-\gamma}\,\xi_0}{\sqrt{e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}(x+y-2(\alpha+\beta)z)\right)+\xi_0}}}, \tag{18}$$

where ξ_0 is an arbitrary constant.

Similarly, in the case of equation (15), from equation (8), we obtain

$$II)\quad Y(\xi) = \frac{X\left(-X^2 + 4(\alpha^2+\beta^2+\gamma)\right)}{4\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}}, \tag{19}$$

from equation (7), we obtain

$$X(\xi) = \pm\frac{2\sqrt{\alpha^2+\beta^2+\gamma}\,e^{\left(\frac{\sqrt{\alpha^2+\beta^2+\gamma}}{\sqrt{2}}\,\xi\right)+\xi_0}}{\sqrt{-1+e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}\,\xi\right)+\xi_0}}} \tag{20}$$

and then the exact solution to Cubic-Quintic nonlinear Schr*ö*dinger equation can be written as

$$\psi(x,y,z) = \pm e^{i(\alpha x+\beta y+\gamma z)}\frac{2\sqrt{\alpha^2+\beta^2+\gamma}\,e^{\left(\frac{\sqrt{\alpha^2+\beta^2+\gamma}}{\sqrt{2}}(x+y-2(\alpha+\beta)z)\right)+\xi_0}}{\sqrt{-1+e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}(x+y-2(\alpha+\beta)z)\right)+\xi_0}}}, \tag{21}$$

where ξ_0 is an arbitrary constant.

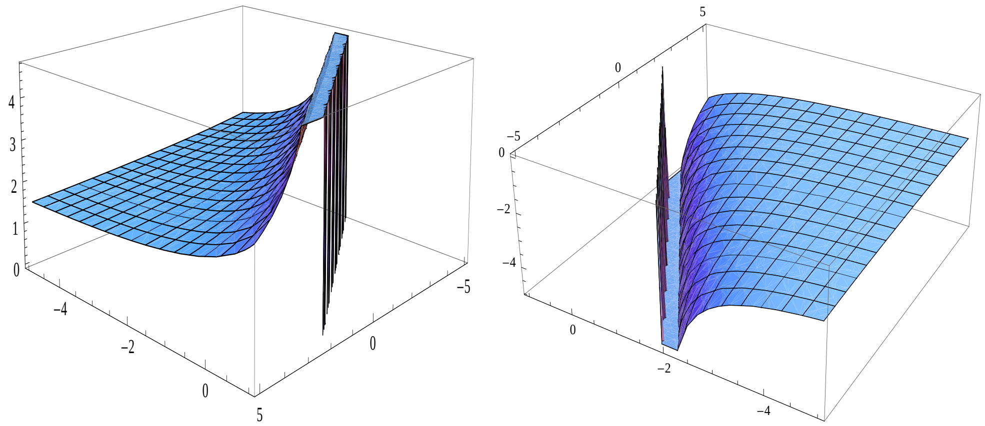


Figure (1,2): Travelling waves solutions of equation (20) is plotted: the bright and dark solitary waves

Figure (1,2) shown that the travelling wave solutions with ($\alpha = 0.1$, $\beta = -0.25$, $\gamma = 0.4$, $z = 0.1$) ; in the interval $[-5, 5]$ and $[-5, 1]$.

Case B: Suppose that m= 2, by comparing with the coefficients of Y^i $(i = 0, 1, 2, 3)$ on both sides of equation (9), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{22}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{23}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X) - a_2(X)\left((\alpha^2 + \beta^2 + \gamma)X - X^3 + \sigma X^5\right), \tag{24}$$

$$a_1[X]\left[\frac{1}{2}(\alpha^2 + \beta^2 + \gamma)X - \frac{1}{2}X^3 + \frac{\sigma}{2}X^5\right] = g(X)a_0(X). \tag{25}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from equation (22) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of g(X), $a_1(X)$ and $a_2(X)$, we conclude that deg(g(X))= 2 only. Suppose that $g(X) = A_1X^2 + B_1X + A_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$a_1(X) = \frac{A_1}{3}X^3 + \frac{B_1}{2}X^2 + A_0X + B_0, \tag{26}$$

$$a_0(X) = d + A_0B_0X + \frac{1}{2}\left(-(\alpha^2 + \beta^2 + \gamma) + A_0^2 + B_0B_1\right)X^2 + \left(\frac{A_1B_0}{3} + \frac{A_0B_1}{2}\right)X^3$$
$$+\left(\frac{1}{4} + \frac{A_0A_1}{3} + \frac{B_1^2}{8}\right)X^4 + \frac{A_1B_1}{6}X^5 + \left(\frac{-\sigma}{6} + \frac{A_1^2}{18}\right)X^6. \tag{27}$$

