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On the Non-Archimedean and Random Approximately General Additive Mappings: Direct and Fixed Point Methods

H. Azadi Kenary[a], M.H. Eghtesadifard[b]

[a]*Department of Mathematics, College of Sciences, Yasouj University, Yasouj 75918-74831, Iran*
[b]*The Instructor of Fars Education Department,Shiraz,Iran*

Abstract. In this paper, we prove the Hyers-Ulam stability of the following generalized additive functional equation

$$\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

where m is a positive integer greater than 3, in various normed spaces.

1. Introduction and Preliminaries

Let Γ^+ denote the set of all probability distribution functions $F:\mathbb{R}\cup[-\infty,+\infty]\to[0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0)=0, F(+\infty)=1$. It is clear that the set $D^+=\{F\in\Gamma^+ : l^-F(-\infty)=1\}$, where $l^-f(x)=\lim_{t\to x^-}f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F\le G$ if and only if $F(t)\le G(t)$ for all $t\in\mathbb{R}$. For any $a\ge 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t)=\begin{cases} 0, & \text{if} \quad t\le a, \\ 1, & \text{if} \quad t>a. \end{cases}.$$

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?*. If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [45] in 1940.

In the next year, Hyers [22] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [33] proved a generalization of Hyers' theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta [20] by replacing the bound $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$.

In 1897, Hensel [21] introduced a normed space which does not have the Archimedean property. It turned

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Email addresses: azadi@yu.ac.ir (H. Azadi Kenary), hadi.eghtesadee.fard@gmail.com (M.H. Eghtesadifard)

out that non-Archimedean spaces have many nice applications [23, 24].
The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([2]–[20], [26]–[43]).
The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$".

Example 1.1. *Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the* p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$ where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact filed.*

Arriola and Beyer [1] investigated the Hyers-Ulam stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ϵ: $|f(x + y) - f(x) - f(y)| \leq \epsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \epsilon$ for all $x \in \mathbb{Q}_p$.
However, the following example shows that the same result of Theorem 1.1 is not true in non-Archimedean normed spaces.

Example 1.2. *Let $p > 2$ and let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$. Then for $\epsilon = 1$, $|f(x + y) - f(x) - f(y)| = 1 \leq \epsilon$ for all $x, y \in \mathbb{Q}_p$. However, the sequences $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ and $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ are not Cauchy. In fact, by using the fact that $|2| = 1$, we have*

$$\left|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right| = |2^{-n} \cdot 2 - 2^{-(n+1)} \cdot 2| = |2^{-n}| = 1$$

and

$$\left|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right| = |2^n.2 - 2^{(n+1)}.2| = |2^{n+1}| = 1$$

for all $x, y \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence these sequences are not convergent in $\mathbb{Q}_p$.

In Sections 2 and 3, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [44].
The reader, can find the definitions of continuous triangular norm, random normed spaces, non-Archimedean field and non-Archimedean normed spaces, respectively , in, [2] and [3] .

Theorem 1.3. *[10, 11] Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty \tag{1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(a) $d(J^n x, J^{n+1}x) < \infty$ for all $n_0 \geq n_0$;
(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(c) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In this paper, we prove the Hyers-Ulam stability of the following functional equation:

$$\sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) = \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i) \tag{2}$$

in non-Archimedean and random normed spaces.
First, we introduce the following lemma due to A. Najati and A. Ramjbar [27] with $n = 3$ in (2).

Lemma 1.4. *Let X and Y be linear spaces. A mapping $f : X \to Y$ satisfies the equation*

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2[f(x)+f(y)+f(z)] \tag{3}$$

for all $x, y, z \in X$ if and only if f is additive.

Secondly , we introduce the following lemma due to J.M. Rassias and H.M. Kim [32].

Lemma 1.5. *Let X and Y be linear spaces and let $m \geq 3$ be a fixed positive integer. A mapping $f : X \to Y$ satisfies the equation*

$$\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)=\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)$$

for all $x_1, x_2, \cdots, x_m \in X$ if and only if f is an additive mapping.

2. Non-Archimedean Stability of Functional Equation (2): Fixed Point Alternative Method

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of functional equation (2) in non-Archimedean normed spaces.
Throughout this section, assume that X is a non-Archimedean normed space and that Y is a complete non-Archimedean normed space. Also $|m-1| \neq 1$.

Theorem 2.1. *Let $\zeta : X^m \to [0, \infty)$ be a function such that there exists $L < 1$ with*

$$|m-1|\zeta\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)\leq L\zeta(x_1,x_2,\cdots,x_m) \tag{4}$$

for all $x_1, x_2, \cdots, x_m \in X$. If $f : X \to Y$ is a mapping satisfying

$$\left\|\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)\right\|\leq\zeta(x_1,x_2,\cdots,x_m) \tag{5}$$

for all $x_1, x_2, \cdots, x_m \in X$, then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\|\leq\frac{|2|L\zeta(x,x,\cdots,x)}{|m||m-1|^2-|m||m-1|^2L}. \tag{6}$$

