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On Ideal Convergence of Sequences of Linear Operators in the Space of Analytic Functions

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Abstract. We investigate the problem of ideal convergence of the sequences of linear operators without the properties of $k-$positivity in the space of analytic functions in a bounded simply connected domain of complex plane.

1. Introduction

Let $A(D)$ denote the space of all analytic functions in a bounded simply connected domain D and let $\phi(z)$ be any function mapping D conformally and one to one on the unit disk. Since the system of functions $\phi^k(z)$, $k = 0, 1, 2, ...$ is a basis in the space $A(D)$ (see [3]), for every $f \in A(D)$ the Taylor expansion of f is given by

$$f(z) = \sum_{k=0}^{\infty} f_k \phi^k(z), \tag{1}$$

where f_k is the Taylor coefficients of f and satisfies

$$\limsup_{k\to\infty} \left|f_k\right|^{\frac{1}{k}} = 1. \tag{2}$$

Note that Taylor coefficients of f are calculated by

$$f_k = \frac{1}{2\pi i} \int_C \frac{f(z)\,\phi'(z)}{\left(\phi(z)\right)^{k+1}} dz, \tag{3}$$

where C is any contour lying in the interior of D. The series (1) under the condition (2) is uniformly convergent if $\left|\phi(z)\right| \le r < 1$. Denoting

$$\left\|f\right\|_{A(D),r} = \max_{\left|\phi(z)\right| \le r < 1} \left|f(z)\right|, \tag{4}$$

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we transfer $A(D)$ in a Fréchet space with the family of norms $\|.\|_{A(D),r}$ depending on r.

Let T_n be a sequence of linear operators acting from $A(D)$ into $A(D)$. Then by (1), for any function $f \in A(D)$ we can write Taylor expansion of $T_n f(z)$ as follows:

$$T_n f(z) = \sum_{k=0}^{\infty} \phi^k(z) \sum_{m=0}^{\infty} f_m T_{k,m}^{(n)}, \tag{5}$$

where $T_{k,m}^{(n)}$ is the Taylor coefficient of $T_n \phi^k(z)$ such that

$$\limsup_{k\to\infty} \left| \sum_{m=0}^{\infty} f_m T_{k,m}^{(n)} \right|^{\frac{1}{k}} = 1.$$

It is well known that the operator T_n is $k-$positive if and only if $T_{k,m}^{(n)}$ is nonnegative for all k, m, n (see [6], [8]). In the present paper, we deal with the sequences of linear operators (5) without properties of $k-$positivity. We note that the definition of $k-$positivity of linear operators acting in the space of analytic functions in the unit disk was introduced by Gadjiev [6] to establish analogues of the famous Korovkin theorem. Very recently, this definition of $k-$positivity given in [6] has been used intensively in different approximation problems (see [2, 7 − 10, 12, 13, 16]).

The statistical convergence was introduced in [4] and has been improved in various areas by a number of authors (see [1] ; [5] ; [11] ; [15] ; [17] ; [18]) . Notice that statistical convergence has been examined in approximation theory by Gadjiev and Orhan [11] . In the paper [11] , the authors firstly proved the Korovkin type theorems on statistical approximation and also gave the definition of order of statistical approximation by positive linear operators. Recently, some different versions of statistical approximation by $k-$positive linear operators were obtained in [2, 16] . Inspired by the recent works on these topics, we investigate the problem of approximation of analytic functions and their derivatives by the sequences of linear operators and their derivatives acting on the space of analytic functions in a simply connected bounded domain without properties of $k-$positivity, via ideal convergence. So, our results are more general than obtained results for k-positive cases.

Now, we recall the concept of ideal convergence first introduced in the paper [14] . Ideal convergence is a generalization of the statistical convergence and it is based on the ideal of subsets of the set $\mathbb{N}$ of positive integers.

Let X be a nonempty set. A class I of subsets of X is said to be an ideal in X provided that
(i) $\phi \in I$,
(ii) if $A, B \in I$, then $A \cup B \in I$,
(iii) if $A \in I$ and $B \subseteq A$, then $B \in I$.

