



Different Types of Quasi Weighted $\alpha\beta$ -Statistical Convergence in Probability

Pratulananda Das^a, Sanjoy Ghosal^b, Sumit Som^a

^aDepartment of Mathematics, Jadaoipur University, Kolkata-700032, West Bengal, India.

^bSchool of Sciences, Netaji Subhas Open University, Kalyani, Nadia-741235, West Bengal, India.

Abstract. The sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be weighted modulus $\alpha\beta$ -statistically convergent in probability to a random variable X [16] if for any $\varepsilon, \delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{T_{\alpha\beta(n)}} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}| = 0$$

where ϕ be a modulus function and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k \forall n \in \mathbb{N}$. In this paper we study a related concept of convergence in which the value $\frac{1}{T_{\alpha\beta(n)}}$ is replaced by $\frac{1}{C_n}$, for some sequence of real numbers $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n > 0 \forall n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} C_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} < \infty$ (like [30]). The results are applied to build the probability distribution for quasi-weighted modulus $\alpha\beta$ -statistical convergence in probability, quasi-weighted modulus $\alpha\beta$ -strongly Cesàro convergence in probability, quasi-weighted modulus $S_{\alpha\beta}$ -convergence in probability and quasi-weighted modulus $N_{\alpha\beta}$ -convergence in probability. If $\{C_n\}_{n \in \mathbb{N}}$ satisfying the condition $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} > 0$, then quasi-weighted modulus $\alpha\beta$ -statistical convergence in probability and weighted modulus $\alpha\beta$ -statistical convergence in probability are equivalent except the condition $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} = 0$. So our main objective is to interpret the above exceptional condition and produce a relational behavior of above mention four convergences.

1. Introduction

The concept of statistical convergence was introduced by Fast [11] and Steinhaus [37] and later on reintroduced by Schonenberg [35] independently and is based on the notion of asymptotic density of the subset of natural numbers. Later on it was further investigated from the sequence space point of view and

2010 Mathematics Subject Classification. 40A35, 40G15, 60B10

Keywords. Quasi-weighted $\alpha\beta$ -statistical convergence, quasi-weighted modulus $\alpha\beta$ -statistical convergence in probability, quasi-weighted modulus $\alpha\beta$ -strong Cesàro convergence in probability, quasi-weighted modulus $S_{\alpha\beta}$ -convergence in probability, quasi-weighted modulus $N_{\alpha\beta}$ -convergence in probability.

Received: 24 February 2015; Revised 10 November 2015; Accepted: 13 November 2015

Communicated by Eberhard Malkowsky

This work is funded by UGC Research, HRDG, India.

Email addresses: pratulananda@yahoo.co.in (Pratulananda Das), sanjoykrghosal@yahoo.co.in (Sanjoy Ghosal), somkardwip@gmail.com (Sumit Som)

linked with summability theorem by Šalát [31], Fridy [13], Connor [8], Fridy and Orhan [14]. In last few years, its several generalizations and applications has been given by many authors like:

- (i) statistical convergence of order α by Çolak [6] (statistical convergence of order α was also independently introduced by Bhunia et al. [4]),
- (ii) pointwise and uniform statistical convergence of order α by Cinar et al. [5],
- (iii) λ -statistical convergence was introduced by Mursaleen [27],
- (iv) λ -statistical convergence of order α by Çolak and Bektaş [7],
- (v) λ -statistical convergence of order α of sequences of functions by Et et al. [10],
- (vi) lacunary statistical convergence of order α by Sengül and Et [36],
- (vii) quasi statistical convergence by Sakaoğlu-Özgüc and Yurdakadim [30],
- (viii) weighted statistical convergence by Karakaya et al. [24, 25] and modified by Mursaleen et al. [28],
- (ix) weighted statistical convergence of order α by Ghosal [19],
- (x) weighted lacunary statistical convergence by Başarir and Konca [2],
- (xi) $\alpha\beta$ -statistical convergence of order γ by Aktuğlu [1],
- (xii) A -statistical limit points via ideals by Gürdal & Sari [21],
- (xiii) ideal convergence in random n -norms space by Gürdal [20, 22] and Gürdal & Huban [23],
- (xiv) lacunary sequence spaces defined by modulus functions by Savaş and Patterson [34] (for more results on this convergence see the paper [32]),
- (xv) statistical convergence in probability by Ghosal [15], and many other, in different fields of mathematics.

In another direction, the history of strong p -Cesàro summability, being longer, is not so clear. As per author's knowledge in [8], it has been shown that if a sequence is strongly p -Cesàro summable (for $0 < p < \infty$) to x , then the sequence must be statistically convergent to the same limit. Both the authors Fast [11] and Schonenberg [35] noted that if a bounded sequence is statistically convergent to x , then it is strongly Cesàro summable to x . In [12], the relation between strongly Cesàro summable and N_θ -convergence was established among other things.

In particular in probability theory, a new type of convergence called statistical convergence in probability was introduced in [15], as follows: Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space \mathcal{W} (for each n) with respect to a given class of events Δ and a given probability function $P : \Delta \rightarrow \mathbb{R}$. Then the sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to be statistically convergent in probability to a random variable $X : \mathcal{W} \rightarrow \mathbb{R}$ if for any $\varepsilon, \delta > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0$$

where the vertical bars denotes the cardinality of the enclosed set. In this case we write $X_n \xrightarrow{(S,P)} X$. The class of all sequences of random variables which are statistically convergent in probability is denoted by (S, P) . For more results on this convergence see the papers [9, 17, 18].

