Filomat 31:5 (2017), 1279–1285
DOI 10.2298/FIL1705279S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On star-K-Hurewicz spaces

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Abstract. A space X is *star-K-Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(K_n : n \in N)$ of compact subsets of X such that for each $x \in X$, $x \in St(K_n, \mathcal{U}_n)$ for all but finitely many n. In this paper, we investigate the relationship between star-K-Hurewicz spaces and related spaces by giving some examples, and also study topological properties of star-K-Hurewicz spaces.

1. Introduction

By a space we mean a topological space. We give definitions of terms which are used in this paper. Let $\mathbb{N}$ denote the set of positive integers. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$.

Let $\mathcal{O}$ be collection of open covers of a space X. Then

The symbol $S_1(\mathcal{O}, \mathcal{O})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{O}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\} \in \mathcal{O}$.

The symbol $S_{fin}(\mathcal{O}, \mathcal{O})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \mathcal{O}$ (see [7,12]).

Kočinac [8,9] introduced star selection hypothesis similar to the previous ones.

(A) The symbol $S^*_{fin}(\mathcal{O}, \mathcal{O})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{O}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{O}$.

(B) The symbol $SS^*_{fin}(\mathcal{O}, \mathcal{O})$ ($SS^*_{comp}(\mathcal{O}, \mathcal{O})$) denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{O}$ there exists a sequence $(K_n : n \in N)$ of finite (resp., compact) subsets of X such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{O}$.

Let Γ be denote the collection of γ-covers of X. An open cover $\mathcal{U}$ of X is said to be a *γ-cover* if each point of X does not belong to at most finitely many elements of $\mathcal{U}$.

Definition 1.1. ([8,9]) A space X is said to be *star-Menger* (*strongly star-Menger, star-K-Menger*) if it satisfies the selection hypothesis $S^*_{fin}(\mathcal{O}, \mathcal{O})$ (resp., $SS^*_{fin}(\mathcal{O}, \mathcal{O})$, $SS^*_{comp}(\mathcal{O}, \mathcal{O})$).

2010 *Mathematics Subject Classification*. Primary 54D20; Secondary 54C10

Keywords. Selection principles, star-Menger, strongly star-Menger, star-K-Hurewicz, star-Hurewicz, strongly star-Hurewicz, strongly starcompact

Received: 24 February 2015; Revised: 22 April 2015; Accepted: 25 April 2015

Communicated by Ljubiša D.R. Kočinac

The author acknowledges the support from National Natural Science Foundation (grant 11271036) of China. A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions

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In 1925, Hurewicz [5](see also [2,6]) introduced the Hurewicz covering property for a space X in the following way:

H: A space X satisfies the *Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\} \in \Gamma$.

Two star versions of the Hurewicz property was introduced in [8, Definition 1.2] (see also [1,10]) and further studied in [1].

SH: A space X satisfies the *star-Hurewicz property*if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \Gamma$.

SSH: A space X satisfies the *strongly star-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in N)$ of finite subsets of X such that $\{St(A_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \Gamma$.

SKH: A space X satisfies the *star-K-Hurewicz property* (see [8]) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in N)$ of compact subsets of X such that $\{St(A_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \Gamma$.

From the above definitions, it is clear that every Hurewicz space is strongly star-Hurewicz, every strongly star-Hurewicz space is star-K-Hurewicz and every star-K-Hurewicz space is star-Hurewicz. But the converses do not hold (see Examples 2.1, 2.4 and 2.7 below).

In [1] and [14] star-Hurewicz and related spaces have been studied. The purpose of this paper is to investigate the relationships between star-K-Hurewicz spaces and related spaces by giving some examples, and also to study topological properties of star-K-Hurewicz spaces.

Throughout this paper, let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [4].

2. Star-K-Hurewicz Spaces

We give some examples showing that the relationship between star-K-Hurewicz spaces and other related spaces. Recall from [3,11] that a space X is said to be *strongly starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite F of X such that $St(F, \mathcal{U}) = X$. Clearly, every strongly starcompact space is strongly star-Hurewicz. It is well known that strongly starcompactness is equivalent to countably compactness for Hausdorff spaces (see [3,11]).

Example 2.1. *There exists a Tychonoff strongly star-Hurewicz space X which is not Menger (hence not Hurewicz).*

Proof. Let $X = [0, \omega_1)$ with the usual order topology. Then X is countably compact. Hence X is strongly star-Hurewicz, since every countably compact space is strongly starcompact and every strongly starcompact space is strongly star-Hurewicz. It is well known that X is not Lindelöf, thus X is not Menger, since every Menger space is Lindelöf. Thus we complete the proof. □

For the next example, we need a lemma from [2].

