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Positive Solutions for a Fractional p-Laplacian Boundary Value Problem

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Abstract. In this paper we study the existence of positive solutions for the fractional p-Laplacian boundary value problem

$$\begin{cases} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = f(t,u(t)), t \in (0,1), \\ u(0) = u'(0) = 0, u'(1) = au'(\xi), D_{0+}^{\alpha}u(0) = 0, D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta), \end{cases}$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, 1/p+1/q = 1, 0 < \xi, \eta < 1, 0 \le a < \xi^{2-\alpha}, 0 \le b < \eta^{\frac{1-\beta}{p-1}}$ and $f \in C([0,1]\times[0,+\infty),[0,+\infty))$. Using the monotone iterative method and the fixed point index theory in cones, we establish two new existence results when the nonlinearity f is allowed to grow $(p-1)$-sublinearly and $(p-1)$-superlinearly at infinity.

1. Introduction

In this paper we discuss the existence of positive solutions for the fractional p-Laplacian boundary value problem

$$\begin{cases} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = f(t,u(t)), t \in (0,1), \\ u(0) = u'(0) = 0, u'(1) = au'(\xi), D_{0+}^{\alpha}u(0) = 0, D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta), \end{cases} \tag{1}$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, 1/p+1/q = 1, 0 < \xi, \eta < 1, 0 \le a < \xi^{2-\alpha}, 0 \le b < \eta^{\frac{1-\beta}{p-1}}$ and $f \in C([0,1]\times[0,+\infty),[0,+\infty))$.

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Fractional differential equations arise naturally for example in physics, chemistry, diffusion and transport theory, chaos and turbulence, viscoelastic mechanics and non-newtonian fluid mechanics; for more details on fractional applications, we refer the reader to [1–3]. There are many papers in the literature on the existence of solutions for fractional boundary value problems; see for example [4–12] and the references therein. In [4], the authors investigated the existence of positive solutions for the fractional differential equation with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, t \in (0,1), \\ u(0) = u'(0) = 0, u(1) = \displaystyle\int_0^1 g(s)u(s)\mathrm{d}s, \end{cases}$$

and obtained an existence result if the following condition is satisfied:

(H_f) there exist $a, \Lambda > 0$ such that $f(t,x) \le f(t,y) \le \Lambda a$, for $0 \le x \le y \le a, t \in [0,1]$.

Note for multi-point boundary value problems the Green's functions may be complicated. Bai [5] considered the fractional three point boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = 0, \beta u(\eta) = u(1), \end{cases} \tag{2}$$

where $\alpha \in (1,2]$, $\beta\eta^{\alpha-1}, \eta \in (0,1)$. The Green's function is

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-\beta\eta^{\alpha-1})}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \le \eta, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-\beta\eta^{\alpha-1})}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 < \eta \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le \eta < 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1, \eta \le s. \end{cases} \tag{3}$$

Note if $\beta = 0$, then (2) reduces to the problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

The Green's function is

$$g(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1} & 0 \le t \le s \le 1. \end{cases} \tag{5}$$

Now if the three point problem (2) is considered as a perturbation of the two point problem (4), we can use (5) to obtain (3), i.e.,

$$G(t,s) = g(t,s) + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} g(\eta,s).$$

This simple idea motivates our study in Section 2.

In this paper we first obtain an existence result with f growing $(p-1)$-sublinearly at infinity. Moreover, we establish an iterative sequence for approximating the solution. Next, using the fixed point index theory, we obtain an existence result with f growing $(p-1)$-superlinearly at infinity.

2. Preliminaries

For convenience, in this section we present some basic definitions and notations from fractional calculus.

Definition 2.1(see [3, page 36-37]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2(see [3, Definition 2.1]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann-Liouville derivative one obtains the following result.