In this paper a new approach has been made to extend the application area by correlating quasi-statistical convergence, $\alpha\beta$ -statistical convergence and weighted-statistical convergence for more complete analysis of sequences of real numbers and to apply the theory of probability distributions. The results are applied to build the probability distribution for quasi-weighted modulus $\alpha\beta$ -statistical convergence in probability, quasi-weighted modulus $\alpha\beta$ -strong Cesàro convergence in probability, quasi-weighted modulus $S_{\alpha\beta}$ -convergence in probability, quasi-weighted modulus $N_{\alpha\beta}$ -convergence in probability have been introduced and the interrelations among them have been investigated. Also their certain basic properties have been studied. It is important to note that the methods of proofs and in particular the examples are not analogous to the real case.

The following definitions and notions will be needed in sequel.

Definition 1.1 (see [30]): A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be quasi-statistically convergent to x if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{C_n} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0$$

where $C_n > 0 \forall n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} C_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{C_n}{n} < \infty$. In this case we write $x_n \xrightarrow{S^q} x$ and the set of all quasi-statistically convergent sequences are denoted by S^q .

Definition 1.2 (see [1]): Let $\{\alpha_n\}_{n \in \mathbb{N}}$ and $\{\beta_n\}_{n \in \mathbb{N}}$ be two sequences of positive real numbers such that

(i) α and β are both non-decreasing,

(ii) $\beta_n \geq \alpha_n \forall n \in \mathbb{N}$,

(iii) $(\beta_n - \alpha_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Then the sequence of real numbers $\{x_n\}_{n \in \mathbb{N}}$ is said to be $\alpha\beta$ -statistically convergent of order γ (where $0 < \gamma \leq 1$) to a real number x if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{(\beta_n - \alpha_n + 1)^\gamma} |\{k \in [\alpha_n, \beta_n] : |x_k - x| \geq \varepsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{S_{\alpha\beta}^\gamma} x$ and the set of all sequences which are $\alpha\beta$ -statistically convergent of order γ is denoted by $S_{\alpha\beta}^\gamma$.

Definition 1.3 (see [19]): Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$ and $T_n = t_1 + t_2 + \dots + t_n \forall n \in \mathbb{N}$. A real sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be weighted statistically convergent of order γ (where $0 < \gamma \leq 1$) to x if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{T_n^\gamma} |\{k \leq T_n : t_k |x_k - x| \geq \varepsilon\}| = 0.$$

In this case we write $x_n \xrightarrow{(S_{\overline{N}, t_n}^\gamma)} x$. The class of all sequences which are weighted statistically convergent of order γ is denoted by $(S_{\overline{N}, t_n}^\gamma)$. For $\gamma = 1$, we say that $\{x_n\}_{n \in \mathbb{N}}$ is weighted statistically convergent to x and this denoted by $x_n \xrightarrow{(S_{\overline{N}, t_n}^1)} x$.

Definition 1.4 (see [24, 28]): Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_n = t_1 + t_2 + \dots + t_n$ where $n \in \mathbb{N}$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$. Then the sequence of real numbers $\{x_n\}_{n \in \mathbb{N}}$ is said to be weighted strongly Cesàro convergent (or strongly (\overline{N}, t_n) -summable) to a real number x if

$$\lim_{n \rightarrow \infty} \frac{1}{T_n} \sum_{k=1}^n t_k |x_k - x| = 0.$$

In this case we write $x_n \xrightarrow{[\overline{N}, t_n]} x$. The set of all strongly (\overline{N}, t_n) -summable real sequences are denoted by $[\overline{N}, t_n]$.

Definition 1.5 (see [26, 29]): A modulus function ϕ is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $\phi(x) = 0$ if and only if $x = 0$,

(ii) $\phi(x + y) \leq \phi(x) + \phi(y)$, for all $x, y > 0$,

(iii) ϕ is increasing,

(iv) ϕ is continuous from the right at zero.

A modulus function may be bounded or unbounded. Savaş [32, 33] and Tripathy & B. Sarma [38] and other authors used modulus function function to construct new sequence spaces. Recently Savaş &

Patterson [34] have defined and studied some sequence spaces by using a modulus function.

Definition 1.6 (see [16]): Let ϕ be a modulus function and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k$, for all $n \in \mathbb{N}$. A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be weighted modulus $\alpha\beta$ -statistically convergent of order γ (where $0 < \gamma \leq 1$) in probability to a random variable $X : \mathcal{W} \rightarrow \mathbb{R}$ if for any $\epsilon, \delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{T_{\alpha\beta(n)}^\gamma} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \epsilon)) \geq \delta\}| = 0.$$

In this case, we write $X_n \xrightarrow{(S_{\alpha\beta}^\gamma, P^\phi, t_n)} X$ and the class of all sequences of random variables which are weighted modulus $\alpha\beta$ -statistically convergent of order γ in probability is denoted by $(S_{\alpha\beta}^\gamma, P^\phi, t_n)$.

