On the Non-Archimedean and Random Approximately General Additive Mappings: Direct and Fixed Point Methods

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Abstract. In this paper, we prove the Hyers-Ulam stability of the following generalized additive functional equation
\[ \sum_{1 \leq i < j \leq m} f \left( \frac{1}{2} x_i + \frac{1}{2} x_j + \sum_{l \neq i,j} x_l \right) = \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i) \]
where \( m \) is a positive integer greater than 3, in various normed spaces.

1. Introduction and Preliminaries

Let \( \Gamma^+ \) denote the set of all probability distribution functions \( F : \mathbb{R} \cup [-\infty, +\infty) \to [0, 1] \) such that \( F \) is left-continuous and nondecreasing on \( \mathbb{R} \) and \( F(0) = 0, F(+\infty) = 1 \). It is clear that the set \( D^+ = \{ F \in \Gamma^+ : l^+ F(-\infty) = 1 \} \), where \( l^+ f(x) = \lim_{t \to x^+} f(t) \), is a subset of \( \Gamma^+ \). The set \( \Gamma^+ \) is partially ordered by the usual point-wise ordering of functions, that is, \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \in \mathbb{R} \). For any \( a \geq 0 \), the element \( H_a(t) \) of \( D^+ \) is defined by
\[ H_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases} \]

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [45] in 1940.

In the next year, Hyers [22] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [33] proved a generalization of Hyers' theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Gavruta [20] by replacing the bound \( \epsilon(\|x\|^p + \|y\|^p) \) by a general control function \( \phi(x, y) \).

In 1897, Hensel [21] introduced a normed space which does not have the Archimedean property. It turned
out that non-Archimedean spaces have many nice applications [23, 24].
The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([2]–[20], [26]–[43]).
The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: “for \(x, y > 0\), there exists \(n \in \mathbb{N}\) such that \(x < ny\).

**Example 1.1.** Fix a prime number \(p\). For any nonzero rational number \(x\), there exists a unique integer \(n_x \in \mathbb{Z}\) such that \(x = \frac{a}{p^n}\), where \(a\) and \(b\) are integers not divisible by \(p\). Then \(|x|_p := p^{-n}\) defines a non-Archimedean norm on \(\mathbb{Q}\). The completion of \(\mathbb{Q}\) with respect to the metric \(d(x, y) = |x - y|_p\) is denoted by \(\mathbb{Q}_p\), which is called the \(p\)-adic number field. In fact, \(\mathbb{Q}_p\) is the set of all formal series \(x = \sum_{k \geq 0} a_k p^k\) where \(|a_k| \leq p - 1\) are integers. The addition and multiplication between any two elements of \(\mathbb{Q}_p\) are defined naturally. The norm \(|\sum_{k \geq 0} a_k p^k|_p = p^{-n}\) is a non-Archimedean norm on \(\mathbb{Q}_p\) and it makes \(\mathbb{Q}_p\) a locally compact field.

Arriola and Beyer [1] investigated the Hyers-Ulam stability of approximate additive functions \(f : \mathbb{Q}_p \to \mathbb{R}\). They showed that if \(f : \mathbb{Q}_p \to \mathbb{R}\) is a continuous function for which there exists a fixed \(c\): 
\(|f(x + y) - f(x) - f(y)| \leq \epsilon\) for all \(x, y \in \mathbb{Q}_p\), then there exists a unique additive function \(T : \mathbb{Q}_p \to \mathbb{R}\) such that 
\(|f(x) - T(x)| \leq \epsilon\) for all \(x \in \mathbb{Q}_p\).

However, the following example shows that the same result of Theorem 1.1 is not true in non-Archimedean normed spaces.

**Example 1.2.** Let \(p > 2\) and let \(f : \mathbb{Q}_p \to \mathbb{Q}_p\) be defined by \(f(x) = 2\). Then for \(\epsilon = 1\), \(|f(x + y) - f(x) - f(y)| = 1 \leq \epsilon\) for all \(x, y \in \mathbb{Q}_p\). However, the sequences \(\left\{\left(\frac{f(2^n x)}{2^n}\right)_{n=1}^{\infty}\right\}\) and \(\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}\) are not Cauchy. In fact, by using the fact that \(|2| = 1\), we have

\[
\left|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}}\right| = \left|2^{-n} \cdot 2 - 2^{-n(n+1)} \cdot 2\right| = |2^{-n}| = 1
\]

and

\[
\left|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right| = \left|2^n \cdot 2 - 2^{n(n+1)} \cdot 2\right| = |2^{n+1}| = 1
\]

for all \(x, y \in \mathbb{Q}_p\) and \(n \in \mathbb{N}\). Hence these sequences are not convergent in \(\mathbb{Q}_p\).