Lemma 2.1(see [6]) Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has a unique solution

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N}, c_i \in \mathbb{R},\ i = 1, 2, \ldots, N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2.2(see [6]) Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N}, \text{ for some } c_i \in \mathbb{R},\ i = 1, 2, \ldots, N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2.3 Let α, ξ, a be as in (1) and $y \in C[0, 1]$. Then solving

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, t \in (0, 1), \\ u(0) = u'(0) = 0, u'(1) = au'(\xi), \end{cases} \tag{6}$$

is equivalent to solving

$$u(t) = \int_0^1 G(t, s) y(s) \mathrm{d}s,$$

where

$$\begin{aligned} G(t, s) &= g_1(t, s) + \frac{a t^{\alpha-1}}{1 - a\xi^{\alpha-1}} g_2(\xi, s), \\ g_1(t, s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \le t \le s \le 1, \end{cases} \\ g_2(t, s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & 0 \le s \le t \le 1, \\ t^{\alpha-2}(1-s)^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{7}$$

Proof. It is enough to consider the case when u is a solution of (2.1). From Definition 2.2 and Lemma 2.2 we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \mathrm{d}s,$$

for some constants $c_i \in \mathbb{R}, i = 1, 2, 3$.

From $u(0) = u'(0) = 0$ we have $c_2 = c_3 = 0$. Hence

$$u(t) = c_1 t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s,$$

and

$$u'(t) = c_1(\alpha-1)t^{\alpha-2} - (\alpha-1)\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s.$$

Consequently, we obtain

$$u'(1) = c_1(\alpha-1) - (\alpha-1)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s,$$

and

$$u'(\xi) = c_1(\alpha-1)\xi^{\alpha-2} - (\alpha-1)\int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s.$$

Then $u'(1) = au'(\xi)$ implies that

$$c_1 - \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s = c_1 a\xi^{\alpha-2} - a\int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s,$$

and

$$c_1 = \frac{1}{1-a\xi^{\alpha-2}}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \frac{a}{1-a\xi^{\alpha-2}}\int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s.$$

As a result,

$$\begin{aligned}
u(t) &= \frac{1}{1-a\xi^{\alpha-2}}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \frac{at^{\alpha-1}}{1-a\xi^{\alpha-2}}\int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s \\
&= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + \frac{at^{\alpha-1}}{1-a\xi^{\alpha-2}}\int_0^1 \frac{\xi^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s \\
&\quad - \frac{at^{\alpha-1}}{1-a\xi^{\alpha-2}}\int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)\mathrm{d}s \\
&= \int_0^1 G(t,s)y(s)\mathrm{d}s.
\end{aligned}$$

This completes the proof. □

Lemma 2.4 Let $\alpha, \beta, \xi, \eta, a, b$ be as in (1) and $y \in C[0,1]$. Then solving

$$\begin{cases} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = y(t), t \in (0,1), \\ u(0) = u'(0) = 0, u'(1) = au'(\xi), D_{0+}^{\alpha}u(0) = 0, D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta), \end{cases} \tag{8}$$

is equivalent to solving

$$u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)y(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

where G is defined in (7) and

$$\begin{aligned} H(t,s) &= h_1(t,s) + \frac{b^{p-1}t^{\beta-1}}{1-b^{p-1}\eta^{\beta-1}} h_1(\eta,s), \\ h_1(t,s) &= \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{9}$$

Proof. It is enough to consider the case when u is a solution of (2.3). From Lemma 2.2 we have

$$I_{0+}^{\beta} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = \phi_p(D_{0+}^{\alpha}u(t)) + c_1 t^{\beta-1} + c_2 t^{\beta-2},$$

for some constants $c_i \in \mathbb{R}, i = 1, 2$. In view of (8), we obtain

$$I_{0+}^{\beta} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = I_{0+}^{\beta} y(t).$$

Also we find

$$\begin{aligned} \phi_p(D_{0+}^{\alpha}u(t)) &= I_{0+}^{\beta} y(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} \\ &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + c_1 t^{\beta-1} + c_2 t^{\beta-2}. \end{aligned}$$

Then $D_{0+}^{\alpha}u(0) = 0$ implies that $c_2 = 0$. Hence,

$$\phi_p(D_{0+}^{\alpha}u(1)) = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + c_1,$$

and

$$\phi_p(D_{0+}^{\alpha}u(\eta)) = \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + c_1 \eta^{\beta-1}.$$

Consequently, $D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta)$ implies that

$$\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + c_1 = b^{p-1} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + c_1 b^{p-1}\eta^{\beta-1},$$

and

$$c_1 = \frac{b^{p-1}}{1-b^{p-1}\eta^{\beta-1}} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s - \frac{1}{1-b^{p-1}\eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s.$$

