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Geometric Characterizations of Quasi-Product Production Models in Economics

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Abstract. In this work, we investigate quasi-product production functions taking the form:

$$L(x_1, \ldots, x_n) = F\Big(\prod_{i=1}^{n} f_i(x_i)\Big).$$

We get a simple geometric classification of quasi-product production functions via studying geometric properties of their associated graphs in Euclidean spaces. Moreover precisely, if their corresponding graphs are flat spaces, a complete classification of quasi-product production functions with an arbitrary number of inputs is obtained.

1. Introduction

In economics, a production function can relate physical output of a production process to physical inputs or factors of production. Economists always using a production function in economic analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

Hence, the production function is one of the key concepts of mainstream neoclassical theories, used to define marginal product and to distinguish allocative efficiency, the defining focus of economics.

In economics, a production function is always defined as *a non-constant positive function with non-vanishing first derivative*.

In 1928, C. W. Cobb and P. H. Douglas [14] first introduced a famous two inputs production function, nowadays called Cobb-Douglas production function, which has the form

$$Y = bL^kC^{1-k},$$

where L denotes the labor input, C is the capital input, b is the total factor productivity and Y is the total production.

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The generalized form of the Cobb-Douglas production function with an arbitrary number of inputs can be expressed as

$$L(x_1, \dots, x_n) = A x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1.1}$$

where $x_i > 0$ $(i = 1, \dots, n)$, A is a positive constant and $\alpha_1, \dots, \alpha_n$ are nonzero constants.

In 1961, K. J. Arrow et al. [1] introduced another famous two inputs production function, nowadays called ACMS production function, written as

$$Q = b(aK^r + (1-a)L^r)^{\frac{1}{r}},$$

where Q is the output, b the factor productivity, a the share parameter, K and L the primary production factors, $r = (s-1)/s$, and $s = 1/(1-r)$ is the elasticity of substitution. Hence it is also called constant elasticity of substitution (CES) production function [15, 16]. Also, the generalized form of CES production function with an arbitrary number of inputs is defined by

$$L(x_1, \dots, x_n) = A\Big(\sum_{i=1}^{n} a_i^{\rho} x_i^{\rho}\Big)^{\frac{\gamma}{\rho}},$$

where a_i, γ, A, ρ are nonzero constants, $A, a_i > 0$ and $\rho < 1$.

A production function $Q = L(x_1, \dots, x_n)$ is said to be γ-homogeneous or homogeneous of degree γ, if given any positive constant t,

$$L(tx_1, \dots, tx_n) = t^{\gamma} L(x_1, \dots, x_n).$$

for some nonzero constant γ. If $\gamma > 1$, the function exhibits *increasing return to scale*, and it exhibits *decreasing return to scale* if $\gamma < 1$. If it is homogeneous of degree 1, it exhibits *constant return to scale*.

Clearly, both of the generalized form of the Cobb-Douglas production function and CES production function are homogeneous production functions.

A production function is called *quasi-sum* (see [5, 8] for details), if the function takes the following form

$$L(x_1, \dots, x_n) = F\Big(\sum_{i=1}^{n} f_i(x_i)\Big),$$

where F is a continuous strict monotone positive function and f_i are continuous strict monotone positive functions.

Similarly, a production function is called *quasi-product*, if the function takes the following form

$$L(x_1, \dots, x_n) = F\Big(\prod_{i=1}^{n} f_i(x_i)\Big),$$

where F is a continuous strict monotone positive function and f_i are continuous strict monotone positive functions.

Note that quasi-sum production functions and quasi-product production functions include all of the generalized Cobb-Douglas production functions and CES production functions.

Recently, Vîlcu et al. showed in [17, 18] that the generalized Cobb-Douglas production functions and generalized CES production functions have constant return to scale if and only if the corresponding hypersurfaces have vanishing Gauss-Kronecker curvature. These results establish an interesting link between some fundamental notions in the theory of production functions and the differential geometry of hypersurfaces in Euclidean spaces.

We note that, in the theory of differential geometry, the study of hypersurfaces with certain curvature properties is of fundamental importance and receives extensive attention by geometers [11].