Lemma 2.2. *A space X is strongly star-Hurewicz iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$, $St(x, \mathcal{U}_n) \cap A_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.*

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X. Recall from [3,11] that a space X is said to be *K-starcompact* if for every open cover $\mathcal{U}$ of X there exists a compact subset F of X such that $St(F, \mathcal{U}) = X$. It is clear that every K-starcompact space is star-K-Hurewicz. For the next example, we need the following lemma.

Lemma 2.3. *Let κ be infinite cardinal and $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of cardinality κ. Then the subspace $X = (\beta D \times [0, \kappa^+)) \cup (D \times \{\kappa^+\})$ of the product space $\beta D \times [0, \kappa^+]$ is star-K-Hurewicz.*

Proof. We show that X is star-K-Hurewicz. We only show that X is K-starcompact, since every K-starcompact space is star-K-Hurewicz. To this end, let $\mathcal{U}$ be an open cover of X. For each $\alpha < \kappa^+$, there exists $U_\alpha \in \mathcal{U}$ such that $\langle d_\alpha, \kappa^+\rangle \in U_\alpha$, then we can find $\beta_\alpha < \kappa^+$ such that $\{d_\alpha\} \times (\beta_\alpha, \kappa^+] \subseteq U_\alpha$. Let $\beta = \sup\{\beta_\alpha : \alpha < \kappa\}$. Then $\beta < \kappa^+$. Let $K_1 = \beta D \times \{\beta + 1\}$. Then K_1 is compact and $U_\alpha \cap K_1 \neq \emptyset$ for each $\alpha < \kappa$. Hence

$$D \times \{\kappa^+\} \subseteq St(K_1, \mathcal{U}).$$

On the other hand, since $\beta D \times [0, \kappa^+)$ is countably compact and consequently $\beta D \times [0, \kappa^+)$ is strongly starcompact, hence there exists a finite subset K_2 of $\beta D \times [0, \kappa^+)$ such that

$$\beta D \times [0, \kappa^+) \subseteq St(K_2, \mathcal{U}).$$

If we put $K = K_1 \cup K_2$, then K is a compact subset of X such that $X = St(K, \mathcal{U})$, which shows that X is K-starcompact. $\square$

Example 2.4. *There exists a Tychonoff star-K-Hurewicz space X which is not strongly star-Hurewicz.*

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let

$$X = (\beta D \times [0, \mathfrak{c}^+)) \cup (D \times \{\mathfrak{c}^+\})$$

be the subspace of the product space $\beta D \times [0, \mathfrak{c}^+]$. Then X is a Tychonoff star-K-Hurewicz space by Lemma 2.3.

Similar to the proof that X is not strongly star-Hurewicz of Example 2.2 [14], we can prove that X is not strongly star-Hurewicz. $\square$

For the next example, we need the following lemmas.

Lemma 2.5. *If X is a σ-compact space, then X is star-Hurewicz.*

Lemma 2.6 is straightforward.

Lemma 2.6. *A space X is star-K-Hurewicz if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of compact subsets of X such that for every $x \in X$, $St(x, \mathcal{U}_n) \cap A_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.*

Example 2.7. *There exists a Hausdorff star-Hurewicz space which is not star-K-Hurewicz.*

Proof. Let

$$A = \{a_\alpha : \alpha < \mathfrak{c}\}, B = \{b_n : n \in \omega\}$$
$$\text{and } Y = \{\langle a_\alpha, b_n\rangle : \alpha < \mathfrak{c}, n \in \omega\},$$

and let

$$X = Y \cup A \cup \{a\} \text{ where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of a point $a_\alpha \in A$ for each $\alpha < \mathfrak{c}$ takes the form

$$U_{a_\alpha}(n) = \{a_\alpha\} \cup \{\langle a_\alpha, b_m\rangle : m > n\} \text{ for } n \in \omega$$

and a basic neighborhood of a point a takes the form

$$U_a(F) = \{a\} \cup \cup\{\langle a_\alpha, b_n\rangle : a_\alpha \in A \setminus F, n \in \omega\} \text{ for a countable subset } F \text{ of } A.$$

Clearly, X is a Hausdorff space by the construction of the topology of X. However, X is not regular, since the point a can not be separated from the closed subset A by disjoint open subsets of X.