Proof. Putting $x_1 = \cdots = x_m = x$ in (5), we have

$$\left\|\frac{m!}{2!(m-2)!}f((m-1)x)-\frac{m(m-1)^2}{2}f(x)\right\|\leq\zeta(x,x,\cdots,x) \tag{7}$$

for all $x \in X$. Replacing x by $\frac{x}{m-1}$ in (7), we obtain

$$\begin{aligned}\left\|(m-1)f\left(\frac{x}{m-1}\right)-f(x)\right\| &\leq \frac{|2|}{|m^2-m|}\zeta\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right) \\ &\leq \frac{|2|L\zeta(x,x,\cdots,x)}{|m^2-m||m-1|}.\end{aligned} \tag{8}$$

for all $x \in X$. Consider the set $S^* := \{g : X \to Y\}$ and the generalized metric d^* in S^* defined by

$$d^*(g,h) = \inf\left\{\mu \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq \mu\zeta(x,x,\cdots,x),\ \forall x \in X\right\}, \tag{9}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S^*, d^*) is complete (see [26], Lemma 2.1). Now, we consider a linear mapping $J^* : S^* \to S^*$ such that

$$J^*h(x) := (m-1)h\left(\frac{x}{m-1}\right) \tag{10}$$

for all $x \in X$. Let $g,h \in S^*$ be arbitrary. Denote $\epsilon = d^*(g,h)$. We will show that $d^*(Jg, Jh) \leq L\epsilon$. Since $\|g(x) - h(x)\| \leq \epsilon\zeta(x,x,\cdots,x)$ for all $x \in X$, we get

$$\begin{aligned}
\|J^*g(x) - J^*h(x)\| &= \left\|(m-1)g\left(\frac{x}{m-1}\right) - (m-1)h\left(\frac{x}{m-1}\right)\right\| \\
&\leq |m-1|\epsilon\zeta\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right) \\
&\leq |m-1|\epsilon\frac{L\zeta(x,x,\cdots,x)}{|m-1|}
\end{aligned}$$

for all $x \in X$. Thus $d^*(g,h) = \epsilon$ implies that $d^*(J^*g, J^*h) \leq L\epsilon$. This means that $d^*(J^*g, J^*h) \leq Ld^*(g,h)$ for all $g,h \in S^*$. It follows from (8) that $d^*(f, J^*f) \leq \dfrac{|2|L}{|m^2-m||m-1|}$. By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J^*, that is,

$$A\left(\frac{x}{m-1}\right) = \frac{A(x)}{m-1} \tag{11}$$

for all $x \in X$. The mapping A is a unique fixed point of J^* in the set $\Omega = \{h \in S^* : d^*(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (11) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x) - A(x)\| \leq \mu\zeta(x,x,\cdots,x)$ for all $x \in X$.
(2) $d^*(J^{*n}f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right) = A(x)$$

for all $x \in X$.
(3) $d^*(f,A) \leq \dfrac{d^*(f,J^*f)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d^*(f,A) \leq \frac{|2|L}{|m||m-1|^2 - |m||m-1|^2L}. \tag{12}$$

This implies that the inequality (6) holds. By (5), we have

$$\begin{aligned}
&\left\|\sum_{1\leq i<j\leq m} A\left(\frac{x_i+x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i)\right\| \\
&= \lim_{n\to\infty}\left\|(m-1)^n\left[\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2(m-1)^n} + \sum_{l=1,k_l\neq i,j}^{n-2}\frac{x_{k_l}}{(m-1)^n}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\right]\right\| \\
&\leq \lim_{n\to\infty}|m-1|^n\zeta\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right) \\
&\leq \lim_{n\to\infty}|m-1|^n \cdot \frac{L^n\zeta(x_1,x_2,\cdots,x_m)}{|m-1|^n}
\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $n \geq 1$ and so

$$\left\| \sum_{1 \leq i < j \leq m} A\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l} \right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} A(x_i) \right\| = 0$$

for all $x_1, x_2, \cdots, x_m \in X$. On the other hand

$$(m-1)A\left(\frac{x}{m-1}\right) - A(x) = \lim_{n\to\infty} (m-1)^{n+1} f\left(\frac{x}{(m-1)^{n+1}}\right) - \lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right) = 0.$$

Therefore, the mapping $A : X \to Y$ is additive. This completes the proof. $\square$

Corollary 2.2. *Let $\theta \geq 0$ and p be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\left\| \sum_{1 \leq i < j \leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{m-2} x_{k_l} \right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i) \right\| \leq \theta \left(\sum_{i=1}^{m} \|x_i\|^p \right) \tag{13}$$

for all $x_1, x_2, \cdots, x_m \in X$. Then the limit $A(x) = \lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{m|2||m-1|\theta\|x\|^p}{|m|(|m-1|^{p+2} - |m-1|^3)}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 if we take $\zeta(x_1, x_2, \cdots, x_m) = \theta\left(\sum_{i=1}^{m} \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. In fact, if we choose $L = |m-1|^{1-p}$, then we get the desired result. $\square$

Theorem 2.3. *Let $\zeta : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\zeta(x_1, x_2, \cdots, x_m) \leq |m-1| L \zeta\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right) \tag{14}$$

for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ be a mapping satisfying (5). Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{|2|\zeta(x, x, \cdots, x)}{|m||m-1|^2 - |m||m-1|^2 L}. \tag{15}$$

Proof. It follows from (7) that

$$\left\| f(x) - \frac{f((m-1)x)}{m-1} \right\| \leq \frac{|2|\zeta(x, x, \cdots, x)}{|m||m-1|^2} \tag{16}$$

for all $x \in X$. Let (S^*, d^*) be the generalized metric space defined in the proof of Theorem 2.1.
Now, we consider a linear mapping $J : S^* \to S^*$ such that

$$Jh(x) := \frac{1}{m-1} f((m-1)x) \tag{17}$$

for all $x \in X$. Let $g, h \in S^*$ be arbitrary. Denote $\epsilon = d^*(g, h)$. We will show that $d^*(Jg, Jh) \leq L\epsilon$. Since $\|g(x) - h(x)\| \leq \epsilon\zeta(x, x, \cdots, x)$ for all $x \in X$, we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| = \left\| \frac{g((m-1)x)}{m-1} - \frac{h((m-1)x)}{m-1} \right\| &\leq \frac{\epsilon\zeta\left((m-1)x, (m-1)x, \cdots, (m-1)x\right)}{|m-1|} \\ &\leq \frac{|m-1|L\zeta(x, x, \cdots, x)}{|m-1|} \end{aligned}$$