Definition 1.7 (see [16]): Let ϕ be a modulus function and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k \rightarrow \infty$ as $n \rightarrow \infty$. Then the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be weighted modulus $\alpha\beta$ -strongly Cesàro summable of order γ (where $0 < \gamma \leq 1$) in probability to a random variable X if for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{T_{\alpha\beta(n)}^\gamma} \sum_{k \in [\alpha_n, \beta_n]} t_k \phi(P(|X_k - X| \geq \epsilon)) = 0.$$

In this case, $X_n \xrightarrow{(N_{\alpha\beta}^\gamma, P^\phi, t_n)} X$ and the class of all sequences of random variables which are weighted modulus $\alpha\beta$ -strongly Cesàro summable of order γ in probability is denoted by $(N_{\alpha\beta}^\gamma, P^\phi, t_n)$.

Definition 1.8 (see [16]): Let ϕ be a modulus function and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$, and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k$, for all $n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be weighted modulus $S_{\alpha\beta}$ -convergent of order γ in probability (where $0 < \gamma \leq 1$) to a random variable X if for every $\epsilon, \delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{T_{\alpha\beta(n)}^\gamma} |\{k \in I_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \epsilon)) \geq \delta\}| = 0,$$

where $I_{\alpha\beta(n)} = (T_{[\alpha(n)]}, T_{[\beta(n)]}]$ and $[x]$ denotes the greatest integer not greater than x . In this case we write $X_n \xrightarrow{(WS_{\alpha\beta}^\gamma, P^\phi, t_n)} X$. The class of all sequences of random variables which are weighted modulus $S_{\alpha\beta}$ -convergent of order γ in probability is denoted by $(WS_{\alpha\beta}^\gamma, P^\phi, t_n)$.

Definition 1.9 (see [16]): Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k \rightarrow \infty$, as $n \rightarrow \infty$ and ϕ be a modulus function. The sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be weighted modulus $N_{\alpha\beta}$ -convergent of order γ in probability (where $0 < \gamma \leq 1$) to a random variable X if for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{T_{\alpha\beta(n)}^\gamma} \sum_{k \in I_{\alpha\beta(n)}} t_k \phi(P(|X_k - X| \geq \epsilon)) = 0.$$

In this case, $X_n \xrightarrow{(WN_{\alpha\beta}^\gamma, P^\phi, t_n)} X$ and the class of all sequences of random variables which are weighted modulus $N_{\alpha\beta}$ -convergent of order γ in probability is denoted by $(WN_{\alpha\beta}^\gamma, P^\phi, t_n)$.

2. Main Results

We first introduce the following definitions.

Definition 2.1. Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k$, for all $n \in \mathbb{N}$. Then the sequence of real numbers $\{x_n\}_{n \in \mathbb{N}}$ is said to be quasi-weighted $\alpha\beta$ -statistically convergent to x if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k |x_k - x| \geq \varepsilon\}| = 0$$

where $C_n > 0$, for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} C_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} < \infty$. In this case we write $x_n \xrightarrow{(S_{\alpha\beta}^q, t_n)} x$. The class of all quasi-weighted $\alpha\beta$ -statistically convergent sequences are denoted by $(S_{\alpha\beta}^q, t_n)$.

Remark 2.2. (i) For $t_n = 1, \alpha_n = 1, \beta_n = n, \forall n \in \mathbb{N}$ then quasi-weighted $\alpha\beta$ -statistical convergence coincides with quasi-statistical convergence [30].

(ii) For $t_n = 1, \alpha_n = 1, \beta_n = n, C_n = n^\gamma$ (where $0 < \gamma \leq 1$), $\forall n \in \mathbb{N}$ then quasi-weighted $\alpha\beta$ -statistical convergence coincides with statistical convergence of order γ [4, 7].

(iii) For $t_n = 1, \alpha_n = n - \lambda_n + 1, \beta_n = n$ and $C_n = \lambda_n^\gamma, \forall n \in \mathbb{N}$ then quasi-weighted $\alpha\beta$ -statistical convergence coincides with λ -statistical convergence of order γ [7, 10].

(iv) For $\alpha_n = 1, \beta_n = n$ and $C_n = T_n^\gamma$ (where $T_{\alpha\beta(n)} = T_n = \sum_{k \in [1, n]} t_k$), $\forall n \in \mathbb{N}$ then quasi-weighted $\alpha\beta$ -statistical convergence coincides with weighted statistical convergence of order γ [19, 24, 28].

(v) For $\alpha_n = n - \lambda_n + 1, \beta_n = n$ and $C_n = T_{\lambda_n}$ (where $T_{\alpha\beta(n)} = T_{\lambda_n} = \sum_{k \in [n - \lambda_n + 1, n]} t_k$) $\forall n \in \mathbb{N}$ then quasi-weighted $\alpha\beta$ -statistical convergence coincides with weighted λ -statistical convergence [3].

(vi) For $C_n = T_{\alpha\beta(n)}^\gamma$ (where $0 < \gamma \leq 1$) $\forall n \in \mathbb{N}$ then quasi-weighted $\alpha\beta$ -statistical convergence coincides with weighted $\alpha\beta$ -statistical convergence of order γ [16] and so on.

It is obvious that if $x_n \xrightarrow{(S_{\alpha\beta}^q, t_n)} x$ and $x_n \xrightarrow{(S_{\alpha\beta}^q, t_n)} y$ then $x = y$.

Throughout the paper we assume that $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers such that $C_n > 0, \forall n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} C_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} < \infty$.