Therefore, a natural question in economic analysis is to study some production functions via geometric properties of their associated graph hypersurfaces in Euclidean spaces.

Concerning this topic, some important contributions have been made recently by some authors, see [2-10, 13, 19-21] and references therein.

Interestingly, it is proved in [12] that a homogeneous production function with an arbitrary number of inputs defines a flat hypersurface if and only if either it has constant return to scale or it is a multinomial production function.

Also, Chen proved the following results in [5]:

Theorem 1.1 ([5]) *A twice differentiable quasi-sum production function with more that two factors is quasi-linear if and only if its production hypersurfaces is a flat space.*

Theorem 1.2 ([5]) *Let $f(x, y)$ be a twice differentiable quasi-sum production function. Then the production surface of f is flat if and only if, up to translations, f is one of the following functions:*

1. *a quasi-linear production function;*
2. *a Cobb-Douglas production function, i.e. $f = ax_1^c x_2^{1-c}$ for some nonzero constants a, r with $r \neq 1$;*
3. *an ACMS functions given by $f = (ax_1^{\frac{1}{c}} + bx_2^{\frac{1}{c}})^c$;*
4. *$f = a\ln(be^{rx} + ce^{sy})$ for some nonzero constants a, b, c, r, s.*

Motivated by Chen's work mentioned above, in this paper we study the geometric conditions of quasi-product productions as graphs in Euclidean space. We give a classification of quasi-product productions, provided its corresponding graph hypersurfaces are flat space.

2. Basic Theory of Hypersurfaces in Differential Geometry

It is well known that each production function $L(x_1, \ldots, x_n)$ can be identified with a graph of a non-parametric hypersurface of an Euclidean $(n + 1)$-space $\mathbb{E}^{n+1}$ given by

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, L(x_1, \ldots, x_n)). \tag{2.1}$$

This hypersurface is called as *a production hypersurface*.

Let M^n be a orientable hypersurface in an $(n+1)$-dimension Euclidean space. Since M^n is orientable, the Gauss map v can be defined globally by

$$v : M^n \to S^n \subset \mathbb{E}^{n+1},$$

which maps M^n to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The Gauss map is a continuous map such that $v(p)$ is a unit normal vector $\xi(p)$ of M^n at point p.

The differential dv of the Gauss map v can be used to define an extrinsic curvature. It is well known that the shape operator S_p and dv can be related by:

$$g(S_p u, w) = g(dv(u), w),$$

where $u, w \in T_pM$ and g is the metric tensor on M^n.

Moreover, the second fundamental form h is related with the shape operator S_p by

$$g(S_p u, w) = g(h(u, w), \xi(p))$$

for $u, w \in T_pM$.

Denote the partial derivatives $\frac{\partial L}{\partial x_i}, \frac{\partial^2 L}{\partial x_i \partial x_j}, \ldots$, etc. by $L_i, L_{ij}, \ldots$, etc. Put

$$W = \sqrt{1 + \sum_{i=1}^{n} F_i^2}. \tag{2.2}$$

Let us recall some well-known results for a graph of hypersurface (2.1) in $\mathbb{E}^{n+1}$ from [11, 4].

Proposition 2.1. *For a production hypersurface of* $\mathbb{E}^{n+1}$ *defined by*

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, L(x_1, \ldots, x_n)),$$

the following statements hold:

1. *The unit normal* ξ *is given by:*

$$\xi = \frac{1}{W}(-L_1, \ldots, -L_n, 1).$$

2. *The coefficient* $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ *of the metric tensor is given by:*

$$g_{ij} = \delta_{ij} + L_i L_j,$$

where $\delta_{ij} = 1$ *if* $i = j$*, otherwise 0.*

3. *The Gauss-Kronecker curvature G is*

$$G = \frac{\det h_{ij}}{\det g_{ij}} = \frac{\det L_{ij}}{W^{n+2}}. \tag{2.3}$$

4. *The sectional curvature* K_{ij} *with respect to the plane section spanned by* $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\}$ *is given by*

$$K_{ij} = \frac{L_{ii}L_{jj} - L_{ij}^2}{W^2(1 + L_i^2 + L_j^2)}. \tag{2.4}$$

In particular, when $n = 2$*, the sectional curvature* K_{12} *is the Gauss-Kronecker curvature G or also known as the Gauss curvature.*

5. *The Riemann curvature tensor R and the metric tensor satisfy*

$$g\Big(R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\Big) = \frac{L_{il}L_{jk} - L_{ik}L_{jl}}{W^4}. \tag{2.5}$$

3. Quasi-Product Production Hypersurfaces

In the following, we study quasi-product production hypersurfaces. Under the assumption that hypersurfaces is flat, we obtain a complete classification.