In Sections 2 and 3, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [44].
The reader can find the definitions of continuous triangular norm, random normed spaces, non-Archimedean field and non-Archimedean normed spaces, respectively, in [2] and [3].

**Theorem 1.3.** [10, 11] Let \((X, d)\) be a complete generalized metric space and \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then, for all \(x \in X\), either

\[
d(\sum_{1 \leq i < j \leq m} f\left(\frac{x_i + x_j}{2}\right) + \sum_{i=1, k \neq i}^{m-2} x_k) = \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f(x_i)
\]

for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that
(a) \(d(f^n x, f^{n+1} x) < \infty\) for all \(n \geq n_0\);
(b) the sequence \(\{f^n x\}\) converges to a fixed point \(y^*\) of \(J\);
(c) \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X : d(f^n x, y) < \infty\}\);
(d) \(d(y, y^*) \leq \frac{1}{1-L} d(y, f y)\) for all \(y \in Y\).

In this paper, we prove the Hyers-Ulam stability of the following functional equation:

\[
\sum_{1 \leq i < j \leq m} f\left(\frac{x_i + x_j}{2}\right) + \sum_{i=1, k \neq i}^{m-2} x_k = \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f(x_i)
\]
in non-Archimedean and random normed spaces. First, we introduce the following lemma due to A. Najati and A. Ramjbar [27] with \( n = 3 \) in (2).

**Lemma 1.4.** Let \( X \) and \( Y \) be linear spaces. A mapping \( f : X \to Y \) satisfies the equation
\[
f\left(\frac{x + y}{2} + z\right) + f\left(\frac{x + z}{2} + y\right) + f\left(\frac{y + z}{2} + x\right) = 2[f(x) + f(y) + f(z)]
\] (3)
for all \( x, y, z \in X \) if and only if \( f \) is additive.

Secondly, we introduce the following lemma due to J.M. Rassias and H.M. Kim [32].

**Lemma 1.5.** Let \( X \) and \( Y \) be linear spaces and let \( m \geq 3 \) be a fixed positive integer. A mapping \( f : X \to Y \) satisfies the equation
\[
\sum_{1 \leq i < j \leq m} f\left(x_i + x_j + \sum_{l=1, l \neq i, j}^{m-2} x_l\right) = \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f(x_i)
\]
for all \( x_1, x_2, \ldots, x_m \in X \) if and only if \( f \) is an additive mapping.


In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of functional equation (2) in non-Archimedean normed spaces. Throughout this section, assume that \( X \) is a non-Archimedean normed space and that \( Y \) is a complete non-Archimedean normed space. Also \( |m - 1| \neq 1 \).

**Theorem 2.1.** Let \( \zeta : X^m \to [0, \infty) \) be a function such that there exists \( L < 1 \) with
\[
|m - 1|L\zeta\left(\frac{x}{m - 1}, \frac{x}{m - 1}, \ldots, \frac{x}{m - 1}\right) \leq L\zeta(x_1, x_2, \ldots, x_m)
\] (4)
for all \( x_1, x_2, \ldots, x_m \in X \). If \( f : X \to Y \) is a mapping satisfying
\[
\left\|\sum_{1 \leq i < j \leq m} f\left(x_i + x_j + \sum_{l=1, l \neq i, j}^{m-2} x_l\right) - \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f(x_i)\right\| \leq \zeta(x_1, x_2, \ldots, x_m)
\] (5)
for all \( x_1, x_2, \ldots, x_m \in X \), then there is a unique additive mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \frac{2L\zeta(x_1, x_2, \ldots, x)}{|m||m - 1|^2 - |m||m - 1|^2L}.
\] (6)

**Proof.** Putting \( x_1 = \ldots = x_m = x \) in (5), we have
\[
\left\|\frac{m!}{2!(m-2)!} f((m - 1)x) - \frac{m(m - 1)^2}{2} f(x)\right\| \leq \zeta(x, x, \ldots, x)
\] (7)
for all \( x \in X \). Replacing \( x \) by \( \frac{x}{m - 1} \) in (7), we obtain
\[
\left\|(m - 1)f\left(\frac{x}{m - 1}\right) - f(x)\right\| \leq \frac{2}{|m^2 - m|} \zeta\left(\frac{x}{m - 1}, \frac{x}{m - 1}, \ldots, \frac{x}{m - 1}\right)
\]
\[
\leq \frac{2L\zeta(x_1, x_2, \ldots, x)}{|m^2 - m||m - 1|}.
\] (8)
for all \( x \in X \). Consider the set \( S' := \{ g : X \to Y \} \) and the generalized metric \( d' \) in \( S' \) defined by

\[
d'(g, h) = \inf \{ \mu \in \mathbb{R}^+ : ||g(x) - h(x)|| \leq \mu \zeta(x, x, \cdots, x), \forall x \in X \},
\]