An ideal is called nontrivial if $X \notin I$. Also, a nontrivial ideal in X is called admissible if $\{x\} \in I$ for each $x \in X$. Let I be a nontrivial ideal in $\mathbb{N}$. A sequence $x := (x_n)$ is ideal convergent (or $I-$convergent) to a number L if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I$. Note that taking $I = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, then $I-$convergence coincides with $A-$statistical convergence, where A is a nonnegative regular summability matrix and $\delta_A(K)$ denotes the A density of K. Moreover, if we choose $A = C_1$, the Cesáro matrix of order one, then we obtain the statistical convergence. Besides, if I is the class of all finite subsets of $\mathbb{N}$, then $I-$convergence reduces to the ordinary convergence.

2. Ideal Approximation by Linear Operators

In this section, we present some sufficient conditions for approximation of analytic functions belonging to some appropriate subspace of $A(D)$. For this aim firstly we will give the following definition introduced in [10] to compute the degree of ideal convergence of sequences. Recall that in the paper [11] , this definition is given for statistical convergence of sequences of linear positive operators.

Definition 2.1. *Let I be an admissible ideal in $\mathbb{N}$. Then, one says that a sequence (x_n) is ideal convergent to a number L with degree $0 < \beta \leq 1$ if, for every $\varepsilon > 0$,*

$$\left\{n \in \mathbb{N} : \frac{|x_n - L|}{n^{1-\beta}} \geq \varepsilon\right\} \in I.$$

In this case, we write

$$x_n - L = I - o\left(n^{-\beta}\right), \qquad \text{as } n \to \infty.$$

We note here that if we choose $\beta = 1$ in Definition 2.1, we immediately obtain the ideal convergence of (x_n) to L.

Also, the proof of our main theorems require the following lemma.

Lemma 2.2. *Let I be an admissible ideal in $\mathbb{N}$ and let (f_n) be a sequence of analytic functions on D with the Taylor coefficients $f_{n,k}$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, for $0 < \beta \leq 1$,*

$$\left\|f_n\right\|_{A(D),r} = I - o\left(n^{-\beta}\right), \qquad \text{as } n \to \infty$$

if and only if

$$\left|f_{n,k}\right| \leq \varepsilon_n \left(1 + \delta_n\right)^k, \tag{6}$$

where

$$\lim_{n\to\infty} \delta_n = 0 \text{ and } \varepsilon_n = I - o\left(n^{-\beta}\right) \text{ as } n \to \infty. \tag{7}$$

Proof. Suppose that (6) holds. Then, using (4) , we may write that

$$\left\|f_n\right\|_{A(D),r} \leq \varepsilon_n \sum_{k=0}^{\infty} \left(1 + \delta_n\right)^k r^k$$

for any $r < 1$. Since $\lim_{n\to\infty} \delta_n = 0$, we have

$$\left\|f_n\right\|_{A(D),r} \leq \frac{\varepsilon_n}{1 - r\left(1 + \delta_n\right)}.$$

Taking into account that $\lim_{n\to\infty} \frac{1}{1-r(1+\delta_n)} = \frac{1}{1-r}$ is finite, there exists a positive constant $K(r)$ such that for every $n \in \mathbb{N}$

$$\frac{1}{1 - r\left(1 + \delta_n\right)} \leq K(r).$$

Hence, it follows that

$$\left\|f_n\right\|_{A(D),r} \leq \varepsilon_n K(r)$$

for every $n \in \mathbb{N}$. Now, for a given $\varepsilon > 0$, we define the following sets

$$\begin{aligned} U &:= \left\{n \in \mathbb{N} : \frac{\left\|f_n\right\|_{A(D),r}}{n^{1-\beta}} \geq \varepsilon\right\}, \\ V &:= \left\{n \in \mathbb{N} : \frac{\varepsilon_n K(r)}{n^{1-\beta}} \geq \varepsilon\right\}. \end{aligned}$$

By the condition $\varepsilon_n = I - o\left(n^{-\beta}\right)$ as $n \to \infty$, we obtain $V \in I$. Since $U \subseteq V$, using the definition of ideal, we can write $U \in I$, which is desired result.

