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Generalized Asymptotically $\mathcal{I}$-Lacunary Statistical Equivalent of Order α for Sequences of Sets

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Abstract. In this paper we present, for sequences of sets, the concept of generalized asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α, $(0 < \alpha \leq 1)$, to multiple L, where $\mathcal{I}$ is an ideal of the subset of $\mathbb{N}$. In addition to this definition, inclusion theorems are also presented. The study leaves some interesting open problems.

1. Introduction

Before continuing with this paper we present some definitions and preliminaries.
The concept of $\mathcal{I}-$convergence was introduced by Kostyrko et al. in a metric space [7]. Later it was further studied by Dems [3], Savas ([2], [12],[15][16],[17], [18] [19]) and many others. $\mathcal{I}-$convergence is a generalization form of statistical convergence, which was introduced by Fast (see [4]) and that is based on the notion of an ideal of the subset of positive integers $\mathbb{N}$.

The following definitions and notions will be needed.

Definition 1.1. *A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$,

Definition 1.2. *A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:*
(a) $\phi \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F$, $A \subset B$ implies $B \in F$,

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 1.3. *A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.*

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Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 1.4. *A real sequence $x = (x_k)$ is said to be $\mathcal{I}-$ convergent to $L \in \mathbb{R}$ if and only if for each $\varepsilon > 0$ the set*

$$A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. The number L is called the $\mathcal{I}-$limit of the sequence x, (see, [7]).

Remark 1.5. *If we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbf{N} : A \text{ is a finite subset }\}$, then the corresponding convergence coincides with the usual convergence.*

In 1993, Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices.

Definition 1.6 ([8]). *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

and it is denoted by $x \sim y$.

n 2003, Patterson defined asymptotically statistical equivalent sequences by combining definitions 1.5 and 1.6 as follows:

Definition 1.7 (Patterson, [10]). *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n} \left\{ \text{thenumberof } k < n : \left| \frac{x_k}{y_k} \right| - L \right\} \geq= 0$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent if $L = 1$.

More investigations in this direction and more applications of asymptotically statistical equivalent can be found in [11, 13, 14] where many important references can be found.
Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Definition 1.8 (see, [1]). *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if*

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

In [9], statistical convergence of sequences of sets was given by Nuray and Rhoades as follows:

Definition 1.9. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

In another direction, a new type of convergence called Wijsman lacunary statistical convergence was introduced as follows.

Definition 1.10 (Ulusu & Nuray, [21]). *Let (X,ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A if $\{d(x,A_k)\}$ is lacunary statistically convergent to $d(x,A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim_W = A$ or $A_k \to A(WS_\theta)$.

Definition 1.11 ([6]). *Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$−convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$ the set, $A(x,\varepsilon) = \{k \in \mathbb{N} : |d(x,A_k) - d(x,A)| \geq \varepsilon\}$ belongs to $\mathcal{I}$. In this case we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$.*

Remark 1.12. *If we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}$, then the corresponding convergence coincides with the usual Wijsman convergence.*

Definition 1.13 ([6]). *Let (X,ρ) be a metric space and θ be lacunary sequence. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$−lacunary statistical convergent to A or $S_\theta(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $A_k \to A\left(S_\theta\left(\mathcal{I}_W\right)\right)$.

Quite recently, Ulusu and Savas [23] gave an extension on asymptotically lacunary statistical equivalent set sequences and they investigated some relations between strongly asymptotically lacunary equivalent set sequences and strongly Cesàro asymptotically equivalent set sequences.

Definition 1.14. *Let (X,ρ) be a metric space, θ be lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly Wijsman asymptotically $\mathcal{I}$− lacunary equivalent of multiple L if for each $\varepsilon > 0$ and each $x \in X$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $\{A_k\} \overset{N_\theta^{L(p)}[\mathcal{I}_W]}{\sim} \{B_k\}$ and simply strongly Wijsman asymptotically $\mathcal{I}$− lacunary equivalent if $L = 1$.