for all $x \in X$. Thus $d^*(g,h) = \epsilon$ implies that $d^*(Jg, Jh) \leq L\epsilon$. This means that $d^*(Jg, Jh) \leq Ld^*(g,h)$ for all $g,h \in S$. It follows from (16) that

$$d^*(f, Jf) \leq \frac{|2|}{|m||m-1|^2}. \tag{18}$$

By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$A\left((m-1)x\right) = (m-1)A(x) \tag{19}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S^* : d^*(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (19) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x)-A(x)\| \leq \mu\zeta(x,x,\cdots,x)$ for all $x \in X$.
(2) $d^*(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n} = A(x)$ for all $x \in X$.
(3) $d^*(f,A) \leq \frac{d^*(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d^*(f,A) \leq \frac{|2|}{|m||m-1|^2 - |m||m-1|^2 L}. \tag{20}$$

This implies that the inequality (15) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and p be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying (13). Then the limit $A(x) = \lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\|f(x) - A(x)\| \leq \frac{m|2m-2|\theta\|x\|^p}{|m||m-1|^2(|m-1| - |m-1|^p)}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 if we take $\zeta(x_1, x_2, \cdots, x_m) = \theta\left(\sum_{i=1}^{m} \|x_i\|^p\right)$ for all $x_1, x_2, \cdots, x_m \in X$. In fact, if we choose $L = |m-1|^{p-1}$, then we get the desired result. □

3. Non-Archimedean stability of the functional equation (2): direct method

In this section, we prove the Hyers-Ulam stability of the functional equation (2) in non-Archimedean space. Throughout this section, assume that G is an additive semigroup and that X is a complete non-Archimedean space.

Theorem 3.1. *Let $\zeta : G^m \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} |m-1|^n \zeta\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right) = 0 \tag{21}$$

for all $x_1, x_2, \cdots, x_m \in G$. Suppose that, for any $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty} \max_{0\leq k<n} \left\{|m-1|^k \zeta\left(\frac{x}{(m-1)^{k+1}}, \frac{x}{(m-1)^{k+1}}, \cdots, \frac{x}{(m-1)^{k+1}}\right)\right\} \tag{22}$$

exists and $f : G \to X$ is a mapping satisfying

$$\left\| \sum_{1\leq i<j\leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i)\right\| \leq \zeta(x_1, x_2, \cdots, x_m). \tag{23}$$

Then, for all $x \in G$, $A(x) := \lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ exists and satisfies the

$$\|f(x) - T(x)\| \leq \frac{|2|\Psi(x)}{|m^2 - m|}. \tag{24}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty}\max_{j\leq k<n+j}\left\{|m-1|^k\zeta\left(\frac{x}{(m-1)^{k+1}}, \frac{x}{(m-1)^{k+1}}, \cdots, \frac{x}{(m-1)^{k+1}}\right)\right\} = 0, \tag{25}$$

then T is the unique additive mapping satisfying (24).

Proof. By (8), we get

$$\left\|(m-1)f\left(\frac{x}{m-1}\right) - f(x)\right\| \leq \frac{|2|}{|m^2-m|}\zeta\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right). \tag{26}$$

for all $x \in G$. Replacing x by $\dfrac{x}{(m-1)^n}$ in (26), we obtain

$$\begin{aligned}&\left\|(m-1)^{n+1}f\left(\frac{x}{(m-1)^{n+1}}\right) - (m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\| \\ &\leq \frac{|2||m-1|^n}{|m^2-m|}\zeta\left(\frac{x}{(m-1)^{n+1}}, \frac{x}{(m-1)^{n+1}}, \cdots, \frac{x}{(m-1)^{n+1}}\right).\end{aligned} \tag{27}$$

Thus, it follows from (21) and (27) that the sequence $\left\{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\}_{n\geq 1}$ is a Cauchy sequence. Since X is complete, it follows that $\left\{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\}_{n\geq 1}$ is convergent. Set $T(x) := \lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$. By induction, one can show that

$$\begin{aligned}&\left\|(m-1)^n f\left(\frac{x}{(m-1)^n}\right) - f(x)\right\| \\ &\leq \frac{|2|}{|m^2-m|}\max_{0\leq k<n}\left\{|m-1|^k\zeta\left(\frac{x}{(m-1)^{k+1}}, \frac{x}{(m-1)^{k+1}}, \cdots, \frac{x}{(m-1)^{k+1}}\right)\right\}\end{aligned} \tag{28}$$

for all $n \geq 1$ and $x \in G$. By taking $n \to \infty$ in (28) and using (22), one obtains (24). By (21) and (23), we get

$$\begin{aligned}&\left\|\sum_{1\leq i<j\leq m} A\left(\frac{x_i + x_j}{2} + \sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i)\right\| \\ &= \lim_{n\to\infty}\left\|(m-1)^n\Big[\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2(m-1)^n} + \sum_{l=1,k_l\neq i,j}^{n-2}\frac{x_{k_l}}{(m-1)^n}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\Big]\right\| \\ &\leq \lim_{n\to\infty}|m-1|^n\zeta\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right) \\ &= 0\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in G$ and $n \geq 1$. Therefore, the mapping $T : G \to X$ satisfies (2).