Therefore,

$$\begin{aligned} \phi_p(D_{0+}^{\alpha}u(t)) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + \frac{b^{p-1}t^{\beta-1}}{1-b^{p-1}\eta^{\beta-1}} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s \\ &\quad - \frac{1}{1-b^{p-1}\eta^{\beta-1}} \int_0^1 \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s \\ &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s - \int_0^1 \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s + \frac{b^{p-1}t^{\beta-1}}{1-b^{p-1}\eta^{\beta-1}} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s \\ &\quad - \frac{b^{p-1}t^{\beta-1}}{1-b^{p-1}\eta^{\beta-1}} \int_0^1 \frac{\eta^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\mathrm{d}s \\ &= -\int_0^1 H(t,s)y(s)\mathrm{d}s. \end{aligned}$$

Also we have

$$D_{0+}^{\alpha}u(t)+\phi_q\left(\int_0^1 H(t,s)y(s)\mathrm{d}s\right)=0.$$

Note Lemma 2.3 and the boundary conditions $u(0)=u'(0)=0, u'(1)=au'(\xi)$, so we have

$$u(t)=\int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)y(\tau)\mathrm{d}\tau\right)\mathrm{d}s.$$

This completes the proof. □

Lemma 2.5 The functions G,H have the following properties:

(i) $G,H\in C([0,1]\times[0,1],[0,+\infty))$ and $G(t,s),H(t,s)>0$ for $t,s\in(0,1)$,

(ii) $G(t,s)\le\delta_1 t^{\alpha-1}$ for $t,s\in[0,1]$, where $\delta_1:=\frac{1}{\Gamma(\alpha)}\left[1+\frac{a\xi^{\alpha-3}}{1-a\xi^{\alpha-1}}\right]>0$.

(iii) $\delta_2 t^{\alpha-1}s(1-s)^{\alpha-2}\le G(t,s)\le\delta_1 s(1-s)^{\alpha-2}$ for $t,s\in[0,1]$, where $\delta_2:=\frac{a(\alpha-2)\xi^{\alpha-2}(1-\xi)}{\Gamma(\alpha)(1-a\xi^{\alpha-1})}$.

Proof. From [7–10] we have $g_1,g_2,h_1\in C([0,1]\times[0,1],[0,+\infty))$ and $g_1(t,s),h_1(t,s)>0$ for $t,s\in(0,1)$, so G,H have these properties.

From [10, Lemma 4] we have

$$g_1(t,s)\le\frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-2},\ \ g_1(t,s)\le\frac{1}{\Gamma(\alpha)}t^{\alpha-1},\ \text{for } t,s\in[0,1],$$

and

$$\frac{(\alpha-2)t^{\alpha-2}(1-t)s(1-s)^{\alpha-2}}{\Gamma(\alpha)}\le g_2(t,s)\le\frac{1}{\Gamma(\alpha)}t^{\alpha-3}s(1-s)^{\alpha-2}\le\frac{1}{\Gamma(\alpha)}t^{\alpha-3},\ \text{for } t,s\in[0,1].$$

Consequently,

$$\begin{aligned}
G(t,s)&=g_1(t,s)+\frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}}g_2(\xi,s)\le\frac{1}{\Gamma(\alpha)}\left[1+\frac{a\xi^{\alpha-3}}{1-a\xi^{\alpha-1}}\right]t^{\alpha-1},\\
G(t,s)&=g_1(t,s)+\frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}}g_2(\xi,s)\le\frac{1}{\Gamma(\alpha)}\left[1+\frac{a\xi^{\alpha-3}}{1-a\xi^{\alpha-1}}\right]s(1-s)^{\alpha-2},\\
G(t,s)&=g_1(t,s)+\frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}}g_2(\xi,s)\ge\frac{a(\alpha-2)\xi^{\alpha-2}(1-\xi)}{\Gamma(\alpha)(1-a\xi^{\alpha-1})}t^{\alpha-1}s(1-s)^{\alpha-2}.
\end{aligned}$$

This completes the proof. □

Let $E:=C[0,1]$, $\|u\|:=\max_{t\in[0,1]}|u(t)|$, $P:=\{u\in E: u(t)\ge 0,\forall t\in[0,1]\}$. Then $(E,\|\cdot\|)$ is a real Banach space and P is a cone on E. We let $B_\rho:=\{u\in E:\|u\|<\rho\}$ for $\rho>0$ in the sequel. Define $A:P\to P$ by

$$(Au)(t)=\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s.$$

Then, by Lemma 2.4 the existence of solutions for (1) is equivalent to the existence of fixed points for the operator A. Furthermore, in view of the continuity G,H and f, we can use the Ascoli-Arzela theorem to show that A is a completely continuous operator.