Remark 1.15. *If we take $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}$,, then the strongly Wijsman asymptotically $\mathcal{I}$− lacunary equivalent coincides with the strongly Wijsman asymptotically lacunary equivalent,(see,[23]) .*

In this paper, we shall generalize the above definition to order α and also some inclusion theorems are proved.

2. Main Results

In this section we shall give some new definitions and also examine some inclusion relations.

Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$, we define $d(x;A_k,B_k)$ as follows:

$$d(x;A_k,B_k) = \begin{cases} \dfrac{d(x,A_k)}{d(x,B_k)} & , \quad x \notin A_k \cup B_k \\ \\ L & , \quad x \in A_k \cup B_k. \end{cases}$$

Definition 2.1. *Let (X,ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α, where $0 < \alpha \leq 1$, to multiple L provided that for any $\epsilon > 0$ and $\delta > 0$*

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x;A_k,B_k) - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I},$$

(denoted by $\{A_k\} \overset{S_\theta^L(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\}$) and simply asymptotically $\mathcal{I}_\mathcal{W}$-lacunary statistical equivalent of order α if $L = 1$. Furthermore, let $S_\theta^L(\mathcal{I}_\mathcal{W})^\alpha$ denote the set of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{S_\theta^L(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\}$.

Remark 2.2. *For $\mathcal{I} = \mathcal{I}_{fin}$, Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α coincides with the Wijsman asymptotically lacunary statistical equivalent of order α. For an arbitrary ideal I and for $\alpha = 1$ it coincides with Wijsman asymptotically lacunary statistical equivalent.*

When $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes only Wijsman asymptotically lacunary statistical equivalent, (see, [22]).

Definition 2.3. *Let (X,ρ) be a metric space, θ be lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly Wijsman asymptotically $\mathcal{I}-$ lacunary equivalent of order α, where $0 < \alpha \leq 1$, to multiple L provided that for each $\varepsilon > 0$ and for each $x \in X$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x;A_k,B_k) - L|^{p_k} \geq \varepsilon \right\} \in \mathcal{I}.$$

In this situation we write $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I})^\alpha}{\sim} \{B_k\}$.

If we take $p_k = p$ for all $k \in \mathbb{N}$ we write $\{A_k\} \overset{N_\theta^{Lp}(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\}$ instead of $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\}$. For $\alpha = 1$ the above definition coincides with strongly Wijsman asymptotically $\mathcal{I}-$ lacunary equivalent of multiple L. Further, let $N_\theta^{L(p)}(\mathcal{I}_\mathcal{W})^\alpha$ denote the set of A_k and B_k such that $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\}$.

Definition 2.4. *Let (X,ρ) be a metric space and $p = (p_k)$ be a sequence of positive real numbers. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically $\mathcal{I}-$ Cesáro equivalent of order α, where $0 < \alpha \leq 1$, to multiple L, if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^{n} |d(x;A_k,B_k) - L|^{p_k} \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $\{A_k\} \overset{\sigma^{(p)}(\mathcal{I})^\alpha}{\sim} \{B_k\}$ and simply strongly Wijsman asymptotically $\mathcal{I}-$ Cesáro equivalent of order α if $L = 1$. For $\alpha = 1$ the above definition coincides with strongly Wijsman asymptotically $\mathcal{I}-$ Cesáro equivalent.

The first set of theorems address some standard questions from summability theory. First we establish a relationship between strong summability and statistical convergence.

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence. Then, If $\{A_k\} \overset{N_\theta^{Lp}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ then $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.*

Proof. Let $\{A_k\} \overset{N_\theta^{Lp}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ and $\varepsilon > 0$ is given. Then,

$$\begin{aligned}\sum_{k\in I_r} |d(x;A_k,B_k) - L|^p &\geq \sum_{k\in I_r \;\&\; |d(x;A_k,B_k)-L|\geq\varepsilon} |d(x;A_k,B_k) - L|^p \\ &\geq \varepsilon^p \left|\{k \in I_r : |d(x;A_k,B_k) - L| \geq \varepsilon\}\right|\end{aligned}$$

and so

$$\frac{1}{\varepsilon^p h_r^\alpha} \sum_{k\in I_r} |d(x;A_k,B_k) - L|^p \geq \frac{1}{h_r^\alpha} \left|\{k \in I_r : |d(x;A_k,B_k) - L| \geq \varepsilon\}\right|$$