Now we like to introduce the definition of quasi-weighted modulus $\alpha\beta$ -statistical convergence in probability of a sequence of random variables as follows:

Definition 2.3. Let ϕ be a modulus function and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k$, for all $n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be quasi-weighted modulus $\alpha\beta$ -statistically convergent in probability to a random variable X (where $X : \mathcal{W} \rightarrow \mathbb{R}$) if for any $\varepsilon, \delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}| = 0.$$

In this case we write $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} X$ and the class of all quasi-weighted modulus $\alpha\beta$ -statistically convergent sequences in probability is denoted by $(S_{\alpha\beta}^q, P^\phi, t_n)$.

Theorem 2.4. If $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} X$ and $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} Y$ then $P\{X = Y\} = 1$.

Proof. If possible let $P\{X = Y\} \neq 1$. Then there exists two positive real numbers ε, δ such that $\phi(P(|X - Y| \geq \varepsilon)) > \delta$ and $\liminf_{n \rightarrow \infty} t_n > \delta$.

Then for large value of n we get

$$\begin{aligned} \eta < \frac{T_{\alpha\beta(n)}}{C_n} &= \frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X - Y| \geq \varepsilon)) \geq \delta^2\}| \\ &\leq \frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \frac{\varepsilon}{2})) \geq \frac{\delta^2}{2}\}| \\ &\quad + \frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - Y| \geq \frac{\varepsilon}{2})) \geq \frac{\delta^2}{2}\}| \end{aligned}$$

where η be a positive real number (existence of η is guaranteed by the conditions of the sequence $\{C_n\}_{n \in \mathbb{N}}$). Which is impossible because the right hand side tends to zero as $n \rightarrow \infty$. Hence the result.

Theorem 2.5. If $\{X_n\}_{n \in \mathbb{N}}$ is quasi-weighted modulus $\alpha\beta$ -statistically convergent in probability to X then it is weighted modulus $\alpha\beta$ -statistically convergent in probability to X .

Proof. The proof is parallel to that of Lemma 1.1 in [30] and therefore omitted.

Theorem 2.6. Let $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers satisfying the condition $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} > 0$ and $\{X_n\}_{n \in \mathbb{N}}$ is weighted modulus $\alpha\beta$ -statistically convergent in probability to X then it is quasi-weighted modulus $\alpha\beta$ -statistically convergent in probability to X .

Proof. The proof is parallel to that of Lemma 1.3 in [30] and therefore omitted.

If $\{C_n\}_{n \in \mathbb{N}}$ satisfying the condition $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} > 0$ then quasi-weighted modulus $\alpha\beta$ -statistical convergence in probability is equivalent to weighted modulus $\alpha\beta$ -statistical convergence in probability. So the problem is quite interesting if $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} = 0$.

The following example shows that there is a sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables which is weighted modulus $\alpha\beta$ -statistically convergent in probability to a random variable X but it is not quasi-weighted modulus $\alpha\beta$ -statistically convergent in probability to X .

Example 2.7. Let the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\}, & \text{with p.m.f } P(X_n = -1) = P(X_n = 0), \text{ if } n = m^2, \text{ where } m \in \mathbb{N}, \\ \{0, 1\}, & \text{with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^4}, P(X_n = 1) = \frac{1}{n^4}, \text{ if } n \neq m^2, \text{ where } m \in \mathbb{N}. \end{cases}$$

Let $t_n = 2n, \alpha_n = n, \beta_n = n^2, C_n = n, \forall n \in \mathbb{N}$ and $\phi(x) = \sqrt{x}, \forall x \in [0, \infty) \Rightarrow T_{\alpha\beta(n)} = n^4 + n, \forall n \in \mathbb{N}$ and $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} = 0$.

For $0 < \varepsilon < 1$, we get

$$P(|X_n - 0| \geq \varepsilon) = \begin{cases} 1, & \text{if } n = m^2, \text{ where } m \in \mathbb{N}, \\ \frac{1}{n^4}, & \text{if } n \neq m^2, \text{ where } m \in \mathbb{N}. \end{cases}$$

Now let $0 < \delta < 1$, then

$$\frac{1}{T_{\alpha\beta(n)}} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \leq \frac{\sqrt{n^4 + n}}{(n^4 + n)} \leq \frac{\sqrt{2}n^2}{n^4} \leq \frac{\sqrt{2}}{n^2}$$

and

$$\frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \geq \frac{\sqrt{n^4 + n} - 1}{n} \geq n \sqrt{1 + \frac{1}{n^3}} - \frac{1}{n}.$$

This shows that $\{X_n\}_{n \in \mathbb{N}}$ is weighted modulus $\alpha\beta$ -statistically convergent in probability to a random variable 0 but it is not quasi-weighted modulus $\alpha\beta$ -statistically convergent in probability to 0.

Next example shows that there is a sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables which is quasi-weighted $\alpha\beta$ -statistically convergent in probability to a random variable X but it is not quasi-weighted modulus $\alpha\beta$ -statistically convergent in probability to X .