where \( \inf \emptyset = +\infty \). It is easy to show that \((S', d')\) is complete (see [26], Lemma 2.1). Now, we consider a linear mapping \( J' : S' \to S' \) such that

\[
J'h(x) := (m - 1)h\left( \frac{x}{m - 1} \right)
\]

for all \( x \in X \). Let \( g, h \in S' \) be arbitrary. Denote \( \epsilon = d'(g, h) \). We will show that \( d'(Jg, Jh) \leq \epsilon L \). Since \( ||g(x) - h(x)|| \leq \epsilon \zeta(x, x, \cdots, x) \) for all \( x \in X \), we get

\[
||Jg(x) - Jh(x)|| = \|(m - 1)g\left( \frac{x}{m - 1} \right) - (m - 1)h\left( \frac{x}{m - 1} \right)\|
\]

\[
\leq |m - 1|\epsilon \zeta\left( \frac{x}{m - 1}, \frac{x}{m - 1}, \cdots, \frac{x}{m - 1} \right)
\]

\[
\leq |m - 1|\epsilon \zeta(x, x, \cdots, x)
\]

for all \( x \in X \). Thus \( d'(g, h) = \epsilon \) implies that \( d'(Jg, Jh) \leq \epsilon L \). This means that \( d'(Jg, Jh) \leq \epsilon Ld'(g, h) \) for all \( g, h \in S' \). It follows from (8) that \( d'(f, f') \leq \frac{|2L|}{|m||m - 1|^2 - |m - 1|^2} \). By Theorem 1.3, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( f' \), that is,
   \[
   A\left( \frac{x}{m - 1} \right) = \frac{A(x)}{m - 1}
   \]

   for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( f' \) in the set \( \Omega = \{ h \in S' : d'(g, h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (11) such that there exists \( \mu \in (0, \infty) \) satisfying \( ||f(x) - A(x)|| \leq \mu \zeta(x, x, \cdots, x) \) for all \( x \in X \).

2. \( d'(f^n f, A) \to 0 \) as \( n \to \infty \). This implies the equality

   \[
   \lim_{n \to \infty} (m - 1)^n f\left( \frac{x}{(m - 1)^n} \right) = A(x)
   \]

   for all \( x \in X \).

3. \( d'(f, A) \leq \frac{d'(f, f')}{{1 - L}} \) with \( f \in \Omega \), which implies the inequality

   \[
   d'(f, A) \leq \frac{|2L|}{|m||m - 1|^2 - |m - 1|^2}.
   \]

This implies the inequality (6) holds. By (5), we have

\[
\left\| \sum_{1 \leq i < j \leq m} A\left( \frac{x_i + x_j}{2} + \sum_{l=1, l \neq i, j}^{m-2} x_l \right) - \frac{(m - 1)^2}{2} \sum_{i=1}^{m} A(x_i) \right\|
\]

\[
= \lim_{n \to \infty} \left\| (m - 1)^n \left[ \sum_{1 \leq i < j \leq m} f\left( \frac{x_i + x_j}{2(m - 1)^n} + \sum_{l=1, l \neq i, j}^{m-2} \frac{x_l}{(m - 1)^n} \right) - \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f\left( \frac{x_i}{(m - 1)^n} \right) \right] \right\|
\]

\[
\leq \lim_{n \to \infty} |m - 1|^n \epsilon \zeta\left( \frac{x_1}{(m - 1)^n}, \frac{x_2}{(m - 1)^n}, \cdots, \frac{x_m}{(m - 1)^n} \right) \]

\[
\leq \lim_{n \to \infty} |m - 1|^n \frac{\epsilon \zeta(x_1, x_2, \cdots, x_m)}{|m - 1|^n}
\]
for all $x_1, x_2, \cdots, x_m \in X$ and $n \geq 1$ and so
\[
\left\| \sum_{1 \leq i < j \leq m} A \left( \frac{x_i + x_j}{2} + \sum_{l=1, l \neq i, j}^{m-2} x_l \right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} A(x_i) \right\| = 0
\]
for all $x_1, x_2, \cdots, x_m \in X$. On the other hand
\[
(m-1)A \left( \frac{x}{m-1} \right) - A(x) = \lim_{n \to \infty} (m-1)^{n+1} f \left( \frac{x}{(m-1)^n} \right) - \lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right) = 0.
\]
Therefore, the mapping $A : X \to Y$ is additive. This completes the proof. \hfill \Box

**Corollary 2.2.** Let $\theta \geq 0$ and $p$ be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying
\[
\left\| \sum_{1 \leq i < j \leq m} f \left( \frac{x_i + x_j}{2} + \sum_{l=1, l \neq i, j}^{m-2} x_l \right) - \frac{(m-1)^2}{2} \sum_{i=1}^{m} f(x_i) \right\| \leq \theta \left( \sum_{i=1}^{m} \|x_i\|^p \right)
\]
for all $x_1, x_2, \cdots, x_m \in X$. Then the limit $A(x) = \lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that
\[
\| f(x) - A(x) \| \leq \frac{m^2 |m-1| \theta \|x\|^p}{m(m-1)^{p+2} - |m-1|^2}
\]
for all $x \in X$.