Now we show that X is star-Hurewicz. To this end, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Without loss of generality, we assume that $\mathcal{U}_n$ consists of basic open sets of X for each $n \in \mathbb{N}$. For each

$n \in \mathbb{N}$, since $\mathcal{U}_n$ is an open cover of X, there exists $U_n \in \mathcal{U}_n$ such that $a \in U_n$. By assumption, there exists a countable subset F_n of A such that $U_n = U_a(F_n)$. By the definition of the topology of X, thus we have

$$(A \setminus F_n) \cup U_n \subseteq St(U_n, \mathcal{U}_n).$$

For each $a_\alpha \in \cup_{n\in\mathbb{N}} F_n$, let

$$B_{a_\alpha} = \{a_\alpha\} \cup \{\langle a_\alpha, b_n\rangle : n \in \omega\}.$$

Then B_{a_α} is a compact subset of X by the definition of the topology of X. Let $B = \bigcup_{a_\alpha \in \cup_{n\in\mathbb{N}} F_n} B_{a_\alpha}$. Then B is σ-compact, since F_n is countable for each $n \in \mathbb{N}$. Let $U = U_a(\cup_{n\in\mathbb{N}} F_n)$. Then $X = B \cup (A \setminus \cup_{n\in\mathbb{N}} F_n) \cup U$. By Lemma 2.5, B is star-Hurewicz. Then for the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in B$, $x \in St(\cup\mathcal{V}'_n, \mathcal{U}_n)$ for all but finitely many $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \mathcal{V}'_n \cup \{U_n\}$. Then the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X is star-Hurewicz. In fact, for each $x \in X$, if $x \in (A \setminus \cup_{n\in\mathbb{N}} F_n) \cup U$, then $x \in St(U_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$; if $x \in B$, then $x \in St(\cup\mathcal{V}'_n, \mathcal{U}_n)$ for all but finitely many $n \in \mathbb{N}$.

Next we show that X is not star-K-Hurewicz. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{a_\alpha\} \cup \{\langle a_\alpha, b_n\rangle : n \in \omega\} \text{ and } U = U_a(\emptyset).$$

Then U_α is open in X by the construction of the topology of X and

$$U_\alpha \cap U_{\alpha'} = \emptyset \text{ for } \alpha \neq \alpha'.$$

For $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{U\}.$$

Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. We only show that for the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists $x \in X$ such that $St(x, \mathcal{U}_n) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$, for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X by Lemma 2.6. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of X. For each $n \in \mathbb{N}$, since K_n is compact, then there exists $\alpha_n < \mathfrak{c}$ such that $K_n \cap U_\alpha = \emptyset$ for each $\alpha > \alpha_n$. Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. If we pick $\beta > \alpha'$, then $U_\beta \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Since U_β is the only element of $\mathcal{U}_n$ containing the point a_β for each $n \in \mathbb{N}$, then $St(a_\beta, \mathcal{U}_n) = U_\beta$ for each $n \in \mathbb{N}$, which shows that X is not star-K-Hurewicz. Thus we complete the proof. □

Remark 2.8. Since every star-K-Hurewicz space is star-K-Menger, thus the space X of Example 2.7 is not star-K-Menger. The author does not know if there exists a regular or Tychonoff star-Hurewicz space which is not star-K-Hurewicz.

In [1] it was shown that a paracompact Hausdorff space X is star-Hurewicz if and only if X is Hurewicz. Thus we have the following theorem.

Theorem 2.9. *Let X be a paracompact Hausdorff space. Then the following are equivalent:*
(1) X is Hurewicz;
(2) X is strongly star-Hurewicz;
(3) X is star-K-Hurewicz;
(4) X is star-Hurewicz.

In the following, we study topological properties of star-K-Hurewicz spaces. The space X of the proof of Example 2.4 shows that a closed subset of a Tychonoff star-K-Hurewicz space X need not be star-K-Hurewicz, since $D \times \{\mathfrak{c}^+\}$ is a discrete closed subset of cardinality $\mathfrak{c}$. Now we give an example showing that a regular-closed subset of a Tychonoff star-K-Hurewicz space X need not be star-K-Hurewicz. Here a subset A of a space X is said to be *regular-closed* in X if $cl_X int_X A = A$.

For the next example, we need the following lemma.