Substituting a_0, a_1 and g(X) in the last equation in equation (25) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Mathematica, we obtain

$$d = 0,\ \sigma = \frac{3}{16(\alpha^2 + \beta^2 + \gamma)},\ A_0 = \sqrt{2}\sqrt{\alpha^2 + \beta^2 + \gamma}, \tag{28}$$

$$A_1 = -\frac{3}{2\sqrt{2}\sqrt{\alpha^2 + \beta^2 + \gamma}},\ B_0 = B_1 = 0$$

and

$$d = 0,\ \sigma = \frac{3}{16(\alpha^2 + \beta^2 + \gamma)},\ A_0 = -\sqrt{2}\sqrt{\alpha^2 + \beta^2 + \gamma}, \tag{29}$$

$$A_1 = \frac{3}{2\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}}, \; B_0 = B_1 = 0,$$

where α, β and γ are arbitrary constant. Using the conditions equation (28) into equation (8), we get

$$I) \quad Y(\xi) = \frac{X\left(-X^2 + 4(\alpha^2+\beta^2+\gamma)\right)}{4\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}}, \tag{30}$$

combining equation (30) with equation (7), we obtain

$$X(\xi) = \pm \frac{2\sqrt{\alpha^2+\beta^2+\gamma}e^{\left(\frac{\sqrt{\alpha^2+\beta^2+\gamma}}{\sqrt{2}}\xi\right)+\xi_0}}{\sqrt{-1+e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}\xi\right)+\xi_0}}} \tag{31}$$

and then the exact solution to Cubic-Quintic nonlinear Schrödinger equation can be written as

$$\psi(x,y,z) = \pm e^{i(\alpha x+\beta y+\gamma z)} \frac{2\sqrt{\alpha^2+\beta^2+\gamma}e^{\left(\frac{\sqrt{\alpha^2+\beta^2+\gamma}}{\sqrt{2}}(x+y-2(\alpha+\beta)z)\right)+\xi_0}}{\sqrt{-1+e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}(x+y-2(\alpha+\beta)z)\right)+\xi_0}}}, \tag{32}$$

where ξ_0 is an arbitrary constant. Similarly, in the case of equation (29), from equation (8), we obtain

$$II) \quad Y(\xi) = \frac{X\left(X^2 - 4(\alpha^2+\beta^2+\gamma)\right)}{4\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}} \tag{33}$$

and from equation (7), we obtain that

$$X(\xi) = \pm \frac{2\sqrt{-\alpha^2-\beta^2-\gamma}\xi_0}{\sqrt{e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}\xi\right)+\xi_0}}} \tag{34}$$

and then the exact solution to Cubic-Quintic nonlinear Schrödinger equation can be written as

$$\psi(x,y,z) = \pm e^{i(\alpha x+\beta y+\gamma z)} \frac{2\sqrt{-\alpha^2-\beta^2-\gamma}\xi_0}{\sqrt{e^{\left(\sqrt{2}\sqrt{\alpha^2+\beta^2+\gamma}(x+y-2(\alpha+\beta)z)\right)+\xi_0}}}, \tag{35}$$

where ξ_0 is an arbitrary constant.