In order to prove the necessity, we choose δ_n tending to zero so slowly such that

$$\varepsilon_n := \max_{|\phi(z)|=\frac{1}{1+\delta_n}<1} \left|f_n(z)\right| = I - o\left(n^{-\beta}\right), \text{ as } n \to \infty.$$

By using (3), it follows that

$$\begin{aligned}\left|f_{n,k}\right| &\leq \frac{1}{2\pi} \int\limits_{|\phi(z)|=\frac{1}{1+\delta_n}<1} \frac{\left|f_n(z)\right|\left|\phi'(z)\right|}{\left|\phi(z)\right|^{k+1}} |dz| \\ &\leq \varepsilon_n \int\limits_{|u|=\frac{1}{1+\delta_n}<1} \frac{|du|}{|u|^{k+1}} = \varepsilon_n (1+\delta_n)^k,\end{aligned}$$

whence the result. □

Let $g_k \geq 1$ be an increasing sequence of real numbers, $\limsup\limits_{k\to\infty} g_k^{\frac{1}{k}} = 1$ and let $A_g(D)$ be the subspace of functions in $A(D)$ with Taylor coefficients f_k satisfying the inequality $\left|f_k\right| \leq M_f g_k$ $(k = 0, 1, 2...)$, where M_f is a constant depending only on f. Also, assume that g_k satisfies the condition

$$\limsup_{k\to\infty} \left(\sqrt{g_k} - \sqrt{g_{k-1}}\right)^{\frac{1}{k}} = 1. \tag{8}$$

Then, we can state the first main result of this section.

Theorem 2.3. *Let I be an admissible ideal in $\mathbb{N}$ and let (T_n) be a sequence of linear operators from $A(D)$ into $A(D)$. If there exist sequences ε_n and δ_n satisfying (7) such that the following inequalities*

$$\left|\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1\right| < \varepsilon_n (1+\delta_n)^k \tag{9}$$

$$\left|\sum_{m=0}^{\infty} \left|T_{k,m}^{(n)}\right| - 1\right| < \varepsilon_n (1+\delta_n)^k \tag{10}$$

$$\left|\sum_{m=0}^{\infty} \sqrt{g_m}\left|T_{k,m}^{(n)}\right| - \sqrt{g_k}\right| < \varepsilon_n (1+\delta_n)^k \tag{11}$$

$$\left|\sum_{m=0}^{\infty} g_m\left|T_{k,m}^{(n)}\right| - g_k\right| < \varepsilon_n (1+\delta_n)^k \tag{12}$$

hold, then for any function $f \in A_g(D)$ and $0 < \beta \leq 1$, we have

$$\left\|T_n f - f\right\|_{A(D),r} = I - o\left(n^{-\beta}\right) \text{ as } n \to \infty.$$

Proof. By (1) and (5), for each function $f \in A_g(D)$, we can write

$$T_n f(z) - f(z) = \sum_{k=0}^{\infty} \phi^k(z) \sum_{m=0}^{\infty} (f_m - f_k) T_{k,m}^{(n)} + \sum_{k=0}^{\infty} \phi^k(z) f_k \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1\right).$$

Using (4) , the last equality gives that for any $r < 1$

$$\left\|T_n f - f\right\|_{A(D),r} \leq \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty} \left|f_m - f_k\right| \left|T_{k,m}^{(n)}\right| + \sum_{k=0}^{\infty} r^k \left|f_k\right| \left|\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1\right|. \tag{13}$$

By simple calculations, we easily obtain

$$\left|f_m - f_k\right| \leq 8M_f \frac{g_k^3}{\Delta_k^2(g)} \left(\sqrt{g_m} - \sqrt{g_k}\right)^2 \tag{14}$$

for all $m, k \in \mathbb{N}_0$, where

$$\Delta_k(g) = \min\left\{\sqrt{g_k} - \sqrt{g_{k-1}};\ \sqrt{g_{k+1}} - \sqrt{g_k}\right\}. \tag{15}$$

On the other hand, from the inequalities $(10 - 12)$, it is easily verified that

$$\begin{aligned}\sum_{m=0}^{\infty} \left(\sqrt{g_m} - \sqrt{g_k}\right)^2 \left|T_{k,m}^{(n)}\right| &< \varepsilon_n (1 + \delta_n)^k \left(1 + \sqrt{g_k}\right)^2 \\ &< 4\varepsilon_n (1 + \delta_n)^k g_k.\end{aligned}$$