We first deal with the case of quasi-product productions with the corresponding production surfaces being flat in $\mathbb{E}^3$.

Theorem 3.1. *Let* $L(x_1, x_2) = F(f_1(x_1)f_2(x_2))$ *be a quasi-product production function. If the graph of L in* $\mathbb{E}^3$ *is flat, then, up to translations, L is given by one of the following functions:*

1. $L = F\Big(\exp(c_1x_1 + c_2x_2)\Big), \quad c_1, c_2 \in \mathbb{R}/0;$
2. *a Cobb-Douglas production function, i.e.* $L = ax_1^c x_2^{1-c}, \quad a \in \mathbb{R}^+$ *and* $c \in \mathbb{R}/\{0, 1\}$;
3. $L = c_1x_1 + c_2\ln(f_2(x_2)), \quad c_1, c_2 \in \mathbb{R}/0$, *where* f_2 *satisfies* $f_2f_2'' \neq f_2'^2$;
4. $L = a\Big(\ln f_1(x_1) + \ln f_2(x_2)\Big)^b, \quad a \in \mathbb{R}^+, b \in \mathbb{R}/\{0, 1\}$, *where* f_1 *and* f_2 *satisfy differential equations* $\frac{(b-1)f_1'^2}{f_1f_1''-f_1'^2} + \ln f_1 = c$ *and* $\frac{(b-1)f_2'^2}{f_2f_2''-f_2'^2} + \ln f_2 = -c$ *for some constant c, respectively;*
5. $L = c\ln\Big(-\ln f_1(x_1) - \ln f_2(x_2)\Big), c \in \mathbb{R}^+$, *where* $\frac{f_1'^2}{f_1f_1''-f_1'^2} - \ln f_1 = c$ *and* $\frac{f_2'^2}{f_2f_2''-f_2'^2} - \ln f_2 = -c$ *for some constant c, respectively.*

Remark 3.2. *Note that, if* $c = 0$*, the production function in Case (4) becomes*

$$L = \Big(c_1x_1^{\frac{1}{b}} + c_2x_2^{\frac{1}{b}})\Big)^b, \quad c_1, c_2 \in \mathbb{R}, b \in \mathbb{R}/\{0, 1\},$$

which is the ACMS (CES) production function.

Proof. Suppose that L is a quasi-product production function taking the form

$$L(x_1, x_2) = F(f_1(x_1)f_2(x_2)). \tag{3.1}$$

Denote by $u = f_1(x_1)f_2(x_2)$. We assume that $F', f_1', f_2' \neq 0$, where $''\prime''$ denotes the derivative with respect to the variable u.

Hence, we can define a quasi-product production surface M in $\mathbb{E}^3$ by a graph

$$f(x_1, x_2) = (x_1, x_2, L(x_1, x_2)). \tag{3.2}$$

It follows from (3.1) that

$$L_1 = f_1'f_2F', \quad L_2 = f_1f_2'F',$$

$$L_{11} = f_1''f_2F' + f_1'^2f_2^2F'', \tag{3.3}$$

$$L_{12} = f_1'f_2'F' + f_1f_1'f_2f_2'F'', \tag{3.4}$$

$$L_{22} = f_1f_2''F' + f_1^2f_2'^2F''. \tag{3.5}$$

Hence, (3.3-3.5) imply that

$$L_{11}L_{22} - L_{12}^2 = (f_1f_1''f_2f_2'' - f_1'^2f_2'^2)F'^2 + f_1f_2(f_1f_1''f_2'^2 + f_1'^2f_2f_2'' - 2f_1'^2f_2'^2)F'F''. \tag{3.6}$$