Lemma 2.6 Let $P_0:=\{u\in P:\min_{t\in[\theta_1,\theta_2]}u(t)\ge\frac{\delta_2\theta_1^{\alpha-1}}{\delta_1}\|u\|\}$, where $0<\theta_1<\theta_2\le 1$. Then $A(P)\subset P_0$.

Proof. For any $u\in P$, from (iii) of Lemma 2.5 we have

$$\begin{aligned}
(Au)(t)&=\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\\
&\le\delta_1\int_0^1 s(1-s)^{\alpha-2}\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s.
\end{aligned}$$

Also for $t \in [\theta_1, \theta_2]$, we obtain

$$\begin{aligned}(Au)(t) &= \int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \\ &\geq \int_0^1 \delta_2 t^{\alpha-1}s(1-s)^{\alpha-2}\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \\ &\geq \delta_2\theta_1^{\alpha-1}\int_0^1 s(1-s)^{\alpha-2}\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s.\end{aligned}$$

Consequently,

$$(Au)(t) \geq \frac{\delta_2\theta_1^{\alpha-1}}{\delta_1}\delta_1\int_0^1 s(1-s)^{\alpha-2}\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \geq \frac{\delta_2\theta_1^{\alpha-1}}{\delta_1}\|Au\|.$$

This completes the proof. □

Lemma 2.7(see [13, Lemma 2.6]) Let $\theta > 0$ and $\varphi \in P$. Then

$$\left(\int_0^1 \varphi(t)\mathrm{d}t\right)^{\theta} \leq \int_0^1 (\varphi(t))^{\theta}\mathrm{d}t,\ \forall\theta \geq 1,\quad \left(\int_0^1 \varphi(t)\mathrm{d}t\right)^{\theta} \geq \int_0^1 (\varphi(t))^{\theta}\mathrm{d}t,\ \forall 0 < \theta \leq 1.$$

Lemma 2.8(see [14]) Let $R > 0$ and $A : \overline{B}_R \cap P \to P$ a continuous compact operator. If there exists $u_0 \in P \setminus \{0\}$ such that $u - Au \neq \mu u_0$ for all $\mu \geq 0$ and $u \in \partial B_R \cap P$, then $i(A, B_R \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 2.9(see [14]) Let $r > 0$ and $A : \overline{B}_r \cap P \to P$ a continuous compact operator. If $\|Au\| \leq \|u\|$ and $Au \neq u$ for $u \in \partial B_r \cap P$, then $i(A, B_r \cap P, P) = 1$.

Let $p_* = \min\{p-1, 1\}$, $p^* = \max\{p-1, 1\}$, $\gamma(t) = t^{\alpha-1}$ for $t \in [0,1]$, and $t_0 \in (0,1)$ is a given point. For convenience, we put

$$\kappa_1 := 2^{\frac{p^*}{p-1}-1}\int_0^1\int_0^1 H^{\frac{p^*}{p-1}}(s,\tau)\gamma^{p^*}(\tau)\mathrm{d}\tau\mathrm{d}s,\ \kappa_2 := 2^{\frac{p^*}{p-1}-1}\int_0^1\int_0^1 H^{\frac{p^*}{p-1}}(s,\tau)\mathrm{d}\tau\mathrm{d}s.$$

$$\lambda_1 := \frac{1}{\delta_1\sqrt[p^*]{\kappa_1}},\quad \lambda_2 = \sqrt[p_*]{\frac{2}{\int_0^1 G^{p_*}(t_0,s)\int_{\theta_1}^{\theta_2} H^{\frac{p_*}{p-1}}(s,\tau)\mathrm{d}\tau\mathrm{d}s}\frac{\delta_1}{\delta_2\theta_1^{\alpha-1}}},$$

and

$$\lambda_3 := \frac{1}{\left(\delta_1\int_0^1 s(1-s)^{\alpha-2}\left(\int_0^1 H(s,\tau)\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\right)^{p-1}}.$$

We now list our hypotheses:

(H1) $f(t,u) \in C([0,1]\times[0,+\infty), [0,+\infty))$.