To prove the uniqueness property of A, let L be another mapping satisfying (24). Then we have

$$\begin{aligned}
&\left\|A(x)-L(x)\right\| \\
&= \lim_{j\to\infty} |m-1|^j \left\|A\left(\frac{x}{(m-1)^j}\right) - L\left(\frac{x}{(m-1)^j}\right)\right\| \\
&\le \lim_{j\to\infty} |m-1|^j \max\left\{\left\|A\left(\frac{x}{(m-1)^j}\right) - f\left(\frac{x}{(m-1)^j}\right)\right\|, \left\|f\left(\frac{x}{(m-1)^j}\right) - L\left(\frac{x}{(m-1)^j}\right)\right\|\right\} \\
&\le \frac{|2|}{|m^2-m|} \lim_{j\to\infty}\lim_{n\to\infty}\max_{j\le k<n+j}\left\{|m-1|^k \zeta\left(\frac{x}{(m-1)^k}, \frac{x}{(m-1)^k}, \cdots, \frac{x}{(m-1)^k}\right)\right\} \\
&= 0
\end{aligned}$$

for all $x \in G$. Therefore, $A = L$. This completes the proof. □

Corollary 3.2. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi\left(\frac{t}{|m-1|}\right) \le \xi\left(\frac{1}{|m-1|}\right)\xi(t), \quad \xi\left(\frac{1}{|m-1|}\right) < \frac{1}{|m-1|}$$

for all $t \ge 0$. Let $\kappa > 0$ and $f : G \to X$ be a mapping such that

$$\left\|\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2} + \sum_{l=1,k_l\ne i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^m f(x_i)\right\| \le \kappa\left(\sum_{i=1}^m \xi(|x_i|)\right) \tag{29}$$

for all $x_1, x_2, \cdots, x_m \in G$. Then there exists a unique additive mapping $A : G \to X$ such that

$$\|f(x) - A(x)\| \le \frac{m|2|\kappa\xi(|x|)}{|m^2-m||m-1|}.$$

Proof. If we define $\zeta : G^m \to [0,\infty)$ by $\zeta(x_1, x_2, \cdots, x_m) := \kappa\left(\sum_{i=1}^m \xi(|x_i|)\right)$, then we have

$$\begin{aligned}
&\lim_{n\to\infty} |m-1|^n \zeta\left(\frac{x}{(m-1)^n}, \frac{x}{(m-1)^n}, \cdots, \frac{x}{(m-1)^n}\right) \\
&\le \lim_{n\to\infty}\left[|m-1|\xi\left(\frac{1}{|m-1|}\right)\right]^n \left[\kappa\left(\sum_{i=1}^m \xi(|x_i|)\right)\right] = 0
\end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in G$. On the other hand, for all $x \in G$,

$$\begin{aligned}
\Psi(x) &= \lim_{n\to\infty}\max_{0\le k<n}\left\{|m-1|^k \zeta\left(\frac{x}{(m-1)^{k+1}}, \frac{x}{(m-1)^{k+1}}, \cdots, \frac{x}{(m-1)^{k+1}}\right)\right\} \\
&= \zeta\left(\frac{x}{m-1}, \frac{x}{m-1}, \cdots, \frac{x}{m-1}\right) \\
&= \frac{m\kappa\xi(|x|)}{|m-1|}
\end{aligned}$$

exists. Also, we have

$$\begin{aligned}
&\lim_{j\to\infty}\lim_{n\to\infty}\max_{j\le k<n+j}\left\{|m-1|^k \zeta\left(\frac{x}{(m-1)^{k+1}}, \frac{x}{(m-1)^{k+1}}, \cdots, \frac{x}{(m-1)^{k+1}}\right)\right\} \\
&= \lim_{j\to\infty} |m-1|^j \zeta\left(\frac{x}{(m-1)^{j+1}}, \frac{x}{(m-1)^{j+1}}, \cdots, \frac{x}{(m-1)^{j+1}}\right) \\
&= 0.
\end{aligned}$$

Thus, applying Theorem 3.1, we have the conclusion. This completes the proof. □

Theorem 3.3. *Let $\zeta : G^m \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\zeta((m-1)^n x_1, (m-1)^n x_2, \cdots, (m-1)^n x_m)}{|m-1|^n} = 0 \tag{30}$$

for all $x_1, x_2, \cdots, x_m \in G$. Suppose that, for any $x \in G$, the limit

$$\Psi(x) = \lim_{n\to\infty} \max_{0\le k<n} \left\{ \frac{\zeta((m-1)^k x, (m-1)^k x, \cdots, (m-1)^k x)}{|m-1|^{k+1}} \right\} \tag{31}$$

exists and $f : G \to X$ is a mapping satisfying (23), then, the limit $T(x) := \lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in G$ and satisfies the

$$\|f(x) - T(x)\| \le \frac{|2|\Psi(x)}{|m||m-1|}. \tag{32}$$

Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\le k<n+j} \left\{ \frac{\zeta((m-1)^k x, (m-1)^k x, \cdots, (m-1)^k x)}{|m-1|^{k+1}} \right\} = 0, \tag{33}$$

then T is the unique mapping satisfying (32).