Since the production surface (3.1) is flat, it follows from (2.4) and (3.6) that

$$(f_1f_1''f_2f_2'' - f_1'^2f_2'^2)F' + (f_1f_1''f_2'^2 + f_1'^2f_2f_2'' - 2f_1'^2f_2'^2)uF'' = 0. \tag{3.7}$$

We now distinguish the following two cases:

Case A. $F'' = 0$. We have that $F(u) = au + b$ for some real number a, b with $a \neq 0$. Since $F' \neq 0$, (3.7) reduces to

$$f_1f_1''f_2f_2'' = f_1'^2f_2'^2,$$

namely

$$\frac{f_1f_1''}{f_1'^2} \cdot \frac{f_2f_2''}{f_2'^2} = 1.$$

Hence, there exists a nonzero constant c such that

$$\frac{f_1f_1''}{f_1'^2} = c, \quad \frac{f_2f_2''}{f_2'^2} = \frac{1}{c}. \tag{3.8}$$

Case A.1. $c = 1$. By solving (3.8), we get

$$f_1(x_1) = c_3e^{c_1x_1}, \quad f_2(x_2) = c_4e^{c_2x_2} \tag{3.9}$$

for some nonzero real numbers c_i $(i = 1, 2, 3, 4)$. This gives a special case of Case (1) in Theorem 1.

Case A.2. $c \neq 1$. The solutions of equations in (3.8) are given by

$$f_1(x_1) = (c_1x_1 + c_2)^{\frac{1}{1-c}}, \quad f_2(x_2) = (c_3x_2 + c_4)^{\frac{-c}{1-c}} \tag{3.10}$$

for some real numbers c_i $(i = 1, 2, 3, 4)$ with $c_1, c_3 \neq 0$. After suitable translations, we obtain Case (2).

Case B. $F'' \neq 0$. In this case, we have either

$$f_1f_1''f_2f_2'' - f_1'^2f_2'^2 = 0,$$

$$f_1f_1''f_2'^2 + f_1'^2f_2f_2'' - 2f_1'^2f_2'^2 = 0,$$

or

$$f_1f_1''f_2f_2'' - f_1'^2f_2'^2 \neq 0, \tag{3.11}$$

$$f_1f_1''f_2'^2 + f_1'^2f_2f_2'' - 2f_1'^2f_2'^2 \neq 0. \tag{3.12}$$

The former case implies that

$$\frac{f_1 f_1''}{f_1'^2} = \frac{f_2 f_2''}{f_2'^2} = 1. \tag{3.13}$$

Hence, by solving (3.13) we get Case (1) in Theorem 1. In the following, we consider the latter case.

Let us introduce two functions p and q as follows:

$$p(x_1) = \frac{f_1 f_1''}{f_1'^2}, \quad q(x_2) = \frac{f_2 f_2''}{f_2'^2}. \tag{3.14}$$

Then (3.11), (3.12) and (3.7) become

$$\begin{aligned} &pq - 1 \neq 0, \\ &p + q - 2 \neq 0, \\ &(pq-1)F' + (p+q-2)uF'' = 0. \end{aligned} \tag{3.15}$$

Case B.1. $p = 1, q \neq 1$ or $p \neq 1, q = 1$. Without loss of generality, we assume $p = 1, q \neq 1$. Then, we have $f_1(x_1) = c_2 e^{c_1 x_1}$ for some nonzero constants c_1 and c_2. Also, (3.7) yields

$$F' + uF'' = 0, \tag{3.16}$$

which has the solution $F(u) = a \ln u + b$ for some constant a, b with $a \neq 0$. Hence, we obtain Case (3) in Theorem 1.

Case B.2. $p \neq 1, q \neq 1$. We could rewrite (3.15) as

$$-\frac{F'}{uF'' + F'} = \frac{1}{p-1} + \frac{1}{q-1}. \tag{3.17}$$

Differential both hand-sides of equation (3.17) with respect to x_1 and then x_2, we obtain

$$\Big(\frac{F'}{uF'' + F'}\Big)' + u\Big(\frac{F'}{uF'' + F'}\Big)'' = 0. \tag{3.18}$$

By integration on (3.18), we have

$$\frac{F'}{uF'' + F'} = a \ln u + b \tag{3.19}$$

for some constants a and b. Note that $a^2 + b^2 \neq 0$ in (3.19).