(H2) $f(t,u)$ is nondecreasing with respect to u and $f(t,0) \not\equiv 0$ for $t \in [0,1]$.

(H3) $\limsup_{u\to+\infty}\frac{f(t,u)}{u^{p-1}} < \lambda_1^{p-1}$ uniformly on $t \in [0,1]$.

(H4) $\liminf_{u\to+\infty}\frac{f(t,u)}{u^{p-1}} > \lambda_2^{p-1}$ uniformly on $t \in [\theta_1, \theta_2]$.

(H5) there exists $\zeta > 0$ such that $f(t,u) \leq \phi_p(\zeta)\lambda_3$, $\forall 0 \leq u \leq \zeta, t \in [0,1]$.

Example 2.10 (1) Let

$$f(t,u) = e^t + \sum_{i=1}^n m_i u^{\frac{p-1}{i}} \text{ for } t \in [0,1] \text{ and } u \in \mathbb{R}^+,$$

where $m_1 \in (0, \lambda_1^{p-1})$, $m_i \geq 0$ for $i = 2, 3, \ldots, n$.

Let $p = 2$, $\alpha = 2.5$, $\beta = 1.5$, $\xi = 0.5$, $a = 1$ and $b = 0$. Note,

$$\delta_1 = \frac{4}{3\sqrt{\pi}}\frac{2\sqrt{2}+3}{2\sqrt{2}-1},\ \kappa_1 = \int_0^1\int_0^1 H(s,\tau)\gamma(\tau)\mathrm{d}\tau\mathrm{d}s = \frac{5\sqrt{\pi}}{96},$$

and $\lambda_1 \approx 4.5$. Let $m_1 \in (0, 4.5)$. Note (H1)-(H3) hold.

(2) Let $\zeta = 1$. Then $\phi_p(\zeta) = 1$. Let

$$f(t,u) = \sum_{i=1}^{n} m_i u^{i(p-1)} \text{ for } t \in [0,1] \text{ and } u \in \mathbb{R}^+,$$

where m_i are nonnegative numbers such that $\sum_{i=1}^{n} m_i \leq \lambda_3$.

Using the above values for $p, \alpha, \beta, \xi, a, b$, we have

$$\lambda_3 = \left(\delta_1\int_0^1 s(1-s)^{\alpha-2}\int_0^1 H(s,\tau)\mathrm{d}\tau\mathrm{d}s\right)^{-1} = \frac{\beta\Gamma(\beta)}{\delta_1}\left[\frac{\Gamma(\beta+1)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)} - \frac{\Gamma(\beta+2)\Gamma(\alpha-1)}{\Gamma(\alpha+\beta+1)}\right]^{-1} \approx 7.5.$$

Let $\sum_{i=1}^{n} m_i \leq 7.5$. Note (H1), (H4) and (H5) hold.

3. Main Results

Theorem 3.1 Suppose that (H1)-(H3) are satisfied. Then (1) has at least a positive solution u^*. Moreover, there exists a monotone non-increasing sequence $\{u_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} u_n = u^*$, where $u_0(t) = M\gamma(t),\ t \in [0,1]$, ($M$ is defined in the proof), and $u_{n+1} = Au_n$ for $n = 0, 1, 2, \ldots$.

Proof. From (H3) there exist $\varepsilon_1 \in (0, \lambda_1)$ and $c_1 > 0$ such that

$$f(t,u) \leq (\lambda_1 - \varepsilon_1)^{p-1}u^{p-1} + c_1,\ \forall u \in [0,+\infty), t \in [0,1]. \tag{10}$$