**Proof.** The proof follows from Theorem 2.1 if we take $\zeta(x_1, x_2, \cdots, x_m) = \theta \left( \sum_{i=1}^{m} \|x_i\|^p \right)$ for all $x_1, x_2, \cdots, x_m \in X$. In fact, if we choose $L = |m-1|^{2-p}$, then we get the desired result. \hfill \Box

**Theorem 2.3.** Let $\zeta : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with
\[
\zeta(x_1, x_2, \cdots, x_m) \leq |m-1|L \zeta \left( \frac{x_1}{m-1}, \frac{x_2}{m-1}, \cdots, \frac{x_m}{m-1} \right)
\]
for all $x_1, x_2, \cdots, x_m \in X$. Let $f : X \to Y$ be a mapping satisfying (5). Then there is a unique additive mapping $A : X \to Y$ such that
\[
\| f(x) - A(x) \| \leq \frac{2 \zeta \left( x, x, \cdots, x \right)}{|m-1|^2 - m|m-1|^2L}.
\]

**Proof.** It follows from (7) that
\[
\left\| f(x) - \frac{f((m-1)x)}{m-1} \right\| \leq \frac{2 \zeta \left( x, x, \cdots, x \right)}{|m-1|^2}
\]
for all $x \in X$. Let $(S', d')$ be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider a linear mapping $f : S' \to S^*$ such that
\[
f(x) := \frac{1}{m-1} f((m-1)x)
\]
for all $x \in X$. Let $g, h \in S'$ be arbitrary. Denote $e = d'(g, h)$. We will show that $d'(fg, fhl) \leq Le$. Since $\|g(x) - h(x)\| \leq e \zeta \left( x, x, \cdots, x \right)$ for all $x \in X$, we have
\[
\|fg(x) - fhl(x)\| \leq \frac{e \zeta \left( (m-1)x, (m-1)x, \cdots, (m-1)x \right)}{|m-1|^2} \leq \frac{e \zeta \left( (m-1)x, (m-1)x, \cdots, (m-1)x \right)}{|m-1|^2} \leq \frac{|m-1| \zeta \left( x, x, \cdots, x \right)}{|m-1|}
\]
for all \( x \in X \). Thus \( d'(g, h) = \epsilon \) implies that \( d'(g, h) \leq L \epsilon \). This means that \( d'(g, h) \leq Ld'(g, h) \) for all \( g, h \in S \). It follows from \( (16) \) that

\[
d'(f, f) \leq \frac{|2|}{m(m-1)^2}.
\]

By Theorem 1.3, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( f \), that is,
   \[ A((m-1)x) = (m-1)A(x) \]
   for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( f \) in the set \( \Omega = \{ h \in S^* : d'(g, h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying \( (19) \) such that there exists \( \mu \in (0, \infty) \) satisfying \( ||f(x) - A(x)|| \leq \mu \zeta(x, x, \cdots, x) \) for all \( x \in X \).

2. \( d'(f^n f, A) \to 0 \) as \( n \to \infty \). This implies the equality \( \lim_{n \to \infty} f((m-1)^n x) = A(x) \) for all \( x \in X \).

3. \( d'(f, A) \leq \frac{d'(f, f)}{1-L} \) with \( f \in \Omega \), which implies the inequality
   \[
d'(f, A) \leq \frac{|2|}{m(m-1)^2 - |m||m-1|^2 L}.
\]
   This implies that the inequality \( (15) \) holds. The rest of the proof is similar to the proof of Theorem 2.1.

**Corollary 2.4.** Let \( \theta \geq 0 \) and \( p \) be a real number with \( p > 1 \). Let \( f : X \to Y \) be a mapping satisfying \( (13) \). Then the limit \( A(x) = \lim_{n \to \infty} f((m-1)^n x) / (m-1)^n \) exists for all \( x \in X \) and \( A : X \to Y \) is a unique additive mapping such that

\[
||f(x) - A(x)|| \leq \frac{m|2m-2|\theta||x||p}}{|m|-1|\theta||x||p}}
\]

for all \( x \in X \).

**Proof.** The proof follows from Theorem 2.3 if we take \( \zeta(x_1, x_2, \cdots, x_m) = \theta \left( \sum_{i=1}^{m} ||x_i||^p \right) \) for all \( x_1, x_2, \cdots, x_m \in X \).

In fact, if we choose \( L = ||m-1||^{-1} \), then we get the desired result.