Combining respectively (14) , the last inequality and (9) in (13) , we have

$$\left\|T_n f - f\right\|_{A(D),r} \leq 32 M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1 + \delta_n)^k \frac{g_k^4}{\Delta_k^2(g)} + M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1 + \delta_n)^k g_k.$$

Since $\Delta_k^2(g) \leq 2g_k$, it follows

$$\left\|T_n f - f\right\|_{A(D),r} \leq 34 M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1 + \delta_n)^k \frac{g_k^4}{\Delta_k^2(g)}.$$

Taking into account the conditions on g_k and (8) , the series on the right-hand side of the last inequality is convergent for any $0 < r < 1$ since δ_n is infinitely small sequence.

Now, for a given $\varepsilon > 0$, we consider the following sets

$$\begin{aligned} L &:= \left\{n \in \mathbb{N} : \frac{\left\|T_n f - f\right\|_{A(D),r}}{n^{1-\beta}} \geq \varepsilon\right\}, \\ N &:= \left\{n \in \mathbb{N} : \frac{\varepsilon_n}{n^{1-\beta}} A_n(g, r) \geq \varepsilon\right\}, \end{aligned}$$

where

$$A_n(g, r) = 34 M_f \sum_{k=0}^{\infty} r^k (1 + \delta_n)^k \frac{g_k^4}{\Delta_k^2(g)} < \infty.$$

By the condition $\varepsilon_n = I - o\left(n^{-\beta}\right)$ as $n \to \infty$, we get $N \in I$. Since $L \subseteq N$, according to the definition of ideal, we can write $L \in I$. This completes the proof. □

Remark 2.4. *Note here that the conditions* (9) *and* (10) *in Theorem 2.3 don't contain each other and both of these conditions are essential for the proof of main results.*

Now, let us suppose that the sequence g_k has the form

$$g_k = 1 + h_k,$$

where h_k is an increasing sequence. In this case the condition (8) has the form

$$\limsup_{k\to\infty} \left(\sqrt{h_k} - \sqrt{h_{k-1}} \right)^{\frac{1}{k}} = 1. \tag{16}$$

The inequalities (11) and (12) in Theorem 2.3 take the forms

$$\left| \sum_{m=0}^{\infty} \sqrt{1+h_m} \left| T_{k,m}^{(n)} \right| - \sqrt{1+h_k} \right| < \varepsilon_n (1+\delta_n)^k$$

$$\left| \sum_{m=0}^{\infty} (1+h_m) \left| T_{k,m}^{(n)} \right| - (1+h_k) \right| < \varepsilon_n (1+\delta_n)^k .$$

We shall show that in this particular case the conditions given above may be chosen in a simple form as

$$\left| \sum_{m=0}^{\infty} \sqrt{h_m} \left| T_{k,m}^{(n)} \right| - \sqrt{h_k} \right| < \varepsilon_n (1+\delta_n)^k$$

$$\left| \sum_{m=0}^{\infty} h_m \left| T_{k,m}^{(n)} \right| - h_k \right| < \varepsilon_n (1+\delta_n)^k .$$

Then, for this particular case the next theorem can be stated as follows.

Theorem 2.5. *Let I be an admissible ideal in $\mathbb{N}$ and let (T_n) be a sequence of linear operators $T_n : A(D) \to A(D)$. If there exist sequences ε_n and δ_n satisfying* (7) *such that the inequalities (9), (10),*

$$\left| \sum_{m=0}^{\infty} \sqrt{h_m} \left| T_{k,m}^{(n)} \right| - \sqrt{h_k} \right| < \varepsilon_n (1+\delta_n)^k \tag{17}$$

$$\left| \sum_{m=0}^{\infty} h_m \left| T_{k,m}^{(n)} \right| - h_k \right| < \varepsilon_n (1+\delta_n)^k \tag{18}$$

hold, then for any function $f \in A_g(D)$ and $0 < \beta \leq 1$, with $g_k = 1 + h_k$ we have

$$\left\| T_n f - f \right\|_{A(D),r} = I - o\left(n^{-\beta}\right) \text{ as } n \to \infty.$$