Then for any $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\{k \in I_r : |d(x;A_k,B_k) - L| \geq \varepsilon\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k\in I_r} |d(x;A_k,B_k) - L|^p \geq \varepsilon^p\delta\right\} \in \mathcal{I}.$$

Therefore $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$. □

Remark 2.6. *The following condition remain true for $0 < \alpha < 1$ is not clear and we leave it as open problem.*

(1) $\{A_k\} \in l_\infty$, the set of the bounded sequences , and $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

The proof of the following theorem is obtained by using the standard techniques, therefore is omitted.

Theorem 2.7. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $N_\theta^{L(p)}(\mathcal{I}_W)^\alpha \subseteq N_\theta^{L(p)}(\mathcal{I}_W)^\beta$.*

Corollary 2.8. *Let $0 < \alpha \leq 1$ be a positive real number. Then $N_\theta^{L(p)}(\mathcal{I}_W)^\alpha \subseteq N_\theta^{L(p)}(\mathcal{I}_W)$ for each $\alpha \in (0,1]$.*

From the next theorem it will follow at once that strongly Wijsman asymptotically $\mathcal{I}-$ lacunary equivalent of order α implies Wijsman asymptotically $\mathcal{I}-$ lacunary statistical equivalent of order α.

Theorem 2.9. *Let $\theta = (k_r)$ is a lacunary sequence, $\inf_k p_k = h$ and $\sup_k p_k = H$. Then,*

$$\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\} \text{ implies } \{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}.$$

Proof. Assume that $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ and $\varepsilon > 0$. Then,

$$\begin{aligned}\frac{1}{h_r^\alpha} \sum_{k\in I_r} |d(x;A_k,B_k) - L|^{p_k} &= \frac{1}{h_r^\alpha} \sum_{k\in I_r \;\&\; |d(x;A_k,B_k)-L|\geq\varepsilon} |d(x;A_k,B_k)|^{p_k} \\ &+ \frac{1}{h_r^\alpha} \sum_{k\in I_r \;\&\; |d(x;A_k,B_k)-L|<\varepsilon} |d(x;A_k,B_k) - L|^{p_k} \\ &\geq \frac{1}{h_r^\alpha} \sum_{k\in I_r \;\&\; |d(x;A_k,B_k)-L|\geq\varepsilon} |d(x;A_k,B_k) - L|^{p_k} \\ &\geq \frac{1}{h_r^\alpha} \sum_{k\in I_r \;\&\; |d(x;A_k,B_k)-L|\geq\varepsilon} \varepsilon^{p_k} \\ &\geq \frac{1}{h_r^\alpha} \sum_{k\in I_r \;\&\; |d(x;A_k,B_k)-L|\geq\varepsilon} \min\left\{\varepsilon^h, \varepsilon^H\right\} \\ &\geq \frac{1}{h_r^\alpha} \min\left\{\varepsilon^h, \varepsilon^H\right\} \left|\{k \in I_r : |d(x;A_k,B_k) - L| \geq \varepsilon\}\right|\end{aligned}$$

and

$$\left\{ r \in : \frac{1}{h_r^{\alpha}} \left| \{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\} \right| \geq \delta \right\} \subseteq \left\{ r \in : \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} \geq \delta \min\left\{\varepsilon^h, \varepsilon^H\right\} \right\} \in I.$$

Thus we have $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)}{\sim} \{B_k\}$. □

We now investigate the relationship between $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I})^\alpha}{\sim} \{B_k\}$ and $\{A_k\} \overset{\sigma^{(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$. First we offer a criterion for $\{A_k\} \overset{\sigma^{(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ to imply $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

Theorem 2.10. *Let $\mathcal{I}$ be an ideal and $\theta = \{k_r\}$ be a lacunary sequence, then*

$$\{A_k\} \overset{\sigma^{(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\} \text{ implies } \{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}.$$

if $\liminf_r q_r^\alpha > 1$.

