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On the Generalization of a Theorem of Bor

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Abstract. In [6], Bor proved a theorem dealing with absolute Riesz summability factors of infinite series. In this paper, we generalize that result for the absolute matrix summability factors of infinite series. Some new results are also obtained.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_n |\Delta\lambda_n| = \sum_n \mid \lambda_n - \lambda_{n+1} \mid < \infty$. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^\delta(\log n)^\sigma, \sigma \geq 0, \ 0 < \delta < 1\}$ (see [12]). If we take $\sigma = 0$, then we get a quasi-δ-power increasing sequence (see [10]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by (u_n) and (t_n) the nth $(C, 1)$ means of the sequence (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [7], [9])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \ (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

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defines the sequence (v_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid v_n - v_{n-1} \mid^k < \infty. \tag{4}$$

In the special case $p_n = 1$ for all values of n $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability.
Given a normal matrix $A = (a_{nv})$, we associate two lower semi-matrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \ n, v = 0, 1, ... \tag{5}$$

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}, \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \ n = 1, 2, ... \tag{6}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \ n = 0, 1, ... \tag{7}$$

and

$$A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \widehat{a}_{nv} a_v. \tag{8}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \overline{\Delta} A_n(s) \mid^k < \infty, \tag{9}$$

where

$$\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

In the special case, for $a_{nv} = p_v/P_n$, $|A, p_n|_k$ summability is the same as $\mid \bar{N}, p_n \mid_k$ summability.

2. The Known Result

Quite recently, Bor has proved the following theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series.

Theorem 2.1 ([6]). Let $(\lambda_n) \in \mathcal{BV}$ and let (X_n) be a quasi-f-power increasing sequence for some δ $(0 < \delta < 1)$ and $\sigma \geq 0$. Suppose that there exists sequences (β_n) and (λ_n) such that

$$\mid \Delta \lambda_n \mid \leq \beta_n, \tag{10}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} n \mid \Delta \beta_n \mid X_n < \infty, \tag{12}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{13}$$

If

$$\sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{14}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{15}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{16}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.
It should be noted that if we take $\sigma = 0$, then we get a result which was proved in [4].

3. The Main Result

The aim of this paper is to generalize Theorem 2.1 by using absolute matrix summability factors. Now, we shall prove the following theorem.
Theorem 3.1. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \; n = 0, 1, 2, ..., \tag{17}$$

$$a_{n-1,v} \geq a_{nv}, \; for \; n \geq v + 1 \tag{18}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{19}$$

$$na_{nn} = O(1) \tag{20}$$

$$\hat{a}_{n,v+1} = O\left(v|\Delta_v \hat{a}_{nv}|\right). \tag{21}$$

Let $(\lambda_n) \in \mathcal{BV}$ and let (X_n) be a quasi-f-power increasing sequence for some δ $(0 < \delta < 1)$ and $\sigma \geq 0$. If the conditions (10)-(16) are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid A, p_n \mid_k$, $k \geq 1$.
It should be noted that if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.1.
We require the following lemmas for the proof of our theorem.
Lemma 3.2 ([3]). If the conditions (15) and (16) are satisfied, then $\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right)$.
Lemma 3.3 ([5]). Except for the condition $(\lambda_n) \in \mathcal{BV}$ under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem, we have the following;

$$nX_n \beta_n = O(1), \tag{22}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{23}$$

Proof of Theorem 3.1. Let (T_n) be the A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then, we have

$$T_n - T_{n-1} = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v v \lambda_v}{v^2 p_v}.$$

Using Abel's transformation, we get that

$$\begin{aligned} T_n - T_{n-1} &= \sum_{v=1}^{n-1} \Delta_v \left(\hat{a}_{nv} \frac{P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{v=1}^{n} v a_v \\ &= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \frac{P_v}{p_v}(v+1) \frac{\lambda_v}{v^2} t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}(v+1) \frac{t_v}{v^2} \frac{P_v}{p_v} \Delta\lambda_v \\ &+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1}(v+1) t_v \Delta\left(\frac{P_v}{v^2 p_v}\right) + \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,r}\mid^k < \infty, \quad for \quad r=1,2,3,4. \tag{24}$$

When $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$ and so we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,1}\mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|t_v|^k\right)\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}(a_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v\hat{a}_{nv}|\\
&= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k a_{vv} = O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k\frac{p_v}{P_v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{p_r}{P_r}|t_r|^k + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)\mid\lambda_m\mid X_m\\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)\mid\lambda_m\mid X_m = O(1)
\end{aligned}$$

as $m\to\infty$, by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3.
Now, by using (15), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\mid\hat{a}_{n,v+1}\mid\mid\Delta\lambda_v\mid\mid t_v\mid\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}v|\Delta_v\hat{a}_{nv}||\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}(v|\Delta_v\hat{a}_{nv}|)^k(\beta_v)^k|t_v|^k\right)\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}(v\beta_v)(v\beta_v)^{k-1}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}(a_{nn})^{k-1}|\Delta_v\hat{a}_{nv}|\\
&= O(1)\sum_{v=1}^{m}v\beta_v|t_v|^k a_{vv}(v\beta_v)^{k-1}\\
&= O(1)\sum_{v=1}^{m}v^{k-1}v\beta_v\frac{1}{v^k}\mid t_v\mid^k = O(1)\sum_{v=1}^{m}v\beta_v\frac{\mid t_v\mid^k}{v}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{\mid t_r\mid^k}{r}+O(1)m\beta_m\sum_{v=1}^{m}\frac{\mid t_v\mid^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\mid (v+1)\Delta\beta_v-\beta_v\mid X_v+O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1}v\mid\Delta\beta_v\mid X_v+O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)m\beta_m X_m=O(1),\ as\ m\to\infty,
\end{aligned}
$$

by virtue of hypotheses of Theorem 3.1 and Lemma 3.3.
Now, since $\Delta\left(\frac{P_v}{v^2p_v}\right)=O\left(\frac{1}{v^2}\right)$, by Lemma 3.2, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}||t_v|\frac{1}{v}\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_{v+1}|^k|t_v^k|\right)\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}(a_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_{v+1}|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v\hat{a}_{nv}|\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^k|t_v|^k a_{vv}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{r=1}^{v}\frac{|t_r|^k}{r}+O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{|t_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|X_v+O(1)|\lambda_{m+1}|X_m\\
&= O(1)\sum_{v=1}^{m}\beta_v X_v+O(1)|\lambda_{m+1}|X_{m+1}=O(1)
\end{aligned}
$$

as $m\to\infty$ by (10), (13), (14), (15), (20) and (21).
Finally, as in $T_{n,3}$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,4}\mid^k &= O(1)\sum_{n=1}^{m}n^{k-1}\frac{1}{n^k}\mid\lambda_n\mid^{k-1}\mid\lambda_n\mid\mid t_n\mid^k\\
&= O(1)\sum_{n=1}^{m}\frac{\mid t_n\mid^k}{n}=O(1),\quad as\quad m\to\infty.
\end{aligned}
$$

This completes the proof of Theorem 3.1. If we take $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n, then we obtain a new result concerning the $\mid C,1\mid_k$ summability factors of infinite series.

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