Proof. Let $f \in A_g(D)$. By (4), it is obvious that the inequality (13) holds. Also, since $f \in A_g(D)$, by making rearrangements we immediately obtain

$$\left| f_m - f_k \right| \leq 4M_f \frac{g_k^2}{\Delta_k^2(h)} \left(\sqrt{h_m} - \sqrt{h_k} \right)^2 \left[1 + \Delta_k^2(h) \right],$$

where $\Delta_k(h)$ is given as at the above (15). Considering $1 + \Delta_k^2(h) \leq g_k$, we easily arrive to

$$\left| f_m - f_k \right| \leq 4M_f \frac{g_k^3}{\Delta_k^2(h)} \left(\sqrt{h_m} - \sqrt{h_k} \right)^2 . \tag{19}$$

Also, by using (10) , (17) and (18) we may write that

$$\sum_{m=0}^{\infty}\left(\sqrt{h_m}-\sqrt{h_k}\right)^2\left|T_{k,m}^{(n)}\right| <\varepsilon_n\left(1+\delta_n\right)^k\left(1+\sqrt{h_k}\right)^2$$

$$<4\varepsilon_n\left(1+\delta_n\right)^k g_k. \tag{20}$$

Thus, applying the inequalities (19), (20) and (9) in (13) we conclude that

$$\left\|T_n f-f\right\|_{A(D),r}\leq 16M_f\varepsilon_n\sum_{k=0}^{\infty}r^k\left(1+\delta_n\right)^k\frac{g_k^4}{\Delta_k^2(h)}+M_f\varepsilon_n\sum_{k=0}^{\infty}r^k\left(1+\delta_n\right)^k g_k,$$

which implies

$$\left\|T_n f-f\right\|_{A(D),r}\leq 17M_f\varepsilon_n\sum_{k=0}^{\infty}r^k\left(1+\delta_n\right)^k\frac{g_k^4}{\Delta_k^2(h)}.$$

Since by the conditions on g_k and (16) , we observe that the last series converges for any $0<r<1$. Here, the remain of the proof is similar with those in Theorem 2.3. □

Now, we will give a general result on approximation in $A(D)$.

Theorem 2.6. *Let I be an admissible ideal in $\mathbb{N}$ and let b_k be an increasing sequence of positive numbers such that $\limsup_{k\to\infty} b_k^{\frac{1}{k}}=1$ and g_k is defined as above. If the sequence of linear operators $T_n: A(D)\to A(D)$ satisfies the conditions (9),*

$$\left|\sum_{m=0}^{\infty}\frac{g_m}{1+b_m}\left|T_{k,m}^{(n)}\right|-\frac{g_k}{1+b_k}\right|<\varepsilon_n\left(1+\delta_n\right)^k \tag{21}$$

$$\left|\sum_{m=0}^{\infty}\frac{\sqrt{b_m}}{1+b_m}g_m\left|T_{k,m}^{(n)}\right|-\frac{\sqrt{b_k}}{1+b_k}g_k\right|<\varepsilon_n\left(1+\delta_n\right)^k \tag{22}$$

$$\left|\sum_{m=0}^{\infty}\frac{b_m}{1+b_m}g_m\left|T_{k,m}^{(n)}\right|-\frac{b_k}{1+b_k}g_k\right|<\varepsilon_n\left(1+\delta_n\right)^k, \tag{23}$$

where ε_n and δ_n are the same as (7) *, then for any function $f\in A_g(D)$ and $0<\beta\leq 1$, we have*

$$\left\|T_n f-f\right\|_{A(D),r}=I-o\left(n^{-\beta}\right)\ \textit{as}\ n\to\infty.$$

Proof. Assume that the inequalities $(21-23)$ and (9) hold. Obviously, by (4) , we may write the inequality (13) for any function $f\in A(D)$. Since $g_k\geq 1$ for each $k=0,1,2...$, we obtain

$$\left|f_m-f_k\right|\leq 2M_f g_k g_m.$$

By direct computations, we can write

$$\left|f_m-f_k\right|\leq 8M_f g_k\frac{1+b_k}{\Delta_k^2(b)}\frac{\left(\sqrt{b_m}-\sqrt{b_k}\right)^2}{1+b_m}g_m, \tag{24}$$

where $\Delta_k(b)$ is given as (15). On the other hand, we conclude from the conditions $(21-23)$ that

$$\sum_{m=0}^{\infty} \frac{\left(\sqrt{b_m}-\sqrt{b_k}\right)^2}{1+b_m} g_m \left|T_{k,m}^{(n)}\right| < \varepsilon_n (1+\delta_n)^k \left(1+\sqrt{b_k}\right)^2. \tag{25}$$