Proof. Suppose first that $\liminf_r q_r^\alpha > 1$. Then there exists $\sigma > 0$ such that $q_r^\alpha \geq 1 + \sigma$ for sufficiently large r which implies that

$$\frac{h_r^\alpha}{k_r^\alpha} \geq \frac{\sigma}{1+\sigma}.$$

Let $\varepsilon > 0$ be given. Now observe that

$$\begin{aligned}
\frac{1}{k_r^\alpha} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L|^{p_k} &\geq \frac{1}{k_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} \\
&\geq \frac{h_r^\alpha}{k_r^\alpha} \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} \\
&\geq \left(\frac{\sigma}{1+\sigma}\right) \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k}.
\end{aligned}$$

Thus we have

$$\frac{1}{k_r^\alpha} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon$$

implies

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon\left(\frac{1+\sigma}{\sigma}\right).$$

So we can conclude that

$$\left\{ r \in : \frac{1}{k_r^\alpha} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon \right\} \subset \left\{ r \in : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon\left(\frac{1+\sigma}{\sigma}\right) \right\}.$$

Finally, since the set defined in the first inclusion is in the filter $\mathcal{F}(\mathcal{I})$, then the set defined in the second inclusion is also in the filter. This proves the theorem. □

Remark 2.11. *The converse of this result is not clear for $\alpha < 1$ and we leave it as an open problem.*

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(I)$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(I)$.

Theorem 2.12. *For a lacunary sequence θ satisfying the above condition, we have*

$$\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\} \text{ implies } \{A_k\} \overset{\sigma^{(p)}(\mathcal{I}_\mathcal{W})^\alpha}{\sim} \{B_k\}$$

$$\text{if } \sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^\alpha}{(k_{r-1})^\alpha} = B(say) < \infty.$$

Proof. If $\limsup q_r < \infty$ then there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$. Let $\{A_k\} \overset{N_\theta^{L(p)}(\mathcal{I}_\mathcal{W})}{\sim} \{B_k\}$ and define the sets T and R such that,

$$T = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon_1 \right\}$$

and

$$R = \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^{n} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon_2 \right\}.$$

Let

$$A_j = \frac{1}{h_j^\alpha} \sum_{k \in I_j} |d(x; A_k, B_k) - L|^{p_k} < \varepsilon_1$$

for all $j \in T$. It is obvious that $T \in F(\mathcal{I})$. Choose n is any integer with $k_{r-1} < n < k_r$ where $r \in T$.

$$\begin{aligned}
&\tfrac{1}{n^\alpha} \sum_{k=1}^{n} |d(x; A_k, B_k - L|^{p_k} \leq \tfrac{1}{k_{r-1}^\alpha} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L|^{p_k} \\
&= \tfrac{1}{k_{r-1}^\alpha} \sum_{k \in I_1} |d(x; A_k, B_k) - L|^{p_k} + \sum_{k \in I_2} |d(x; A_k, B_k - L|^{p_k} + \ldots + \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k}) \\
&= \tfrac{k_1^\alpha}{k_{r-1}^\alpha} \left(\tfrac{1}{h_1^\alpha} \sum_{k \in I_1} |d(x; A_k, B_k) - L|^{p_k} \right) + \tfrac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} \left(\tfrac{1}{h_2^\alpha} \sum_{k \in I_2} |d(x; A_k, B_k - L|^{p_k} \right) + \ldots + \tfrac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \left(\tfrac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L|^{p_k} \right) \\
&= \tfrac{k_1^\alpha}{k_{r-1}^\alpha} A_1 + \tfrac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} A_2 + \ldots + \tfrac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} A_r \leq \sup_{j \in C} A_j . \sup_r \sum_{i=0}^{r-1} \tfrac{(k_{i+1} - k_i)^\alpha}{k_{r-1}^\alpha} < \varepsilon_1 B
\end{aligned}$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{B}$ and in view of the fact that $\bigcup \{n : k_{r-1} < n < k_r, r \in T\} \subset R$ where $T \in F(\mathcal{I})$ it follows from our assumption on θ that the set R also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. □

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