Moreover, if $a \neq -1, 0$, by solving (3.19), we get

$$F(u) = c\Big(a \ln u + b\Big)^{\frac{1+a}{a}} + d, \tag{3.20}$$

where $a, c \in \mathbb{R}/0$ and $b, d \in \mathbb{R}$. Taking into account (3.17) and up to suitable translation, we have Case (4) in Theorem 1.

Similarly, if $a = -1$, after integration we obtain Case (5) in Theorem 1.

In the case $a = 0$, the solution of (3.19) gives $F = c_1 u^{\frac{1}{b}} + c_2$ for some constants c_1, b, c_2 and $c_1, b \neq 0$. In this case, (3.17) means that p, q are nonzero constants. Hence, by integration on (3.14) and applying (3.17) we obtain Case (2) as well , which finishes the proof of Theorem 1. □

In the following, we deal with the general case of quasi-product production function with n-inputs for $n > 2$.

Theorem 3.3. *Let $L(x_1, \dots, x_n) = F\Big(\prod_{i=1}^{n} f_i(x_i)\Big)$ be a quasi-product production function. If the graph of L is flat, then, up to translations, L is given by one of the following functions:*

1. $L = F\Big(\exp\Big(\sum_{i=1}^{n} c_i x_i\Big)\Big), \quad c_i \in \mathbb{R}/0;$
2. $L = c_1 \ln(f_1(x_1)) + c_2 x_2 + \cdots + c_n x_n, \quad c_i \in \mathbb{R}/0,$ *where f_1 satisfies $f_1 f_1'' \neq f_1'^2$;*

3. $L = a\sqrt{x_1 \cdots x_n}, \quad a \in \mathbb{R}^+$.

Proof. Let L be a n-inputs quasi-product production function. Then L corresponds a graph in Euclidean space $\mathbb{E}^{n+1}$,

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, L(x_1, \ldots, x_n)). \tag{3.21}$$

Now suppose that L has the following form

$$L(x_1, \ldots, x_n) = F\Big(\prod_{i=1}^{n} f_i(x_i)\Big). \tag{3.22}$$

Denote by $u = \prod_{i=1}^{n} f_i(x_i)$. Note that $F', f_i' \neq 0$ $(i = 1, \ldots, n)$, where "$'$" denotes the derivative with respect to the variable u.

A direct computation shows that

$$\begin{aligned} L_i &= \frac{f_i'}{f_i} u F', \quad i = 2, \ldots, n, \\ L_{ii} &= \frac{f_i''}{f_i} u F' + \Big(\frac{f_i'}{f_i}\Big)^2 u^2 F'', \quad i = 2, \ldots, n, && (3.23) \\ L_{ij} &= \frac{f_i' f_j'}{f_i f_j} u(F' + uF''), \quad i, j = 2, \ldots, n, \text{ and } i \neq j. && (3.24) \end{aligned}$$

According to the assumption that the sectional curvature K_{ij} is vanishing, we have

$$L_{ii} L_{jj} = L_{ij}^2, \quad i \neq j. \tag{3.25}$$

Substituting (3.23) and (3.24) into (3.25), we get

$$\Big(\frac{f_i'' f_j''}{f_i f_j} - \frac{f_i'^2 f_j'^2}{f_i^2 f_j^2}\Big) u^2 F'^2 + \Big(\frac{f_i'' f_j'^2}{f_i f_j^2} + \frac{f_i'^2 f_j''}{f_i^2 f_j} - 2\frac{f_i'^2 f_j'^2}{f_i^2 f_j^2}\Big) u^3 F' F'' = 0,$$

Since $F' \neq 0$, the above equation reduces to

$$\Big(\frac{f_i'' f_j''}{f_i f_j} - \frac{f_i'^2 f_j'^2}{f_i^2 f_j^2}\Big) + \Big(\frac{f_i'' f_j'^2}{f_i f_j^2} + \frac{f_i'^2 f_j''}{f_i^2 f_j} - 2\frac{f_i'^2 f_j'^2}{f_i^2 f_j^2}\Big) u \frac{F''}{F'} = 0. \tag{3.26}$$