Using the inequalities (24), (25) and (9), we have

$$\left\|T_n f - f\right\|_{A(D),r} \le 8M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1+\delta_n)^k g_k \frac{1+b_k}{\Delta_k^2(b)} \left(1+\sqrt{b_k}\right)^2 + M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1+\delta_n)^k g_k,$$

which finally implies

$$\left\|T_n f - f\right\|_{A(D),r} \le 9M_f \varepsilon_n \sum_{k=0}^{\infty} r^k (1+\delta_n)^k g_k \frac{1+b_k}{\Delta_k^2(b)} \left(1+\sqrt{b_k}\right)^2.$$

Since the conditions on g_k and b_k, the series on the right-hand side of the last inequality converges for any $0 < r < 1$. Therefore, applying similar reasoning as in the proof of Theorem 2.3, we get the desired conclusion. □

Note here that for example in Theorem 2.6, choosing $g_k = 1 + b_k$, we clearly get a theorem on convergence in the subspace of $A(D)$ of functions with Taylor coefficients satisfying $|f_k| \le M_f (1+b_k)$.

3. Ideal Approximation by Derivatives of Linear Operators

In this section, we obtain some theorems on simultaneous approximation of analytic functions by the sequences of linear operators and their derivatives, by using the concept of ideal convergence. Before stating our main theorems, we need the following auxiliary results.

Lemma 3.1. *Let I be an admissible ideal in $\mathbb{N}$ and let (f_n) be a sequence of analytic functions on D with the Taylor coefficients $f_{n,k}$ for each $n \in \mathbb{N}$ and $k, p \in \mathbb{N}_0$. Then, for $0 < \beta \le 1$,*

$$\left\|f_n^{(p)}\right\|_{A(D),r} = I - o\left(n^{-\beta}\right), \quad \text{as } n \to \infty \tag{26}$$

if and only if

$$\left|f_{n,k+p}\right| \le \frac{k!}{(k+p)!} \varepsilon_n (1+\delta_n)^{k+p}, \tag{27}$$

where ε_n and δ_n are as in (7).

Proof. Assume that the condition (26) holds. Taking into account the Taylor expansion of f, it is obvious that

$$f_n^{(p)}(z) = \sum_{k=0}^{\infty} (k+1)(k+2)\ldots(k+p) f_{n,k+p} \phi^k(z). \tag{28}$$

Also, by Lemma 2.2, there exist sequences ε_n and δ_n satisfying (7) such that

$$(k+1)(k+2)\ldots(k+p)\left|f_{n,k+p}\right| \le \varepsilon_n (1+\delta_n)^k,$$

which implies

$$\left|f_{n,k+p}\right| \leq \frac{k!}{(k+p)!}\varepsilon_n (1+\delta_n)^{k+p}.$$

Now, assume that (27) holds. Using (4), we get

$$\begin{aligned}\left\|f_n^{(p)}(z)\right\|_{A(D),r} &\leq \sum_{k=0}^{\infty}(k+1)\ldots(k+p)\left|f_{n,k+p}\right| r^k \\ &\leq \varepsilon_n (1+\delta_n)^p \sum_{k=0}^{\infty}(1+\delta_n)^k r^k \\ &= \varepsilon_n \frac{(1+\delta_n)^p}{1-(1+\delta_n) r}.\end{aligned}$$

Since $\lim\limits_{n\to\infty} \frac{(1+\delta_n)^p}{1-r(1+\delta_n)} = \frac{1}{1-r}$ is finite, there exists a positive constant $M(r)$ such that for every $n \in \mathbb{N}$

$$\frac{(1+\delta_n)^p}{1-r(1+\delta_n)} \leq M(r).$$

From this we obtain

$$\left\|f_n^{(p)}\right\|_{A(D),r} \leq \varepsilon_n M(r)$$

for every $n \in \mathbb{N}$. Using the same method as in the proof of the sufficiency part of Lemma 2.2, we arrive at the assertion of the lemma. □