Take $M \geq c_1^{\frac{1}{p-1}}\varepsilon_1^{-1}\sqrt[p^*]{\frac{\kappa_2}{\kappa_1}}$, where ε_1, c_1 are defined in (10) and let $u_0 = M\gamma$. Hence,

$$\begin{aligned}
[(AM\gamma(t))(t)]^{p^*} &= \left[\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,M\gamma(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\right]^{p^*} \\
&\leq \left[\int_0^1 \delta_1\gamma(t)\left(\int_0^1 H(s,\tau)f(\tau,M\gamma(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\right]^{p^*} \\
&\leq \delta_1^{p^*}[\gamma(t)]^{p^*}\int_0^1\left(\int_0^1 H(s,\tau)f(\tau,M\gamma(\tau))\mathrm{d}\tau\right)^{\frac{p^*}{p-1}}\mathrm{d}s \\
&\leq \delta_1^{p^*}[\gamma(t)]^{p^*}\int_0^1\int_0^1 H^{\frac{p^*}{p-1}}(s,\tau)\left[(\lambda_1-\varepsilon_1)^{p-1}(M\gamma(\tau))^{p-1}+c_1\right]^{\frac{p^*}{p-1}}\mathrm{d}\tau\mathrm{d}s \\
&\leq 2^{\frac{p^*}{p-1}-1}\delta_1^{p^*}[\gamma(t)]^{p^*}\int_0^1\int_0^1 H^{\frac{p^*}{p-1}}(s,\tau)\left[(\lambda_1-\varepsilon_1)^{p^*}(M\gamma(\tau))^{p^*}+c_1^{\frac{p^*}{p-1}}\right]\mathrm{d}\tau\mathrm{d}s \\
&= \delta_1^{p^*}(\lambda_1-\varepsilon_1)^{p^*}M^{p^*}[\gamma(t)]^{p^*}\kappa_1 + c_1^{\frac{p^*}{p-1}}\delta_1^{p^*}[\gamma(t)]^{p^*}\kappa_2.
\end{aligned}$$

Then we have

$$\begin{aligned}
(AM\gamma(t))(t) &\leq \left[\delta_1^{p^*}(\lambda_1-\varepsilon_1)^{p^*}M^{p^*}[\gamma(t)]^{p^*}\kappa_1 + c_1^{\frac{p^*}{p-1}}\delta_1^{p^*}[\gamma(t)]^{p^*}\kappa_2\right]^{\frac{1}{p^*}} \\
&\leq \left(\delta_1(\lambda_1-\varepsilon_1)M\sqrt[p^*]{\kappa_1} + c_1^{\frac{1}{p-1}}\delta_1\sqrt[p^*]{\kappa_2}\right)\gamma(t) \\
&\leq M\gamma(t).
\end{aligned}$$

This implies that

$$u_1 = Au_0 \leq u_0.$$

Also we have from (H2),

$$\begin{aligned} u_2(t) = (Au_1)(t) &= \int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,u_1(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \\ &\leq \int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,u_0(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \\ &= (Au_0)(t) = u_1(t). \end{aligned}$$

By induction, $u_{n+1} \leq u_n, n = 0,1,2,\ldots$. Also $0 \leq u_n(t) \leq M\gamma(t) \leq M$ for $t \in [0,1]$ and $n = 0,1,2,\ldots$. From the monotone bounded theorem we can take the limit as $n \to \infty$ in $u_{n+1} = Au_n$ and we obtain $u^* = Au^*$. Furthermore, because the zero function is not a solution of the problem (1), u^* is a positive solution for (1). This completes the proof. □

Theorem 3.2 Suppose that (H1), (H4) and (H5) are satisfied. Then (1) has at least a positive solution.

Proof. From (H4) there exist $\varepsilon_2 > 0$ and $c_2 > 0$ such that

$$f(t,u) \geq (\lambda_2 + \varepsilon_2)^{p-1}u^{p-1} - c_2, \forall u \in [0,+\infty), t \in [\theta_1,\theta_2]. \tag{11}$$

From (11) we have

$$(\lambda_2 + \varepsilon_2)^{p_*}u^{p_*} = ((\lambda_2 + \varepsilon_2)^{p-1}u^{p-1})^{\frac{p_*}{p-1}} \leq (f(t,u) + c_2)^{\frac{p_*}{p-1}} \leq f^{\frac{p_*}{p-1}}(t,u) + c_2^{\frac{p_*}{p-1}}.$$

Hence,

$$f^{\frac{p_*}{p-1}}(t,u) \geq (\lambda_2 + \varepsilon_2)^{p_*}u^{p_*} - c_2^{\frac{p_*}{p-1}}. \tag{12}$$