Proof. By (7), we have

$$\left\| f(x) - \frac{f((m-1)x)}{m-1} \right\| \le \frac{|2|\zeta(x, x, \cdots, x)}{|m||m-1|^2} \tag{34}$$

for all $x \in G$. Replacing x by $(m-1)^n x$ in (34), we obtain

$$\left\| \frac{f((m-1)^n x)}{(m-1)^n} - \frac{f((m-1)^{n+1} x)}{(m-1)^{n+1}} \right\| \le \frac{|2|\zeta((m-1)^n x, \cdots, (m-1)^n x)}{|m||m-1|^{n+2}}. \tag{35}$$

Thus it follows from (30) and (35) that the sequence $\left\{ \frac{f((m-1)^n x)}{(m-1)^n} \right\}_{n\ge1}$ is convergent. Set $T(x) := \lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$. On the other hand, it follows from (35) that

$$\begin{aligned}
&\left\| \frac{f((m-1)^p x)}{(m-1)^p} - \frac{f((m-1)^q x)}{(m-1)^q} \right\| \\
&= \left\| \sum_{k=p}^{q-1} \frac{f((m-1)^k x)}{(m-1)^k} - \frac{f((m-1)^{k+1} x)}{(m-1)^{k+1}} \right\| \\
&\le \max \left\{ \left\| \frac{f((m-1)^k x)}{(m-1)^k} - \frac{f((m-1)^{k+1} x)}{(m-1)^{k+1}} \right\| \ : \ p \le k < q-1 \right\} \\
&\le \frac{|2|}{|m||m-1|} \max \left\{ \frac{\zeta((m-1)^k x, (m-1)^k x, \cdots, (m-1)^k x)}{|m-1|^{k+1}} : p \le k < q \right\}
\end{aligned}$$

for all $x \in G$ and all integers $p, q \ge 0$ with $q > p \ge 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (31), we obtain (32).

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

Corollary 3.4. *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi(|m-1|t) \leq \xi(|m-1|)\xi(t), \quad \xi(|m-1|) < |m-1|$$

for all $t \geq 0$. Let $\kappa > 0$ and $f : G \to X$ be a mapping satisfying (29). Then there exists a unique additive mapping $A : G \to X$ such that

$$\|f(x) - A(x)\| \leq \frac{|2|\kappa\,[\xi(|x|)]^m}{|m||m-1|^2}.$$

Proof. If we define $\zeta : G^m \to [0,\infty)$ by $\zeta(x_1, x_2, \cdots, x_m) := \kappa\left(\prod_{i=1}^{m} \xi(|x_i|)\right)$ and apply Theorem 3.3, then we get the conclusion. □

4. Random Stability of Functional Equation (2): Fixed Point Method

Throughout this section, using the fixed point alternative approach, we prove Hyers-Ulam stability of functional equation (2) in random normed spaces.

Theorem 4.1. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^m to D^+ ($\Phi(x_1, \cdots, x_m)$ is denoted by $\Phi_{x_1,\cdots,x_m}$) such that there exists $0 < \alpha < \frac{1}{m-1}$ such that*

$$\Phi_{(m-1)x_1,(m-1)x_2,\cdots,(m-1)x_m}(t) \leq \Phi_{x_1,x_2,\cdots,x_m}(\alpha t) \tag{36}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying

$$\mu_{\sum\limits_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2} + \sum\limits_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum\limits_{i=1}^{m} f(x_i)}(t) \geq \Phi_{x_1,x_2,\cdots,x_m}(t) \tag{37}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then, for all $x \in X$

$$A(x) := \lim_{n\to\infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)$$

exists and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,x,\cdots,x}\left(\frac{((m^2-m) - m(m-1)^2\alpha)t}{2\alpha}\right) \tag{38}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $x_1 = \cdots = x_m = x$ in (37), we obtain

$$\mu_{\frac{m(m-1)}{2} f((m-1)x) - \frac{m(m-1)^2}{2} f(x)}(t) \geq \Phi_{x,,x\cdots,x}(t) \tag{39}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf_{u\in(0,\infty)} \left\{\mu_{g(x)-h(x)}(ut) \geq \Phi_{x,x,\cdots,x}(t),\ \forall x \in X,\ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [26], Lemma 2.1).
Now, we consider a linear mapping $J : (S,d) \to (S,d)$ such that $Jh(x) := (m-1)h\left(\frac{x}{m-1}\right)$ for all $x \in X$.
First, we prove that J is a strictly contractive mapping with the Lipschitz constant $(m-1)\alpha$.

In fact, let $g,h \in S$ be such that $d(g,h) < \epsilon$. Then we have $\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,x,\cdots,x}(t)$ for all $x \in X$ and $t > 0$ and so

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}((m-1)\alpha\epsilon t) &= \mu_{(m-1)g(\frac{x}{m-1})-(m-1)h(\frac{x}{m-1})}((m-1)\alpha\epsilon t) \\ &= \mu_{g(\frac{x}{m-1})-h(\frac{x}{m-1})}(\alpha\epsilon t) \\ &\geq \Phi_{\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}}(\alpha t) \\ &\geq \Phi_{x,x,\cdots,x}(t) \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) < \epsilon$ implies that $d(Jg,Jh) < (m-1)\alpha\epsilon$. This means that $d(Jg,Jh) \leq (m-1)\alpha d(g,h)$ for all $g,h \in S$. It follows from (39) that

$$\begin{aligned} \mu_{f(x)-(m-1)f(\frac{x}{m-1})}(t) &\geq \Phi_{\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}}\left(\frac{m(m-1)t}{2}\right) \\ &\geq \Phi_{x,x,\cdots,x}\left(\frac{m(m-1)t}{2\alpha}\right). \end{aligned} \tag{40}$$