### 3. Non-Archimedean stability of the functional equation (2): direct method

In this section, we prove the Hyers-Ulam stability of the functional equation \( (2) \) in non-Archimedean space. Throughout this section, assume that \( G \) is an additive semigroup and that \( X \) is a complete non-Archimedean space.

**Theorem 3.1.** Let \( \zeta : C^m \to [0, +\infty) \) be a function such that

\[
\lim_{n \to \infty} |m-1|^n \zeta \left( \frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n} \right) = 0
\]

for all \( x_1, x_2, \cdots, x_m \in G \). Suppose that, for any \( x \in G \), the limit

\[
\psi(x) = \lim_{n \to \infty} \max_{0 \leq k < n} \left( |m-1|^k \zeta \left( \frac{x}{(m-1)^k}, \frac{x}{(m-1)^k}, \cdots, \frac{x}{(m-1)^k} \right) \right)
\]

exists and \( f : G \to X \) is a mapping satisfying

\[
\left\| \sum_{1 \leq i < j \leq m} f \left( \frac{x_i + x_j}{2} + \sum_{l=1, l \neq i, j}^{m-2} x_l \right) - \frac{(m-1)^2}{2} \sum_{l=1}^{m} f(x_l) \right\| \leq \zeta(x_1, x_2, \cdots, x_m).
\]

(23)
Then, for all \( x \in G \), \( A(x) := \lim_{n \to \infty} (m - 1)^n f \left( \frac{x}{(m - 1)^n} \right) \) exists and satisfies the

\[
\| f(x) - T(x) \| \leq \frac{2|\Psi(x)|}{|m^2 - m|}.
\] (24)

Moreover, if

\[
\lim \lim_{j \to \infty} \max_{k \geq c_n+j} \left\{ m - 1 \right\} \zeta \left( \frac{x}{(m - 1)^{k+1}}, \frac{x}{(m - 1)^{k+1}}, \ldots, \frac{x}{(m - 1)^{k+1}} \right) = 0,
\] (25)

then \( T \) is the unique additive mapping satisfying (24).

Proof. By (8), we get

\[
\left\| (m - 1)f \left( \frac{x}{m - 1} \right) - f(x) \right\| \leq \frac{2|\Psi(x)|}{|m^2 - m|} \zeta \left( \frac{x}{m - 1}, \frac{x}{m - 1}, \ldots, \frac{x}{m - 1} \right).
\] (26)

for all \( x \in G \). Replacing \( x \) by \( \frac{x}{(m - 1)^n} \) in (26), we obtain

\[
\left\| (m - 1)^{n+1}f \left( \frac{x}{(m - 1)^n} \right) - (m - 1)^n f \left( \frac{x}{(m - 1)^n} \right) \right\|
\leq \frac{2|\Psi(1)|}{|m^2 - m|} \zeta \left( \frac{x}{(m - 1)^{n+1}}, \frac{x}{(m - 1)^{n+1}}, \ldots, \frac{x}{(m - 1)^{n+1}} \right).
\] (27)

Thus, it follows from (21) and (27) that the sequence \( \left\{ (m - 1)^n f \left( \frac{x}{(m - 1)^n} \right) \right\}_{n \geq 1} \) is a Cauchy sequence. Since \( X \) is complete, it follows that \( \left\{ (m - 1)^n f \left( \frac{x}{(m - 1)^n} \right) \right\}_{n \geq 1} \) is convergent. Set \( T(x) := \lim_{n \to \infty} (m - 1)^n f \left( \frac{x}{(m - 1)^n} \right) \).

By induction, one can show that

\[
\left\| (m - 1)^n f \left( \frac{x}{(m - 1)^n} \right) - f(x) \right\| \leq \frac{2|\Psi(1)|}{|m^2 - m|} \max_{0 \leq k \leq n} \left\{ m - 1 \right\} \zeta \left( \frac{x}{(m - 1)^{k+1}}, \frac{x}{(m - 1)^{k+1}}, \ldots, \frac{x}{(m - 1)^{k+1}} \right)
\] (28)

for all \( n \geq 1 \) and \( x \in G \). By taking \( n \to \infty \) in (28) and using (22), one obtains (24). By (21) and (23), we get

\[
\left\| \sum_{1 \leq i < j \leq m} A \left( \frac{x_i + x_j}{2} + \sum_{i=1 \neq k}^{m-2} x_k \right) - \frac{(m - 1)^2}{2} \sum_{i=1}^{m} A(x_i) \right\|
\leq \lim_{n \to \infty} \left\| (m - 1)^n \left\| \sum_{1 \leq i < j \leq m} f \left( \frac{x_i + x_j}{2(m - 1)^n} + \sum_{i=1 \neq k}^{m-2} \frac{x_k}{(m - 1)^n} - \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f \left( \frac{x_i}{(m - 1)^n} \right) \right\| \right\|
\leq \lim_{n \to \infty} \left\| (m - 1)^n \zeta \left( \frac{x_1}{(m - 1)^n}, \frac{x_2}{(m - 1)^n}, \ldots, \frac{x_m}{(m - 1)^n} \right) \right\|
= 0
\]