Lemma 3.2. *Let I be an admissible ideal in $\mathbb{N}$ and let (f_n) be a sequence of analytic functions on D. Then, for any $p = 0, 1, 2...$ and $0 < \beta \leq 1$,*

$$\left\|f_n^{(p)}\right\|_{A(D),r} = I - o\left(n^{-\beta}\right), \quad \text{as } n \to \infty$$

if and only if

$$\left\|f_n\right\|_{A(D),r} = I - o\left(n^{-\beta}\right), \quad \text{as } n \to \infty. \tag{29}$$

Proof. Suppose that the condition (29) holds. In view of Lemma 2.2, there exist sequences ε_n and δ_n satisfying the conditions (6) and (7). Using (28), for any $p \in \mathbb{N}_0$ we obtain

$$\left\|f_n^{(p)}\right\|_{A(D),r} \leq \varepsilon_n (1+\delta_n)^p \sum_{k=0}^{\infty}(k+1)\ldots(k+p)(1+\delta_n)^k r^k. \tag{30}$$

Also, since

$$\frac{1}{1-\phi(z)} = \sum_{k=0}^{\infty}\phi^k(z)$$

for $\left|\phi(z)\right| = r < 1$, a simple computation shows that

$$\frac{p!}{\left(1-\phi(z)\right)^{p+1}} = \sum_{k=0}^{\infty}(k+1)\ldots(k+p)\phi^k(z)$$

for any $p \in \mathbb{N}_0$ with $\phi'(z) \neq 0$.

By applying the last equality in (30), we arrive at

$$\left\| f_n^{(p)} \right\|_{A(D),r} \leq \varepsilon_n (1+\delta_n)^p \frac{p!}{(1-(1+\delta_n)r)^{p+1}}.$$

The rest of the proof follows from the sufficiency part of Lemma 2.2. □

We recall that the Lemmas 3.1 and 3.2 were proved in the paper [7] for statistical convergence of analytic functions in the unit disk by the sequences of $k-$positive linear operators.

Now, let T_n be a sequence of linear operators acting from $A(D)$ into $A(D)$. Then, by (5) and (28), we obtain that for each $p \in \mathbb{N}_0$,

$$\frac{d^p}{dz^p} T_n f(z) := T_n^{(p)} f(z) = \sum_{k=0}^{\infty} \phi^k(z)(k+1)\ldots(k+p) \sum_{m=0}^{\infty} T_{k+p,m}^{(n)} f_m.$$

In view of Lemma 3.2, we immediately obtain the following results which are correspondingly the corollaries of Theorems 2.3, 2.5 and 2.6.

Proposition 3.3. *Let I be an admissible ideal in $\mathbb{N}$ and let (T_n) be a sequence of linear operators $T_n : A(D) \to A(D)$. If there exist sequences ε_n and δ_n satisfying* (7) *such that the inequalities* (9), (10), (11) *and* (12) *hold, then for any function $f \in A_g(D)$ and $0 < \beta \leq 1$, we have*

$$\left\| T_n^{(p)} f - f^{(p)} \right\|_{A(D),r} = I - o\left(n^{-\beta}\right) \text{ as } n \to \infty.$$

Proposition 3.4. *Let I be an admissible ideal in $\mathbb{N}$ and assume that the hypothesis of Theorem 2.5 or Theorem 2.6 holds. Then for any function $f \in A_g(D)$ and $0 < \beta \leq 1$, we have*

$$\left\| T_n^{(p)} f - f^{(p)} \right\|_{A(D),r} = I - o\left(n^{-\beta}\right) \text{ as } n \to \infty.$$

In conclusion we note that if we take $I = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, then $I-$convergence reduces to $A-$statistical convergence, where $A = \left(a_{jn}\right)$ is a nonnegative regular summability matrix and $\delta_A(K)$ denotes the A density of K. Corresponding results are natural corollaries of Theorems 2.3, 2.5 and 2.6 and Propositions 3.3 and 3.4.

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