4.2. *The variant Boussinesq equation*

Consider the variant Boussinesq system

$$U_t + V_x + UU_x + pU_{xxt} = 0, \quad V_t + (UV)_x + qU_{xxx} = 0. \tag{36}$$

Applying the transformation $U(x,t) = u(\xi)$, $V(x,t) = v(\xi)$, where $\xi = x - kt$, convert equation (36) into a system of ordinary differential equations as

$$-ku' + v' + uu' - kpu''' = 0, \tag{37}$$

$$-kv' + uv' + u'v + qu''' = 0. \tag{38}$$

We can rewrite equation (37) in the form

$$v' + uu' - ku' - kpu''' = 0. \tag{39}$$

Integrating equation (39), we derive

$$v = \alpha + ku - \frac{1}{2}u^2 + kpu'', \tag{40}$$

where α is an integrating constant. Now, inserting equation (40) into equation (38), yields

$$(q - k^2p)u''' + 3kuu' + (\alpha - k^2)u' + kpu'u'' - \frac{1}{2}u^2u' + kpuu''' - u^2u' = 0. \tag{41}$$

Now, integrating of equation (41), gives

$$u'' = \frac{1}{k^2p - q - kpu}(\frac{3k}{2}u^2 - \frac{1}{2}u^3 + (\alpha - k^2)u + \beta), \tag{42}$$

where β is an integrating constant. Introducing new variables $X = u(\xi)$ and $Y = u'$. Convert equation (42) into a system of ODEs

$$X' = Y \ , \ Y' = \frac{1}{k^2p - q - kpu}(\frac{3k}{2}u^2 - \frac{1}{2}u^3 + (\alpha - k^2)u + \beta). \tag{43}$$

According to the first integral method, assume that $X = X(\xi)$ and $Y = Y(\xi)$ are the nontrivial solutions to equation (43) and $Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in **C**[X, Y] such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{44}$$

$a_i(X)$, $i = 0, 1, 2, ..., m$ are polynomials of X, which $a_m(X) \neq 0$. Due to the division theorem, there exists a polynomial $T(X,Y) = g(X) + h(X)Y$ in **C**[X, Y] so that

$$\frac{dQ}{d\xi} = \frac{\partial Q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial Q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y)(\sum_{i=0}^{m} a_i(X)Y^i). \tag{45}$$

Now, suppose that m=1 in equation (44). By equating the coefficients of Y^i, i=0, 1, 2 on both sides of equation (45), one can obtain

$$a_1'(X) = h(X)a_1(X), \tag{46}$$

$$a_0'(X) = g(X)a_1(X) + h(X)a_0(X), \tag{47}$$

$$a_1(X)\left(\frac{1}{k^2p - q - kpu}(\frac{3k}{2}u^2 - \frac{1}{2}u^3 + (\alpha - k^2)u + \beta)\right) = g(X)a_0(X). \tag{48}$$

Since $a_i(X)$ (i=0, 1) are polynomials, then from equation (46) one can deduce that $a_1(X)$ is a constant and h(X)=0. For convenience, we consider $a_1(X) = 1$. Now, by balancing the degree of g(X) and $a_0(X)$, we can conclude that $deg\,(g(X))$=1. Thus, by assuming that g(X)= $A_1X + B_0$ such that $A_1 \neq 0$, from equation (47) we have

$$a_0(X) = \frac{1}{2}A_1X^2 + B_0X + A_0,$$

where A_0 is an integrating constant. Substituting $a_0(X)$, $a_1(X)$ and g(X) in equation (48) and equating the coefficient of each power of X to zero, a system of algebraic equations can be obtained, which after being solved, we arrive at

$$\alpha = -\frac{\beta}{k} \ , \ A_0 = \pm\frac{\beta}{k\sqrt{q}} \ , \ A_1 = \pm\frac{1}{\sqrt{q}} \ , \ B_0 = \mp\frac{k}{\sqrt{q}}. \tag{49}$$

Setting equation (49) in equation (44) yields

$$Y \pm \frac{kX(X-2k)+2\beta}{2k\sqrt{q}} = 0.$$

Now, by combining these equations with equation (43), two first-order ordinary differential equations are derived, which by solving these equations and considering X=$u(\xi)$ and U(x,t)=$u(\xi)$, we obtain