Example 2.8. Let the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\}, & \text{with p.m.f } P(X_n = -1) = P(X_n = 0), \text{ if } n = m^2, \text{ where } m \in \mathbb{N}, \\ \{0, 1\}, & \text{with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^2}, P(X_n = 1) = \frac{1}{n^2}, \text{ if } n \neq m^2, \text{ where } m \in \mathbb{N}. \end{cases}$$

Let $t_n = 2n, \alpha_n = n, \beta_n = n^2, C_n = n^\gamma$ (where $2 < \gamma < 4$), $\forall n \in \mathbb{N}$ and $\phi(x) = \sqrt{x}, \forall x \in [0, \infty)$
 $\Rightarrow T_{\alpha\beta(n)} = n^4 + n, \forall n \in \mathbb{N}$ and $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} = 0$.

For $0 < \varepsilon, \delta < 1$, we get

$$\frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \leq \frac{\sqrt{n^4 + n}}{n^\gamma} \leq \frac{\sqrt{2}n^2}{n^\gamma} \leq \frac{\sqrt{2}}{n^{\gamma-2}}$$

and

$$\frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \geq \frac{n^4}{n^\gamma} = n^{4-\gamma} \geq 1.$$

So $\{X_n\}_{n \in \mathbb{N}} \in (S_{\alpha\beta}^q, P, t_n)$ but not in $(S_{\alpha\beta}^q, P^\phi, t_n)$.

Theorem 2.9. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on \mathbb{R} . If $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} X$ and $P(|X| \geq x_0) = 0$ for some positive real number x_0 , then $g(X_n) \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} g(X)$.

Proof. The proof is parallel to that of Theorem 2.2 in [16] and therefore omitted.

Corollary 2.10. Let $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} x$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then $g(X_n) \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} g(x)$.

Proof. Proof is straight forward, so omitted.

Definition 2.11. Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k \rightarrow \infty$ as $n \rightarrow \infty$ and ϕ be a modulus function. The sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be quasi-weighted modulus $\alpha\beta$ -strongly Cesàro summable in probability to a random variable X if for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{C_n} \sum_{k \in [\alpha_n, \beta_n]} t_k \phi(P(|X_k - X| \geq \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(N_{\alpha\beta}^q, P^\phi, t_n)} X$ and the class of all sequences which are quasi-weighted modulus $\alpha\beta$ -strongly Cesàro summable in probability is denoted by $(N_{\alpha\beta}^q, P^\phi, t_n)$.

In the following, the relationship between $(S_{\alpha\beta}^q, P^\phi, t_n)$ and $(N_{\alpha\beta}^q, P^\phi, t_n)$ is investigated.

The following example shows that the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ in $(N_{\alpha\beta}^q, P^\phi, t_n)$ converges to X but it is not in $(S_{\alpha\beta}^q, P^\phi, t_n)$ converges to X .

Example 2.12. Let $c \in (0, 1)$, $2c < \gamma \leq 4c$ and a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 0\}, & \text{with p.m.f } P(X_n = -1) = \frac{1}{n^{2c}}, P(X_n = 0) = 1 - \frac{1}{n^{2c}}, \text{ if } n = [m^{\frac{1}{c}}], \text{ where } m \in \mathbb{N}, \\ \{0, 1\}, & \text{with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^{3c}}, P(X_n = 1) = \frac{1}{n^{3c}}, \text{ if } n \neq [m^{\frac{1}{c}}], \text{ where } m \in \mathbb{N}. \end{cases}$$

Let $t_n = 2n, \alpha_n = n, \beta_n = n^2, C_n = n^\gamma, \forall n \in \mathbb{N}$ and $\phi(x) = \sqrt{x}, \forall x \in [0, \infty)$ then $T_{\alpha\beta(n)} = n^4 + n, \forall n \in \mathbb{N}$ and $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} = 0$.

For $0 < \varepsilon, \delta < 1$, we get

$$\begin{aligned} \frac{1}{C_n} \sum_{k \in [\alpha_n, \beta_n]} t_k \phi(P(|X_k - 0| \geq \varepsilon)) &\leq \frac{2}{n^\gamma} \{(n^{2c} - n^c + 1) + (\frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{(n^2)^3})\} \\ &\leq \frac{M}{n^{\gamma-2c}} \text{ (where } M \text{ is a positive constant)} \end{aligned}$$

and

$$\frac{1}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \geq \frac{(n^4 + n)^c}{n^\gamma} > \frac{1}{2} n^{4c-\gamma}$$

So $\{X_n\}_{n \in \mathbb{N}} \in (N_{\alpha\beta}^q, P^\phi, t_n)$ but not in $(S_{\alpha\beta}^q, P^\phi, t_n)$.

Theorem 2.13. If $\liminf_{n \rightarrow \infty} t_n > 0, \lim_{n \rightarrow \infty} \frac{\alpha_n}{C_n} = 0$ and $\liminf_{n \rightarrow \infty} \frac{\beta_n}{T_{\alpha\beta(n)}} > 1$, then $(N_{\alpha\beta}^q, P^\phi, t_n) \subset (S_{\alpha\beta}^q, P^\phi, t_n)$.

Proof. Let $X_n \xrightarrow{(N_{\alpha\beta}^q, P^\phi, t_n)} X$ and $\varepsilon, \delta > 0$. Then

$$\frac{1}{C_n} \sum_{k \in [\alpha_n, \beta_n]} t_k \phi(P(|X_k - X| \geq \varepsilon)) \geq \frac{\delta}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}| - \frac{\delta \alpha_n}{C_n},$$

(since $\beta_n \geq T_{\alpha\beta(n)} \forall n \in \mathbb{N}$). Hence the result follows.