for all \( x_1, x_2, \ldots, x_m \in G \) and \( n \geq 1 \). Therefore, the mapping \( T : G \to X \) satisfies (2).
To prove the uniqueness property of \( A \), let \( L \) be another mapping satisfying (24). Then we have
\[
\|A(x) - L(x)\| = \lim_{j \to \infty} |m - 1| \| A \left( \frac{x}{(m - 1)^j} \right) - L \left( \frac{x}{(m - 1)^j} \right) \|
\leq \lim_{j \to \infty} |m - 1| \max_{1 \leq i \leq m} \left\{ \| A \left( \frac{x}{(m - 1)^j} \right) - f \left( \frac{x}{(m - 1)^j} \right) \|, \| f \left( \frac{x}{(m - 1)^j} \right) - L \left( \frac{x}{(m - 1)^j} \right) \| \right\}
\leq \frac{2}{{|m^2 - m|}} \lim_{n \to \infty} \max_{1 \leq k \leq n} \left\{ |m - 1| \kappa \left( \frac{x}{(m - 1)^{j + 1}} \right), \frac{x}{(m - 1)^{j + 1}}, \cdots, \frac{x}{(m - 1)^{j + 1}} \right\}
= 0
\]
for all \( x \in G \). Therefore, \( A = L \). This completes the proof. \( \Box \)

**Corollary 3.2.** Let \( \xi : [0, \infty) \to [0, \infty) \) be a function satisfying
\[
\xi \left( \frac{t}{(m - 1)} \right) \leq \xi \left( \frac{1}{(m - 1)} \right) \xi (t), \quad \xi \left( \frac{1}{(m - 1)} \right) \leq \frac{1}{(m - 1)}
\]
for all \( t \geq 0 \). Let \( \kappa > 0 \) and \( f : G \to X \) be a mapping such that
\[\tag{29}
\left\| \sum_{1 \leq k \leq m} f \left( \frac{x_k + x_j}{2} + \sum_{i=1}^{m-2} x_i \right) - \frac{(m - 1)^2}{2} \sum_{i=1}^{m} f(x_i) \right\| \leq \kappa \left( \sum_{i=1}^{m} \xi (|x_i|) \right)
\]
for all \( x_1, x_2, \cdots, x_m \in G \). Then there exists a unique additive mapping \( A : G \to X \) such that
\[
\|f(x) - A(x)\| \leq \frac{m |2| \kappa \xi (|x|)}{|m^2 - m| |m - 1|}.
\]

**Proof.** If we define \( \zeta : G^m \to [0, \infty) \) by \( \zeta (x_1, x_2, \cdots, x_m) := \kappa \left( \sum_{i=1}^{m} \xi (|x_i|) \right) \), then we have
\[
\lim_{n \to \infty} |m - 1|^n \zeta \left( \frac{x}{(m - 1)^n}, \frac{x}{(m - 1)^n}, \cdots, \frac{x}{(m - 1)^n} \right)
\leq \lim_{n \to \infty} \left[ |m - 1| \zeta \left( \frac{1}{(m - 1)} \right) \right]^n \kappa \left( \sum_{i=1}^{m} \xi (|x_i|) \right) = 0
\]
for all \( x_1, x_2, \cdots, x_m \in G \). On the other hand, for all \( x \in G \),
\[
\Psi (x) = \lim_{n \to \infty} \max_{1 \leq k \leq m} \left\{ |m - 1|^k \zeta \left( \frac{x}{(m - 1)^{k+1}}, \frac{x}{(m - 1)^{k+1}}, \cdots, \frac{x}{(m - 1)^{k+1}} \right) \right\}
= \zeta \left( \frac{x}{m - 1}, \frac{x}{m - 1}, \cdots, \frac{x}{m - 1} \right)
= \frac{m \kappa \xi (|x|)}{|m - 1|}
\]
exists. Also, we have
\[
\lim_{j \to \infty} \max_{1 \leq k \leq m} \left\{ |m - 1|^k \zeta \left( \frac{x}{(m - 1)^{k+1}}, \frac{x}{(m - 1)^{k+1}}, \cdots, \frac{x}{(m - 1)^{k+1}} \right) \right\}
= \lim_{j \to \infty} |m - 1|^j \zeta \left( \frac{x}{(m - 1)^{j+1}}, \frac{x}{(m - 1)^{j+1}}, \cdots, \frac{x}{(m - 1)^{j+1}} \right)
= 0.
\]
Thus, applying Theorem 3.1, we have the conclusion. This completes the proof. \( \Box \)
Theorem 3.3. Let $\zeta : \mathbb{G}^m \to [0, +\infty)$ be a function such that

$$\lim_{n \to \infty} \frac{\zeta((m-1)^n x_1, (m-1)^n x_2, \ldots, (m-1)^n x_m)}{|m-1|^n} = 0$$