Lemma 2.10. *Let κ be infinite cardinal and $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of cardinality κ. Then the subspace $X = (\beta D \times [0, \kappa)) \cup (D \times \{\kappa\})$ of the product space $\beta D \times [0, \kappa]$ is not star-K-Hurewicz.*

Proof. We show that X is not star-K-Hurewicz. For each $\alpha < \kappa$, let $U_\alpha = \{d_\alpha\} \times (\alpha, \kappa]$. Then U_α is open in X and

$$U_\alpha \cap U_{\alpha'} = \emptyset \text{ for each } \alpha \neq \alpha'$$

For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \kappa\} \cup \{\beta D \times [0, \kappa)\}.$$

Then $\mathcal{U}_n$ is an open cover of X. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. It suffices to show that there exists $x \in X$ such that $St(x, \mathcal{U}_n) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$, for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X by Lemma 2.5. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of X. For each $n \in \mathbb{N}$, since K_n is compact and $\{\langle d_\alpha, \kappa\rangle : \alpha < \kappa\}$ is a discrete closed subset of X, the set $K_n \cap \{\langle d_\alpha, \omega\rangle : \alpha < \mathfrak{c}\}$ is finite. Then there exists $\alpha_n < \kappa$ such that

$$K_n \cap \{\langle d_\alpha, \kappa\rangle : \alpha > \alpha_n\} = \emptyset.$$

Let $\alpha' = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\alpha' < \kappa$ and

$$(\bigcup_{n\in\mathbb{N}} K_n) \cap \{\langle d_\alpha, \kappa\rangle : \alpha > \alpha'\} = \emptyset.$$

On the other hand, for each $n \in \mathbb{N}$, let $A_n = \{\alpha : \langle d_\alpha, \kappa\rangle \in K_n\}$. Then A_n is finite, since K_n is compact and $\{\langle d_\alpha, \kappa\rangle : \alpha < \kappa\}$ is discrete and closed in X. Let $K'_n = K_n \setminus \bigcup\{U_\alpha : \alpha \in A_n\}$. Then K'_n is closed in K_n and $K'_n \subseteq \beta D \times \kappa$. Hence $\pi(K'_n)$ is a compact subset of the countably compact space κ, where $\pi : \beta D \times \kappa \to \kappa$ is the projection, thus there exists $\alpha'_n < \kappa$ such that $\pi(K'_n) \cap (\alpha'_n, \kappa) = \emptyset$. Let $\alpha'' = \sup\{\alpha'_n : n \in \mathbb{N}\}$. Then $\alpha'' < \kappa$. If we pick $\beta > \max\{\alpha', \alpha''\}$, then $U_\beta \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Since U_β is the only element of $\mathcal{U}_n$ containing the point $\langle d_\beta, \kappa\rangle$ for each $n \in \mathbb{N}$, then $St(\langle d_\beta, \kappa\rangle, \mathcal{U}_n) = U_\beta$, thus $St(\langle d_\beta, \kappa\rangle, \mathcal{U}_n) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$, which shows that X is not star-K-Hurewicz. □

Example 2.11. *There exists a Tychonoff star-K-Hurewicz space having a regular-closed subspace which is not star-K-Hurewicz.*

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$.

Let S_1 be the same space X in the proof of Example 2.4. Then S_1 is a Tychonoff star-K-Hurewicz space.

Let

$$S_2 = (\beta D \times [0, \mathfrak{c})) \cup (D \times \{\mathfrak{c}\})$$

be the subspace of the product space $\beta D \times [0, \mathfrak{c}]$. By Lemma 2.10, S_2 is not star-K-Hurewicz.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\mathfrak{c}^+\} \to D \times \{\mathfrak{c}\}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \mathfrak{c}^+\rangle$ of S_1 with $\pi(\langle d_\alpha, \mathfrak{c}^+\rangle\})$ of S_2 for every $\alpha < \mathfrak{c}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. It is clear that $\varphi(S_2)$ is a regular-closed subspace of X which is not star-K-Hurewicz, since it is homeomorphic to S_2.

Finally we show that X is star-K-Hurewicz. We only show that X is K-starcompact, since every K-starcompact space is star-K-Hurewicz. To this end, let $\mathcal{U}$ be an open cover of X. Since $\varphi(S_1)$ is homeomorphic to S_1, then $\varphi(S_1)$ is K-starcompact. Thus there exists a compact subset K_1 of $\varphi(S_1)$ such that

$$\varphi(S_1) \subseteq St(K_1, \mathcal{U}).$$

Since $\varphi(\beta D \times [0, \mathfrak{c}))$ is homeomorphic to $\beta D \times [0, \mathfrak{c})$, the set $\varphi(\beta D \times [0, \mathfrak{c}))$ is countably compact, hence it is strongly starcompact. Thus we can find a finite subset K_2 of $\varphi(\beta D \times [0, \mathfrak{c}))$ such that

$$\varphi(\beta D \times [0, \mathfrak{c})) \subseteq St(K_2, \mathcal{U}).$$

If we put $K = K_1 \cup K_2$, then K is a compact subset of X such that $X = St(K, \mathcal{U})$, which shows that X is K-starcompac. □