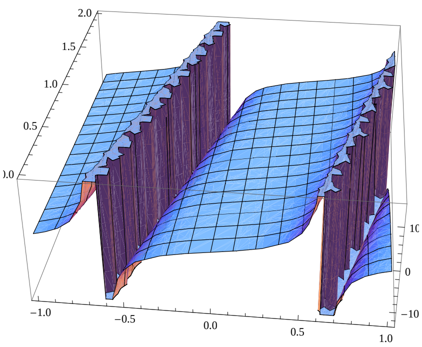
$$U(x,t) = k \pm \frac{\sqrt{-k^3+2\beta}}{\sqrt{k}}\ \tan[\frac{\sqrt{-k^3+2\beta}}{2\sqrt{kq}}(x-kt)+\xi_0], \tag{50}$$

where ξ_0 is an arbitrary constant. Also, by considering the solutions of two first-order differential equations and X=$u(\xi)$ as well as the relations equation (40) and V(x,t)=$v(\xi)$, we will obtain

$$V(x,t) = \mp\frac{(k^3-2\beta)}{2kq}\ \sec^2[\frac{\sqrt{-k^3+2\beta}}{2\sqrt{kq}}(x-kt)+\xi_0] \tag{51}$$

$$.\left(\mp q + \ p\sqrt{k}\sqrt{-k^3+2\beta}\ \tan[\frac{\sqrt{-k^3+2\beta}}{2\sqrt{kq}}(x-kt)+\xi_0]\right),$$

where ξ_0 is an arbitrary constant.



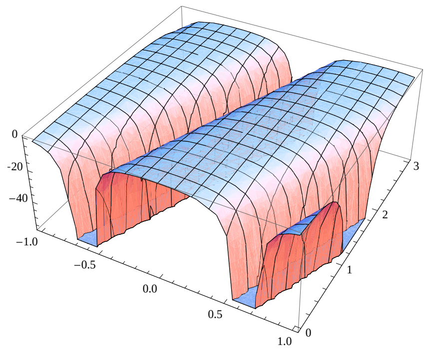


Figure (3) Travelling waves solutions of equation (50) is plotted: periodic solitary waves and Figure (4) Travelling waves solutions of equation (51) is plotted: periodic solitary waves

Figure (3) shown that the travelling wave solutions with ($k = 0.25$, $\beta = 0.5$, $q = 0.16$) ; in the interval $[-1, 1]$ and time in the interval $[0, 2]$.

The soliton solution of equation (50) is stable if:

$$k^2\sqrt{-k^3+2\beta} \neq 0,\ \ \sec^2[\frac{\sqrt{-k^3+2\beta}}{2\sqrt{kq}}] > 0,$$

$$k^3\sqrt{-k^3+2\beta} > \sqrt{kq}(k^3+\beta)\sin[\frac{\sqrt{-k^3+2\beta}}{\sqrt{kq}}] - (2k^3+\beta)\sqrt{-k^3+2\beta}\cos[\frac{\sqrt{-k^3+2\beta}}{\sqrt{kq}}].$$

Figure (4) shown that the travelling wave solutions with ($p = 0$, $k = 0.25$, $\beta = -0.5$, $q = 0.16$) ; in the interval $[-1, 1]$ and time in the interval $[0, 3]$.

The soliton solution of equation (51) is stable if:

$$k \neq 0,\ q \neq 0,\ \ \sec^6[\frac{\sqrt{-k^3+2\beta}}{2\sqrt{kq}}] > 0,$$

$$60(-q^2+kp^2(k^3-2\beta))(k^3+\beta)(k^3-2\beta)-60(q^2+kp^2(k^3-2\beta))(k^3+\beta)(k^3-2\beta)\cos[\frac{\sqrt{-k^3+2\beta}}{\sqrt{kq}}]$$

$$+2\sqrt{kq}\sqrt{-k^3+2\beta}\sin[\frac{\sqrt{-k^3+2\beta}}{\sqrt{kq}}]$$

$$.(-49k^7p^2+75k^3q^2+91k^4p^2\beta+75q^2\beta+14kp^2\beta^2+$$

$$(7k^7p^2+15k^3q^2-13k^4p^2\beta+15q^2\beta-2kp^2\beta^2)(6\cos[\frac{\sqrt{-k^3+2\beta}}{\sqrt{kq}}]+\cos[\frac{2\sqrt{-k^3+2\beta}}{\sqrt{kq}}]))>0$$

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