If one has

$$\frac{f_i'' f_j'^2}{f_i f_j^2} + \frac{f_i'^2 f_j''}{f_i^2 f_j} - 2\frac{f_i'^2 f_j'^2}{f_i^2 f_j^2} = 0, \tag{3.27}$$

then (3.26) gives

$$\frac{f_i'' f_j''}{f_i f_j} - \frac{f_i'^2 f_j'^2}{f_i^2 f_j^2} = 0. \tag{3.28}$$

Combining (3.28) with (3.27) gives

$$\frac{f_i f_i''}{f_i'^2} = 1, \quad i = 1, \ldots, n. \tag{3.29}$$

Solving (3.29) gives Case (1) in Theorem 2.

Hence, we assume that

$$\frac{f_i''f_j'^2}{f_if_j^2}+\frac{f_i'^2f_j''}{f_i^2f_j}-2\frac{f_i'^2f_j'^2}{f_i^2f_j^2}\neq 0. \tag{3.30}$$

Differentiating (3.26) with respect to $x_1, l \neq i, j$, we have

$$\Big(u\frac{F''}{F'}\Big)'u\frac{f_l'}{f_l}=0,$$

that is

$$u\frac{F''}{F'}=c \tag{3.31}$$

for some constant c. If $c \neq -1$, then F is given by

$$F=au^{c+1}+b, \quad a\in\mathbb{R}/0, b\in\mathbb{R}, \tag{3.32}$$

and if $c=-1$, then

$$F=a\ln u+b, \quad a\in\mathbb{R}/0, b\in\mathbb{R}. \tag{3.33}$$

Taking into account (3.31), (3.26) becomes

$$\Big(\frac{f_if_i''}{f_i'^2}\frac{f_jf_j''}{f_j^2}-1\Big)+\Big(\frac{f_if_i''}{f_i'^2}+\frac{f_jf_j''}{f_j^2}-2\Big)c=0. \tag{3.34}$$

Equation (3.34) can also rewritten as

$$\Big(\frac{f_if_i''}{f_i'^2}+c\Big)\Big(\frac{f_jf_j''}{f_j^2}+c\Big)=(c+1)^2. \tag{3.35}$$

When $c=-1$, without loss of generality, we assume $\frac{f_1f_1''}{f_1'^2}\neq 1$. Then, (3.35) implies that

$$\frac{f_if_i''}{f_i'^2}=1, \quad i=2,\dots,n. \tag{3.36}$$

By solving (3.36), we obtain Case (2). This is a quasi-linear production function.

When $c\neq -1$, (3.35) means that

$$\frac{f_if_i''}{f_i'^2}=-2c-1, \quad i=1,\dots,n. \tag{3.37}$$

Solving (3.37) gives

$$f_i=(c_ix_i+d_i)^{\frac{1}{2c+2}}, \quad c_i\in\mathbb{R}/0,\ d_i\in\mathbb{R},\ \ i=1,\dots,n. \tag{3.38}$$

Combining this with (3.32), after suitable translations we get Case (3) in Theorem 2. This is a generalized Cobb-Douglas production function. □

A Riemannian space is called *Ricci-flat* if its Ricci tensor vanishes (cf. [11]). Also, 3-dimensional Ricci-flat manifolds are always flat spaces. As an application of Theorem 2, we have

Corollary 3.4. *Let $L(x_1,x_2,x_3)=F\Big(f_1(x_1)f_2(x_2)f_3(x_3)\Big)$ be a quasi-product production function. If the graph of L is Ricci-flat, then, up to translations, L is given by one of the following three functions:*

1. $L=F\Big(\exp\Big(c_1x_1+c_2x_2+c_3x_3\Big)\Big), \quad c_i\in\mathbb{R}/0;$
2. $L=c_1\ln(f_1(x_1))+c_2x_2+c_3x_3, \quad c_i\in\mathbb{R}/0,$ *where* f_1 *satisfies* $f_1f_1''\neq f_1'^2;$
3. $L=a\sqrt{x_1x_2x_3}, \quad a\in\mathbb{R}^+.$

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