We give a positive result on star-K-Hurewicz spaces:

Theorem 2.12. *An open and closed subset of a star-K-Hurewicz space is star-K-Hurewicz.*

Proof. Let X be a star-K-Hurewicz space and let Y be an open and closed subset of X. To show that Y is star-K-Hurewicz, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y, we have to find a sequence $\{F_n : n \in \mathbb{N}\}$ of compact subsets of Y such that for each $y \in Y$, $y \in St(F_n, \mathcal{U}_n)$ for all but finitely many $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus Y\}.$$

Then $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of open covers of X, so there exists a sequence $\{F'_n : n \in \mathbb{N}\}$ of compact subsets of X such that for each $x \in X$, $x \in St(F_n, \mathcal{V}_n)$ for all but finitely many $n \in \mathbb{N}$, since X is star-K-Hurewicz. For each $n \in \mathbb{N}$, let $F_n = F'_n \cap Y$. Thus $\{F_n : n \in \mathbb{N}\}$ is a sequence of compact subsets of Y, since Y is a closed subset of X. For each $y \in Y$, if $y \in St(F'_n, \mathcal{V}_n)$, then $y \in St(F_n, \mathcal{U}_n)$ by the construction of $\mathcal{U}_n$. Hence the sequence $\{F_n : n \in \mathbb{N}\}$ of compact subsets of Y witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that Y is star-K-Hurewicz. Therefore we complete the proof. □

Since a continuous image of a K-starcompact space is K-starcompact, it is not difficult to show the following result.

Theorem 2.13. *A continuous image of a star-K-Hurewicz space is star-K-Hurewicz.*

Proof. Let $f : X \to Y$ be a continuous mapping from a star-K-Hurewicz space X onto a space Y. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Since X is star-K-Hurewicz, there exists a sequence $(K'_n : n \in N)$ of compact subsets of X such that for each $x \in X$, $x \in St(K'_n, \mathcal{V}_n)$ for all but finitely many n. For each $n \in \mathbb{N}$, let $K_n = f(K'_n)$. Then $(K_n : n \in \mathbb{N})$ is a sequence of compact subsets of Y such that for each $y \in Y$, $y \in St(K_n, \mathcal{U}_n)$ for all but finitely many n. In fact, let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Hence $x \in St(K'_n, \mathcal{V}_n)$ for all but finitely many n. Thus $y = f(x) \in St(f(K'_n), \{U : U \in \mathcal{U}_n\}) = St(K_n, \mathcal{U}_n)$ for all but finitely many n, which shows that Y is star-K-Hurewicz. □

Next we turn to consider preimages. To show that the preimage of a star-K-Hurewicz space under a closed 2-to-1 continuous map need not be star-K-Hurewicz, we use the the Alexandroff duplicate $A(X)$ of a space X. The underlying set $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 0\rangle\})$, where U is a neighborhood of x in X.

Example 2.14. *There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is a star-K-Hurewicz space, but X is not star-K-Hurewicz.*

Proof. Let Y be the same space X in the proof of Example 2.4. As we proved in Example 2.4 above, Y is star-K-Hurewicz. Let X be the Alexandorff duplicate $A(Y)$. Then X is not star-K-Hurewicz. In fact, let $A = \{\langle\langle d_\alpha, \mathfrak{c}^+\rangle, 1\rangle : \alpha < \mathfrak{c}\}$. Then A is an open and closed subset of X with $|A| = \mathfrak{c}$, and each point $\langle\langle d_\alpha, \mathfrak{c}^+\rangle, 1\rangle$ is isolated. Hence $A(X)$ is not star-K-Hurewicz by Theorem 2.12. Let $f : X \to Y$ be the projection. Then f is a closed 2-to-1 continuous map, which completes the proof. □

In [15], the author showed that the preimage of a star-K-Menger space under an open perfect map is star-K-Menger, similarly we can prove the following result:

Theorem 2.15. *Let f be an open perfect map from a space X to a star-K-Hurewicz space Y. Then X is star-K-Hurewicz.*

By Theorem 2.15 we have the following corollary.

Corollary 2.16. *Let X be a star-K-Hurewicz space and Y a compact space. Then $X \times Y$ is star-K-Hurewicz.*

Remark 2.17. Example 2.16 in [13] shows that the product of two star-K-Hurewicz spaces need not be star-K-Hurewicz.

Acknowledgment

The author would like to thank Prof. Rui Li for his kind help and valuable suggestions. He would also like to thank the referees for their careful reading of the paper and a number of valuable suggestions which led to improvements on several places.

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