The following example shows that, the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ in $(S_{\alpha\beta}^q, P^\phi, t_n)$ converges to X but not in $(N_{\alpha\beta}^q, P^\phi, t_n)$ converges to X .

Example 2.14. Let $t_n = n, \alpha_n = 1, \beta_n = n, C_n = \sqrt{n}, \forall n \in \mathbb{N}$ and $\phi(x) = \sqrt{x}, \forall x \in [0, \infty)$ then $T_{\alpha\beta(n)} = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}$ and $\liminf_{n \rightarrow \infty} \frac{C_n}{T_{\alpha\beta(n)}} = 0$.

Consider the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is defined by,

$$X_n \in \begin{cases} \{-1, 1\} & \text{with probability } \frac{1}{2}, \text{ if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N} \\ \{0, 1\} & \text{with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^3}, P(X_n = 1) = \frac{1}{n^3}, \text{ if } n \neq \{T_m\}^{T_m} \\ & \text{for any } m \in \mathbb{N} \end{cases}$$

Let $0 < \varepsilon < 1$, then,

$$P(|X_n - 0| \geq \varepsilon) = \begin{cases} 1 & \text{if } n = \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N} \\ \frac{1}{n^3} & \text{if } n \neq \{T_m\}^{T_m} \text{ for any } m \in \mathbb{N}. \end{cases}$$

This implies $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} 0$.

Now let $H = \{n \in \mathbb{N} : n \neq \{T_m\}^{T_m} \text{ where } m \in \mathbb{N}\}$. Now we have the inequality,

$$\begin{aligned} \sum_{k \in [\alpha_n, \beta_n]} t_k \phi(P(|X_k - 0| \geq \epsilon)) &= \sum_{\substack{k \in [\alpha_n, \beta_n] \\ k \in H}} t_k \phi(P(|X_k - 0| \geq \epsilon)) + \sum_{\substack{k \in [\alpha_n, \beta_n] \\ k \notin H}} t_k \phi(P(|X_k - 0| \geq \epsilon)) \\ &> \sum_{\substack{k \in [\alpha_n, \beta_n] \\ k \in H}} \frac{1}{\sqrt{k}} + \sum_{\substack{k \in [\alpha_n, \beta_n] \\ k \notin H}} 1 \\ &> \sum_{k=1}^n \frac{1}{\sqrt{k}} > \sqrt{n} \quad (\text{Since we know that } \sum_{k=1}^n \frac{1}{\sqrt{k}} > \sqrt{n} \forall n \geq 2) \end{aligned}$$

$$\Rightarrow \frac{1}{C_n} \sum_{k=1}^n t_k \phi(P(|X_k - 0| \geq \epsilon)) > \frac{\sqrt{n}}{\sqrt{n}} = 1.$$

This inequality shows that $\{X_n\}_{n \in \mathbb{N}}$ is not $(N_{\alpha\beta}^q, P^\phi, t_n)$ summable to 0.

Theorem 2.15. Let $\liminf_{n \rightarrow \infty} t_n > 0$ and $\{t_n\}_{n \in \mathbb{N}}$ be a bounded sequence of real numbers such that $\limsup_{n \rightarrow \infty} \frac{\beta_n - \alpha_n}{C_n} < \infty$. Then $(S_{\alpha\beta}^q, P^\phi, t_n) \subset (N_{\alpha\beta}^q, P^\phi, t_n)$.

Proof. Let $X_n \xrightarrow{(S_{\alpha\beta}^q, P^\phi, t_n)} X$ and $t_n \leq M_1, \forall n \in \mathbb{N}$ and $\limsup_{n \rightarrow \infty} \frac{\beta_n - \alpha_n}{C_n} < M_2$, where M_1 and M_2 are positive real numbers. For any $\epsilon, \delta > 0$ setting $H = \{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \epsilon)) \geq \delta\}$. Then

$$\begin{aligned} &\frac{1}{C_n} \sum_{k \in [\alpha_n, \beta_n]} t_k \phi(P(|X_k - X| \geq \epsilon)) \\ &= \frac{1}{C_n} \sum_{k \in [\alpha_n, \beta_n] \cap H} t_k \phi(P(|X_k - X| \geq \epsilon)) + \frac{1}{C_n} \sum_{k \in [\alpha_n, \beta_n] \cap H^c} t_k \phi(P(|X_k - X| \geq \epsilon)) \\ &\leq \frac{M_1 \phi(1)}{C_n} |\{k \leq T_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \epsilon)) \geq \delta\}| + M_2 \delta. \end{aligned}$$

Since δ is arbitrary, so the result follows.

Now we would like to introduce the definitions of quasi-weighted modulus $S_{\alpha\beta}$ -convergence in probability and quasi weighted modulus $N_{\alpha\beta}$ -convergence in probability for a sequence of random variables as follows:

Definition 2.16. Let ϕ be a modulus function and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\liminf_{n \rightarrow \infty} t_n > 0$, and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k, \forall n \in \mathbb{N}$. Then the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be quasi-weighted modulus $S_{\alpha\beta}$ -convergent in probability to X if for every $\epsilon, \delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{C_n} |\{k \in I_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \epsilon)) \geq \delta\}| = 0$$

where $I_{\alpha\beta(n)} = (T_{[\alpha(n)], T_{[\beta(n)]}]$ and $[x]$ denotes the greatest integer not greater than x . In this case we write

$X_n \xrightarrow{(WS_{\alpha\beta}^q, P^\phi, t_n)} X$. The class of all quasi-weighted modulus $S_{\alpha\beta}$ -convergence sequences of random variables

in probability are denoted by $(WS_{\alpha\beta}^q, P^\phi, t_n)$.