In what follows, we shall show that there exists a large positive number $R > \zeta$(ζ is defined in (H5)) such that

$$u - Au \neq \mu u_0 \text{ for all } \mu \geq 0 \text{ and } u \in \partial B_R \cap P, \tag{13}$$

where u_0 is a fixed element in P_0. If not, there exist $\mu \geq 0$ and $u \in \partial B_R \cap P$ such that $u - Au = \mu u_0$, i.e., $u(t) = (Au)(t) + \mu u_0(t)$ for $t \in [0,1]$. Hence $\|u\| = \|Au + \mu u_0\| \geq \|Au\|$. Moreover, note that if $u \in P$, by Lemma 2.6 we have $Au + \mu u_0 \in P_0$ and also $u \in P_0$.

Consequently, from (12), for a fixed point $t_0 \in (0,1)$, we have

$$\begin{aligned} [(Au)(t_0)]^{p_*} &= \left[\int_0^1 G(t_0,s)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s\right]^{p_*} \\ &\geq \int_0^1 G^{p_*}(t_0,s)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{p_*}{p-1}} \mathrm{d}s \\ &\geq \int_0^1 G^{p_*}(t_0,s)\int_0^1 H^{\frac{p_*}{p-1}}(s,\tau)f^{\frac{p_*}{p-1}}(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s \\ &\geq \int_0^1 G^{p_*}(t_0,s)\int_{\theta_1}^{\theta_2} H^{\frac{p_*}{p-1}}(s,\tau)\left[(\lambda_2 + \varepsilon_2)^{p_*}u^{p_*} - c_2^{\frac{p_*}{p-1}}\right]\mathrm{d}\tau\mathrm{d}s \\ &\geq \left[(\lambda_2 + \varepsilon_2)^{p_*}\left(\frac{\delta_2\theta_1^{\alpha-1}}{\delta_1}\right)^{p_*} R^{p_*}\right]\int_0^1 G^{p_*}(t_0,s)\int_{\theta_1}^{\theta_2} H^{\frac{p_*}{p-1}}(s,\tau)\mathrm{d}\tau\mathrm{d}s - c_3, \end{aligned}$$

where $c_3 = c_2^{\frac{p_*}{p-1}} \int_0^1 G^{p_*}(t_0, s) \int_{\theta_1}^{\theta_2} H^{\frac{p_*}{p-1}}(s,\tau)\mathrm{d}\tau\mathrm{d}s$. Therefore, if R is large enough we have

$$\begin{aligned}\|Au\|^{p_*} \geq [(Au)(t_0)]^{p_*} &> \lambda_2^{p_*}\left(\frac{\delta_2\theta_1^{\alpha-1}}{\delta_1}\right)^{p_*} R^{p_*} \int_0^1 G^{p_*}(t_0,s)\int_{\theta_1}^{\theta_2} H^{\frac{p_*}{p-1}}(s,\tau)\mathrm{d}\tau\mathrm{d}s - c_3\\ &= 2R^{p_*} - c_3 \geq R^{p_*} = \|u\|^{p_*},\end{aligned}$$

i.e., $\|Au\| > \|u\|$, and this contradicts $\|u\| \geq \|Au\|$. Thus (13) holds true and Lemma 2.8 yields

$$i(A, B_R \cap P, P) = 0. \tag{14}$$

From (H5) for $u \in \partial B_\zeta \cap P$ we have

$$\begin{aligned}\|Au\| = \max_{t\in[0,1]}(Au)(t) &= \max_{t\in[0,1]}\int_0^1 G(t,s)\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\\ &\leq \zeta\lambda_3^{\frac{1}{p-1}}\delta_1\int_0^1 s(1-s)^{\alpha-2}\left(\int_0^1 H(s,\tau)\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\\ &= \zeta.\end{aligned}$$

Hence, $\|Au\| \leq \|u\|$, for $u \in \partial B_\zeta \cap P$, and Lemma 2.9 implies that

$$i(A, B_\zeta \cap P, P) = 1. \tag{15}$$

Combining (14) and (15) gives

$$i(A, (B_R \setminus \overline{B}_\zeta) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_\zeta \cap P, P) = -1. \tag{16}$$

Consequently the operator A has at least one fixed point on $(B_R \setminus \overline{B}_\zeta) \cap P$, and hence (1) has at least one positive solution. This completes the proof. □

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