(30)

for all $x_1, x_2, \ldots, x_m \in \mathbb{G}$. Suppose that, for any $x \in \mathbb{G}$, the limit

$$\Psi(x) = \lim_{n \to \infty} \max_{0 \leq k \leq n} \left\{ \zeta((m-1)^k x_1, (m-1)^k x_2, \ldots, (m-1)^k x) \right\}$$

(31)

exists and $f : \mathbb{G} \to \mathbb{X}$ is a mapping satisfying (23), then, the limit $T(x) := \lim_{n \to \infty} \frac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in \mathbb{G}$ and satisfies the

$$\|f(x) - T(x)\| \leq \frac{2\|\Psi(x)\|}{\|m\|m-1}.$$  

(32)

Moreover, if

$$\lim_{n \to \infty} \lim_{n \to \infty} \max_{0 \leq k \leq n} \left\{ \zeta((m-1)^k x_1, (m-1)^k x_2, \ldots, (m-1)^k x) \right\} = 0,$$

(33)

then $T$ is the unique mapping satisfying (32).

Proof. By (7), we have

$$\left\| f(x) - \frac{f((m-1)x)}{m-1} \right\| \leq \frac{2\|\zeta(x, x, \ldots, x)\|}{\|m\|m-1^2}$$

(34)

for all $x \in \mathbb{G}$. Replacing $x$ by $(m-1)^n x$ in (34), we obtain

$$\left\| f((m-1)^n x) - f((m-1)^{n+1} x) \right\| \leq \frac{2\|\zeta((m-1)^n x, \ldots, (m-1)^n x)\|}{\|m\|m-1^{n+2}}.$$  

(35)

Thus it follows from (30) and (35) that the sequence $\left\{ \frac{f((m-1)^n x)}{(m-1)^n} \right\}_{n \geq 1}$ is convergent. Set $T(x) := \lim_{n \to \infty} \frac{f((m-1)^n x)}{(m-1)^n}$. On the other hand, it follows from (35) that

$$\left\| \frac{f((m-1)^p x)}{(m-1)^p} - \frac{f((m-1)^{p+1} x)}{(m-1)^{p+1}} \right\| \leq \frac{2}{\|m\|m-1} \max \left\{ \frac{\zeta((m-1)^k x, (m-1)^k x_2, \ldots, (m-1)^k x)}{|m-1|^k+1} : p \leq k < q \right\}$$

for all $x \in \mathbb{G}$ and all integers $p, q \geq 0$ with $q > p \geq 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (31), we obtain (32).

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. $\square$
Corollary 3.4. Let \( \xi : [0, \infty) \to [0, \infty) \) be a function satisfying
\[
\xi((m-1)t) \leq \xi((m-1)\xi(t)), \quad \xi((m-1)) < |m-1|
\]
for all \( t \geq 0 \). Let \( \kappa > 0 \) and \( f : G \to X \) be a mapping satisfying (29). Then there exists a unique additive mapping \( A : G \to X \) such that
\[
\|f(x) - A(x)\| \leq \frac{|\kappa| \left[ \xi(|x|) \right]^m}{m|m-1|^2}.
\]

Proof. If we define \( \zeta : G^m \to [0, \infty) \) by \( \zeta(x_1, x_2, \ldots, x_m) := \kappa \left( \prod_{i=1}^{m} \xi(|x_i|) \right) \) and apply Theorem 3.3, then we get the conclusion. \( \Box \)


Throughout this section, using the fixed point alternative approach, we prove Hyers-Ulam stability of functional equation (2) in random normed spaces.

Theorem 4.1. Let \( X \) be a linear space, \((Y, \mu, T_s)\) be a complete RN-space and \( \Phi \) be a mapping from \( X_m \) to \( D^+ \) \((\Phi(x_1, \cdots, x_m) \) is denoted by \( \Phi_{x_1, \cdots, x_m} \) such that there exists \( 0 < \alpha < \frac{1}{2m} \) such that
\[
\Phi_{(m-1)x_1, (m-1)x_2, \cdots, (m-1)x_m}(t) \leq \Phi_{x_1, x_2, \cdots, x_m}(at) \quad (36)
\]
for all \( x_1, x_2, \cdots, x_m \in X \) and \( t > 0 \). Let \( f : X \to Y \) be a mapping satisfying
\[
\mu \sum_{i=0}^{m-1} \left( \frac{x_{i+1}^{2} + \sum_{k=0}^{m-2} x_{i}^{t}}{m} \right) f(x_{i}) (t) \geq \Phi_{x_1, x_2, \cdots, x_m}(t) \quad (37)
\]
for all \( x_1, x_2, \cdots, x_m \in X \) and \( t > 0 \). Then, for all \( x \in X \)
\[
A(x) := \lim_{m \to \infty} (m-1)^{m} f \left( \frac{x}{(m-1)^{m}} \right)
\]
exists and \( A : X \to Y \) is a unique additive mapping such that
\[
\mu f(x) - A(x) (t) \geq \Phi_{x_1, x_2, \cdots, x_m} \left( \frac{((m^{2} - m) - m(m-1)^{2} \alpha) t}{2m} \right) \quad (38)
\]
for all \( x \in X \) and \( t > 0 \).