It is very obvious that if $X_n \xrightarrow{(WS_{\alpha\beta}^q, P^\phi, t_n)} X$ and $X_n \xrightarrow{(WS_{\alpha\beta}^q, P^\phi, t_n)} Y$ then $P\{X = Y\} = 1$.

Definition 2.17. Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers such that $t_1 > 0$ and $T_{\alpha\beta(n)} = \sum_{k \in [\alpha_n, \beta_n]} t_k \rightarrow \infty$, as $n \rightarrow \infty$ and ϕ be a modulus function. The sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to be quasi-weighted modulus $N_{\alpha\beta}$ -convergent in probability to a random variable X if for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{C_n} \sum_{k \in I_{\alpha\beta(n)}} t_k \phi(P(|X_k - X| \geq \varepsilon)) = 0.$$

In this case, $X_n \xrightarrow{(WN_{\alpha\beta}^q, P^\phi, t_n)} X$ and the class of all quasi-weighted modulus $N_{\alpha\beta}$ -convergence sequences of random variables in probability are denoted by $(WN_{\alpha\beta}^q, P^\phi, t_n)$.

In the following, the relationship between $(WS_{\alpha\beta}^q, P^\phi, t_n)$ and $(WN_{\alpha\beta}^q, P^\phi, t_n)$ is investigated.

Theorem 2.18. Let $\liminf_{n \rightarrow \infty} t_n > 0$. Then $(WN_{\alpha\beta}^q, P^\phi, t_n) \subset (WS_{\alpha\beta}^q, P^\phi, t_n)$ and this inclusion is strict.

Proof. First part of this theorem, let $\varepsilon, \delta > 0$, then

$$\begin{aligned} & \sum_{k \in I_{\alpha\beta(n)}} t_k \phi(P(|X_k - X| \geq \varepsilon)) \\ &= \sum_{k \in I_{\alpha\beta(n)}, t_k \phi(P(|X_k - X| \geq \varepsilon)) \geq \delta} t_k \phi(P(|X_k - X| \geq \varepsilon)) + \sum_{k \in I_{\alpha\beta(n)}, t_k \phi(P(|X_k - X| \geq \varepsilon)) < \delta} t_k \phi(P(|X_k - X| \geq \varepsilon)) \\ &\geq \delta |\{k \in I_{\alpha\beta(n)} : t_k \phi(P(|X_k - X| \geq \varepsilon)) \geq \delta\}|. \end{aligned}$$

For the second part we will give an example.

Let $t_n = n, \alpha(n) = n!, \beta(n) = (n + 1)!, C_n = \sqrt{T_{\alpha\beta(n)}}, \forall n \in \mathbb{N}$ and $\phi(x) = \sqrt{x}, \forall x \in [0, \infty)$ and a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ be defined by,

$$X_n \in \begin{cases} \{-1, 1\}, \text{ with p.m.f } P(X_n = 1) = P(X_n = -1), \text{ if } n \text{ is the first } \lceil \sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})} \rceil \\ \text{integer in the interval } (T_{[\alpha(n)]}, T_{[\beta(n)]}], \\ \{0, 1\}, \text{ with p.m.f } P(X_n = 0) = 1 - \frac{1}{n^3}, P(X_n = 1) = \frac{1}{n^3}, \text{ otherwise.} \end{cases}$$

For $0 < \varepsilon, \delta < 1$, we get

$$\frac{1}{C_n} |\{k \in I_{\alpha\beta(n)} : t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \delta\}| \leq \frac{1}{\sqrt[2]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

For next

$$\frac{1}{C_n} \sum_{k \in I_{\alpha\beta(n)}} t_k \phi(P(|X_k - 0| \geq \varepsilon)) \geq \frac{[\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}]\{[\sqrt[4]{(T_{[\beta(n)]} - T_{[\alpha(n)]})}] + 1\}}{2 \sqrt{(T_{[\beta(n)]} - T_{[\alpha(n)]})}} > \frac{1}{3} > 0$$

Hence the result.

Acknowledgement

We like to thank the respected Referees for his/her careful reading of the paper and for several valuable suggestions which have improved the presentation of the paper.