So $d(f,Jf) \leq \dfrac{2\alpha}{m(m-1)}$. By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{m-1}\right) = \frac{1}{m-1}A(x) \tag{41}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (41) such that there exists $u \in (0,\infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x,\cdots,x}(t)$ for all $x \in X$ and $t > 0$.
(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right) = A(x)$$

for all $x \in X$.
(3) $d(f,A) \leq \dfrac{d(f,Jf)}{1-(m-1)\alpha}$ with $f \in \Omega$, which implies the inequality

$$d(f,A) \leq \frac{2\alpha}{(m^2-m)-m(m-1)^2\alpha}$$

and so

$$\mu_{f(x)-A(x)}\left(\frac{2\alpha t}{(m^2-m)-m(m-1)^2\alpha}\right) \geq \Phi_{x,x,\cdots,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (38) holds. Now, we have

$$\begin{aligned} &\mu_{(m-1)^n\left[\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\right]}(t) \\ &\geq \Phi_{\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}}\left(\frac{t}{(m-1)^n}\right) \end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$ and $n \geq 1$ and so, from (36), it follows that

$$\Phi_{\frac{x_1}{(m-1)^n},\frac{x_2}{(m-1)^n},\cdots,\frac{x_m}{(m-1)^n}}\left(\frac{t}{(m-1)^n}\right) \geq \Phi_{x_1,x_2,\cdots,x_m}\left(\frac{t}{(m-1)^n\alpha^n}\right)$$

Since $\lim_{n\to\infty} \Phi_{x_1,x_2,\cdots,x_m}\left(\frac{t}{(m-1)^n\alpha^n}\right) = 1$ for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$, we have

$$\mu_{\sum_{1\le i<j\le m} A\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{m-2} x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} A(x_i)}(t) = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ satisfies (2).
On the other hand

$$\begin{aligned}
&A((m-1)x) - (m-1)A(x)\\
&= (m-1)\left[\lim_{n\to\infty}(m-1)^{n-1} f\left(\frac{x}{(m-1)^{n-1}}\right) - \lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right]\\
&= 0.
\end{aligned}$$

This completes the proof. □

Corollary 4.2. *Let X be a real normed space, $\theta \ge 0$ and r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{\sum_{1\le i<j\le m} f\left(\frac{x_i+x_j}{2}+\sum_{l=1,k_l\neq i,j}^{n-2} x_{k_l}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)}(t) \ge \frac{t}{t+\theta\left(\sum_{i=1}^m \|x_i\|^r\right)} \tag{42}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then $A(x) = \lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that

$$\mu_{f(x)-A(x)}(t) \ge \frac{((m-1)^{r+1} - (m+1)^2)t}{((m-1)^{r+1} - (m+1)^2))t + 2\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 4.1 if we take $\Phi_{x_1,x_2,\cdots,x_m}(t) = \frac{t}{t+\theta\left(\sum_{i=1}^m \|x_i\|^r\right)}$ for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. In fact, if we choose $\alpha = (m-1)^{-r}$, then we get the desired result. □

Theorem 4.3. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^m to D^+ ($\Phi(x_1, x_2, \cdots, x_m)$ is denoted by $\Phi_{x_1,x_2,\cdots,x_m}$) such that for some $0 < \alpha < m-1$,*

$$\Phi_{\frac{x_1}{m-1},\frac{x_2}{m-1},\cdots,\frac{x_m}{m-1}}(t) \le \Phi_{x_1,x_2,\cdots,x_m}(\alpha t)$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying (37). Then the limit $A(x) := \lim_{n\to\infty}\frac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that for all $x \in X$ and $t > 0$

$$\mu_{f(x)-A(x)}(t) \ge \Phi_{x,x,\cdots,x}\left(\frac{m(m-1)(m-1-\alpha)t}{2}\right). \tag{43}$$

Proof. Putting $x_1 = \cdots = x_m = x$ in (37), we have

$$\mu_{\frac{f((m-1)x)}{m-1}-f(x)}(t) \ge \Phi_{x,x,\cdots,x}\left(\frac{m(m-1)^2 t}{2}\right) \tag{44}$$

for all $x \in X$ and $t > 0$.
Let (S, d) be the generalized metric space defined in the proof of Theorem 4.1. Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that $Jh(x) := \frac{1}{m-1}h((m-1)x)$ for all $x \in X$. It follows from (44) that

$d(f, Jf) \leq \dfrac{2}{m(m-1)^2}$. By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, that is,

$$A((m-1)x) = (m-1)A(x) \tag{45}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that A is a unique mapping satisfying (45) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,x,\cdots,x}(t)$ for all $x \in X$ and $t > 0$.
(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} \dfrac{f((m-1)^n x)}{(m-1)^n} = A(x)$ for all $x \in X$.
(3) $d(f, A) \leq \dfrac{d(f, Jf)}{1 - \frac{\alpha}{m-1}}$ with $f \in \Omega$, which implies the inequality

$$\mu_{f(x)-A(x)}\left(\frac{2t}{m(m-1)(m-1-\alpha)}\right) \geq \Phi_{x,x,\cdots,x}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (43) holds. The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let X be a real normed space, $\theta \geq 0$ and r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying (42). Then the limit $A(x) = \lim_{n\to\infty} \dfrac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{((m-1)^{r+1} - 1)t}{((m-1)^{r+1} - 1)t + 2(m-1)^{r-1}\theta\|x\|^r}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 4.1 if we take $\Phi_{x_1,x_2,\cdots,x_m}(t) = \frac{t}{t+\theta\left(\sum_{i=1}^{m}\|x_i\|^r\right)}$ for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. In fact, if we choose $\alpha = (m-1)^{-r}$, then we get the desired result. □

5. Random Stability of the Functional Equation (2): Direct Method

Throughout this section, using direct method, we prove the Hyers-Ulam stability of the functional equation (2) in random normed spaces.

Theorem 5.1. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\varphi : X^m \to Z$ be a function such that there exists $0 < \alpha < \frac{1}{m-1}$ such that*

$$\mu'_{\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right)}(t) \geq \mu'_{\alpha\varphi(x_1,x_2,\cdots,x_m)}(t) \tag{46}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi\left(\frac{x_1}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right)}\left(\dfrac{t}{(m-1)^n}\right) = 1$ for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping such that

$$\mu_{\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^{m} f(x_i)}(t) \geq \mu'_{\varphi(x_1,x_2,\cdots,x_m)}(t) \tag{47}$$

for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$, then the limit $A(x) = \lim_{n\to\infty}(m-1)^n f\left(\dfrac{x}{(m-1)^n}\right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)(1-(m-1)\alpha)t}{2\alpha}\right) \tag{48}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $x_1 = x_2 = \cdots = x_m = x$ in (47), we obtain

$$\mu_{f(x)-(m-1)f\left(\frac{x}{m-1}\right)}(t) \geq \mu'_{\varphi\left(\frac{x}{m-1},\frac{x}{m-1},\cdots,\frac{x}{m-1}\right)}\left(\frac{m(m-1)t}{2}\right) \tag{49}$$

for all $x \in X$. Replacing x by $\dfrac{x}{(m-1)^n}$ in (49) and using (46), we obtain

$$\begin{aligned}\mu_{(m-1)^{n+1}f\left(\frac{x}{(m-1)^{n+1}}\right)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right)}(t) &\geq \mu'_{\varphi\left(\frac{x}{(m-1)^{n+1}},\frac{x}{(m-1)^{n+1}},\cdots,\frac{x}{(m-1)^{n+1}}\right)}\left(\frac{m(m-1)t}{2(m-1)^n}\right)\\ &\geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2(m-1)^n\alpha^{n+1}}\right).\end{aligned}$$

Since

$$(m-1)^n f\left(\frac{x}{(m-1)^n}\right) - f(x) = \sum_{k=0}^{n-1}(m-1)^{k+1} f\left(\frac{x}{(m-1)^{k+1}}\right) - (m-1)^k f\left(\frac{x}{(m-1)^k}\right)$$

so we have

$$\begin{aligned}&\mu_{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)-f(x)}\left(\sum_{k=0}^{n-1}(m-1)^k\alpha^{k+1}t\right)\\ &= \mu_{\sum_{k=0}^{n-1}(m-1)^{k+1}f\left(\frac{x}{(m-1)^{k+1}}\right)-(m-1)^k f\left(\frac{x}{(m-1)^k}\right)}\left(\sum_{k=0}^{n-1}(m-1)^k\alpha^{k+1}t\right)\\ &\geq T_{k=0}^{n-1}\left(\mu_{(m-1)^{k+1}f\left(\frac{x}{(m-1)^{k+1}}\right)-(m-1)^k f\left(\frac{x}{(m-1)^k}\right)}((m-1)^k\alpha^{k+1}t)\right)\\ &\geq T_{k=0}^{n-1}\left(\mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2}\right)\right)\\ &= \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2}\right).\end{aligned}$$

This implies that

$$\mu_{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)-f(x)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2\sum_{k=0}^{n-1}(m-1)^k\alpha^{k+1}}\right). \tag{50}$$

Replacing x by $\frac{x}{(m-1)^p}$ in (50), we obtain

$$\mu_{(m-1)^{n+p}f\left(\frac{x}{(m-1)^{n+p}}\right)-(m-1)^p f\left(\frac{x}{(m-1)^p}\right)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2\sum_{k=p}^{n+p-1}(m-1)^k\alpha^{k+1}}\right). \tag{51}$$

Since

$$\lim_{p,n\to\infty}\mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2\sum_{k=p}^{n+p-1}(m-1)^k\alpha^{k+1}}\right) = 1,$$

it follows that $\left\{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $A(x) \in Y$ such that $\lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right) = A(x)$. Fix $x \in X$ and put $p = 0$ in (51). Then we obtain

$$\mu_{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)-f(x)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2\sum_{k=0}^{n-1}(m-1)^k\alpha^{k+1}}\right)$$

and so, for any $\epsilon > 0$,

$$\begin{aligned} \mu_{A(x)-f(x)}(t+\epsilon) &\geq T\left(\mu_{A(x)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right)}(\epsilon), \mu_{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)-f(x)}(t)\right) \\ &\geq T\left(\mu_{A(x)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right)}(\epsilon), \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)t}{2\sum_{k=0}^{n-1}(m-1)^k\alpha^{k+1}}\right)\right). \end{aligned} \tag{52}$$

Taking $n \to \infty$ in (52), we get

$$\mu_{A(x)-f(x)}(t+\epsilon) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)(1-(m-1)\alpha)t}{2\alpha}\right). \tag{53}$$

Since ϵ is arbitrary, by taking $\epsilon \to 0$ in (53), we get

$$\mu_{A(x)-f(x)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)(1-(m-1)\alpha)t}{2\alpha}\right).$$

Replacing $x_1, x_2, \cdots, x_m$ by $\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}$, respectively, in (47), we get

$$\begin{aligned} &\mu_{(m-1)^n\left[\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2(m-1)^n}+\sum_{l=1,k_l\neq i,j}^{m-2}\frac{x_{k_l}}{(m-1)^n}\right)-\frac{(m-1)^2}{2}\sum_{i=1}^{m} f\left(\frac{x_i}{(m-1)^n}\right)\right]}(t) \\ &\geq \mu'_{\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right)}\left(\frac{t}{(m-1)^n}\right) \end{aligned}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Since

$$\lim_{n\to\infty}\mu'_{\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right)}\left(\frac{t}{(m-1)^n}\right) = 1,$$

we conclude that A satisfies (2).
On the other hand

$$\begin{aligned} (m-1)A\left(\frac{x}{m-1}\right) - A(x) &= \lim_{n\to\infty}(m-1)^{n+1} f\left(\frac{x}{(m-1)^{n+1}}\right) - \lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right) \\ &= 0. \end{aligned}$$