References

- [1] H. Aktuğlu, Korovkin type approximation theorems proved via $\alpha\beta$ -statistical convergence, *J. Comput. Appl. Math.* 259 (2014) 174-181.
- [2] M. Başarır, S. Konca, On some space of lacunary convergent sequences derived by Nörlund-type mean and weighted lacunary statistical convergence, *Arab J. Math. Sci.* 20 (2) (2014) 250-263.
- [3] C. Belen, S. A. Mohiuddine, Generalized weighted statistical convergence and application, *Appl. Math. Comput.* 219 (2013) 9821-9826.
- [4] S. Bhunia, P. Das, S. Pal, Restricting statistical convergence, *Acta Math. Hungar.* 134 (1-2) (2012) 153-161.
- [5] M. Cinar, M. Karakas, M. Et, On pointwise and uniform statistical convergence of order α for sequences of functions, *Fixed Point Theory Appl.* 33 (2013) 11 pp
- [6] R. Çolak, Statistical convergence of order α , *Modern methods in Analy. and its appl.* New Delhi, India, Anamaya Pub (2010) 121-129.
- [7] R. Çolak, A. C. Bektaş, λ -statistical convergence of order α , *Acta Math. Sci. ser. B Engl. Ed.* 31B (3) (2011) 953-959.
- [8] J. S. Connor, The statistical and strong p -Cesaro convergence sequence, *Analysis* 8 (1988) 47-63.
- [9] P. Das, S. Ghosal, S. Som, Statistical convergence of order α in probability, *Arab J Math. Sci.* 21 (2) (2015) 253-265.
- [10] M. Et, M. Cinar, M. Karakas, On λ -statistical convergence of order α of sequences of functions, *J. Inequal. Appl.* 204 (2013) 8 pp.
- [11] H. Fast, Sur la convergence statistique, *Colloquium Math.* 2 (1951) 241-244.
- [12] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesaro-type summability space, *Proc. London Math. Soc.* 37 (3) (1978) 508-520.
- [13] J. A. Fridy, On statistical convergence, *Analysis* 5 (1985) 301-313.
- [14] J. A. Fridy, C. Orhan, Lacunary statistical convergence, *Pacific J. Math.* 160 (1993) 43-51.
- [15] S. Ghosal, Statistical convergence of a sequence of random variables and limit theorems, *Appl. Math.* 58 (4) (2013) 423-437.
- [16] S. Ghosal, Generalized weighted random convergence in probability, *Appl. Math. Comput.* 249 (2014) 502-509.
- [17] S. Ghosal, \mathcal{I} -statistical convergence of a sequence of random variables in probability, *Afr. Mat.* 25 (3) (2014) 681-692.
- [18] S. Ghosal, S_λ -statistical convergence of a sequence of random variables, *J. Egyptian Math. Soc.* 23 (1) (2015) 85-89.
- [19] S. Ghosal, Weighted statistical convergence of order α and its applications, *J. Egyptian Math. Soc.* 24 (1) (2016) 60-67.
- [20] M. Gürdal, On ideal convergent sequences in 2-normed spaces, *Thai J. Math.* 4 (1) (2006) 85-91.
- [21] M. Gürdal, H. Sari, Extremal A -statistical limit points via ideals, *J. Egyptian Math. Soc.* 22 (1) (2014) 55-58.
- [22] M. Gürdal, A. Sahiner, Ideal convergence in n -normed spaces and some new sequence spaces via n -norm, *J. Fundament. Sci.* 4 (1) (2008) 233-244.
- [23] M. Gürdal, B. M. Huban, On \mathcal{I} -convergence of double sequences in the topology induced by random 2-norms, *Mat. Vesnik* 66 (1) (2014) 73-83.
- [24] V. Karakaya, T. A. Chishti, Weighted statistical convergence, *Iranian J. of Sci. and Techno. Transaction A* 33 (A3) (2009) 219-223.
- [25] V. Karakaya, A. Karaiza, Korovkin type approximation theorems for weighted $\alpha\beta$ -statistical convergence *Bull. Math. Sci.* 5 (2) (2015) 159-169.
- [26] I. J. Maddox, Sequence spaces define by a modulus, *Math. Proc. Cambridge Philos. Soc.* 100 (1) (1986) 161-166.
- [27] M. Mursaleen, λ -statistical convergence, *Math. Slovaca* 50 (1) (2000) 111-115.
- [28] M. Mursaleen, V. Karakaya, M. Erturk, F. Gursoy, Weighted statistical convergence and its application to Korovkin type approximation theorem, *Appl. Math. Comput.* 218 (18) (2012) 9132-9137.
- [29] W. H. Ruckle, FK space in which the sequence of co-ordinate vectors in bounded, *Canad. J. Math.* 25 (1973) 973-978.
- [30] İ. Sakaoğlu-Özgüc, T. Yurdakadim, On quasi-statistical convergence, *Commun. Fac. Sci. Univ. Ank. Series A1* 61 (1) (2012) 11-17.
- [31] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca* 30 (1980) 139-150.
- [32] E. Savaş, On some generalized sequence space defined by a modulus. *Indian J. Pure. Appl. Math.* 38 (1) (1999) 459-464.
- [33] E. Savaş, On some new double lacunary sequences spaces via orlicz function, *J. Comput. Anal. Appl.* 11 (3) (2009) 423-430.
- [34] E. Savaş, R. F. Patterson, Some double lacunary sequence spaces defined by orlicz functions, *Southeast Asian Bull. Math.* 35 (1) (2011) 103-110.
- [35] I. J. Schonenberg, The integrability of certain functions and related summability methods, *Am. Math. Month.* 66 (1959) 361-375.
- [36] H. Sengül, M. Et, On lacunary statistical convergence of order α , *Acta Math. Sci. ser. B Engl. Ed.* 34B (2) (2014) 473-482.
- [37] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloquium Math.* 2 (1951) 73-74.
- [38] B. C. Tripathy, B. Sarma, Double sequence spaces of fuzzy numbers defined by Orlicz function, *Acta Math. Sci. ser. B Engl. Ed.* 31 (1) (2011) 134-140.