This implies that $A : X \to Y$ is an additive mapping.
To prove the uniqueness of the additive mapping A, assume that there exists another additive mapping

$L : X \to Y$ which satisfies (48). Then we have

$$\begin{aligned}
&\mu_{A(x)-L(x)}(t)\\
&= \lim_{n\to\infty} \mu_{(m-1)^n A\left(\frac{x}{(m-1)^n}\right)-(m-1)^n L\left(\frac{x}{(m-1)^n}\right)}(t)\\
&\geq \lim_{n\to\infty} \min\left\{\mu_{(m-1)^n A\left(\frac{x}{(m-1)^n}\right)-(m-1)^n f\left(\frac{x}{(m-1)^n}\right)}\left(\frac{t}{2}\right), \mu_{(m-1)^n f\left(\frac{x}{(m-1)^n}\right)-(m-1)^n L\left(\frac{x}{(m-1)^n}\right)}\left(\frac{t}{2}\right)\right\}\\
&\geq \lim_{n\to\infty} \mu'_{\varphi\left(\frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n}\right)}\left(\frac{m(1-(m-1)\alpha)t}{4(m-1)^{n-1}\alpha}\right)\\
&\geq \lim_{n\to\infty} \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(1-(m-1)\alpha)t}{4(m-1)^{n-1}\alpha^n}\right).
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{m(1-(m-1)\alpha)t}{4(m-1)^{n-1}\alpha^n} = \infty$, we get

$$\lim_{n\to\infty}\lim_{n\to\infty} \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(1-(m-1)\alpha)t}{4(m-1)^{n-1}\alpha^n}\right) = 1.$$

Therefore, it follows that $\mu_{A(x)-L(x)}(t) = 1$ for all $t > 0$ and so $A(x) = L(x)$. This completes the proof. □

Corollary 5.2. *Let X be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let r be a positive real number with $r > 1$, $z_0 \in Z$ and $f : X \to Y$ be a mapping satisfying*

$$\mu_{\sum_{1\leq i<j\leq m} f\left(\frac{x_i+x_j}{2} + \sum_{l=1, k_l\neq i,j}^{m-2} x_{k_l}\right) - \frac{(m-1)^2}{2}\sum_{i=1}^m f(x_i)}(t) \geq \mu'_{\left(\sum_{i=1}^m \|x_i\|^r\right)z_0}(t) \tag{54}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty}(m-1)^n f\left(\frac{x}{(m-1)^n}\right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\|x\|^r z_0}\left(\frac{((m-1)-(m-1)^{2-r})t}{2(m-1)^{-r}}\right)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = (m-1)^{-r}$ and $\varphi : X^m \to Z$ be a mapping defined by $\varphi(x_1, x_2, \cdots, x_m) = \left(\sum_{i=1}^m \|x_i\|^r\right)z_0$. Then, from Theorem 5.1, the conclusion follows. □

Theorem 5.3. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\varphi : X^m \to Z$ be a function such that there exists $0 < \alpha < m-1$ such that*

$$\mu'_{\varphi(x_1,x_2,\cdots,x_m)}(t) \geq \mu'_{\alpha\varphi\left(\frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1}\right)}(t)$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$ and

$$\lim_{n\to\infty} \mu'_{\varphi((m-1)^n x_1, (m-1)^n x_2, \cdots, (m-1)^n x_m)}((m-1)^n x) = 1$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping satisfying (47). Then the limit $A(x) = \lim_{n\to\infty} \frac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{(m^2-m)(m-1-\alpha)t}{2}\right). \tag{55}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $x_1 = \cdots = x_m = x$ in (47), we have

$$\mu_{\frac{f((m-1)x)}{m-1} - f(x)}(t) \geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)^2 t}{2}\right) \tag{56}$$

for all $x \in X$ and $t > 0$. Replacing x by $(m-1)^n x$ in (56), we obtain that

$$\begin{aligned} \mu_{\frac{f((m-1)^{n+1}x)}{(m-1)^{n+1}} - \frac{f((m-1)^n x)}{(m-1)^n}}(t) &\geq \mu'_{\varphi((m-1)^n x,(m-1)x,\cdots,(m-1)^n x)}\left(\frac{m(m-1)^{n+2} t}{2}\right) \\ &\geq \mu'_{\varphi(x,x,\cdots,x)}\left(\frac{m(m-1)^{n+2} t}{2\alpha^n}\right). \end{aligned}$$

The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 5.4. *Let X be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let r be a positive real number with $0 < r < \frac{1}{m}$, $z_0 \in Z$ and $f : X \to Y$ is a mapping satisfying*

$$\mu_{\sum_{1 \leq i < j \leq m} f\left(\frac{x_i + x_j}{2} + \sum_{l=1, k_l \neq i,j}^{n-2} x_{k_l}\right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i)}(t) \geq \mu'_{\left(\prod_{i=1}^{m} \|x_i\|^r\right) z_0}(t) \tag{57}$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty} \dfrac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\|x\|^{mr} z_0}\left(\frac{m((m-1)^{mr+2} - (m-1))t}{2(m-1)^{mr}}\right)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = (m-1)^{-mr}$ and $\varphi : X^m \to Z$ be a mapping defined by

$$\varphi(x_1, x_2, \cdots, x_m) = \left(\prod_{i=1}^{m} \|x_i\|^r\right) z_0.$$

Then, from Theorem 5.3, the conclusion follows. □

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