Proof. Putting \( x_1 = \cdots = x_m = x \) in (37), we obtain
\[
\mu f(x) - A(x) (t) \geq \Phi_{x_1, x_2, \cdots, x_m} (t) \quad (39)
\]
for all \( x \in X \) and \( t > 0 \). Consider the set \( S := [g : X \to Y] \) and the generalized metric \( d \) in \( S \) defined by
\[
d(f, g) = \inf_{t \in [0, \infty)} \left\{ \mu f(x) - h(y)(t) \geq \Phi_{x_1, x_2, \cdots, x_m} (t), \forall x \in X, t > 0 \right\},
\]
where \( \inf \emptyset = +\infty \). It is easy to show that \((S, d)\) is complete (see [26], Lemma 2.1).

Now, we consider a linear mapping \( J : (S, d) \to (S, d) \) such that \( Jh(x) := (m-1)h \left( \frac{x}{(m-1)^{m}} \right) \) for all \( x \in X \).

First, we prove that \( J \) is a strictly contractive mapping with the Lipschitz constant \( (m-1)\alpha \).
In fact, let \( g, h \in S \) be such that \( d(g, h) < \epsilon \). Then we have \( \mu_{f(x)-h(x)}(\epsilon t) \geq \Phi_{x, x, \cdots, x}(t) \) for all \( x \in X \) and \( t > 0 \) and so

\[
\mu_{f(x)-h(x)}((m-1)\alpha t) = \mu_{\mu((m-1)\alpha t)-h((m-1)\alpha t)}((m-1)\alpha t) \\
= \mu_{\mu((m-1)\alpha t)-h((m-1)\alpha t)}(\alpha t) \\
\geq \Phi_{x, x, \cdots, x}(\alpha t) \\
\geq \Phi_{x, x, \cdots, x}(t)
\]

for all \( x \in X \) and \( t > 0 \). Thus \( d(g, h) < \epsilon \) implies that \( d(Jg, Jh) < (m-1)\alpha \epsilon \). This means that \( d(Jg, Jh) \leq (m-1)\alpha d(g, h) \) for all \( g, h \in S \). It follows from (39) that

\[
\mu_{f(x)-(m-1)/((m-1)\alpha t)}(t) \geq \Phi_{x, x, \cdots, x} \left( \frac{m(m-1)t}{2} \right) \\
\geq \Phi_{x, x, \cdots, x} \left( \frac{m(m-1)t}{2\alpha} \right).
\]

(40)

So \( d(f, f) \leq \frac{2\alpha}{m(m-1)} \). By Theorem 1.3, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( f \), that is,

\[
A \left( \frac{x}{m-1} \right) = \frac{1}{m-1} A(x)
\]

(41)

for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( f \) in the set \( \Omega = \{ h \in S : d(g, h) < \infty \} \). This implies that \( A \) is a unique mapping satisfying (41) such that there exists \( u \in (0, \infty) \) satisfying \( \mu_{f(x)-A(x)}(ut) \geq \Phi_{x, x, \cdots, x}(t) \) for all \( x \in X \) and \( t > 0 \).

2. \( d(f, A) \to 0 \) as \( n \to \infty \). This implies the equality

\[
\lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right) = A(x)
\]

for all \( x \in X \).

3. \( d(f, A) \leq \frac{d(f, f)}{1 - (m-1)\alpha} \) with \( f \in \Omega \), which implies the inequality

\[
d(f, A) \leq \frac{2\alpha}{(m^2 - m) - m(m-1)^2 \alpha}
\]

and so

\[
\mu_{f(x)-A(x)} \left( \frac{2\alpha t}{(m^2 - m) - m(m-1)^2 \alpha} \right) \geq \Phi_{x, x, \cdots, x}(t)
\]

for all \( x \in X \) and \( t > 0 \). This implies that the inequality (38) holds. Now, we have

\[
\mu_{(m-1)^n} \left[ \sum_{x_1, x_2, \cdots, x_m \in X} f \left( \frac{x_1}{(m-1)^n}, \frac{x_2}{(m-1)^n}, \cdots, \frac{x_m}{(m-1)^n} \right) \right] (t) \\
\geq \Phi_{x_1, x_2, \cdots, x_m} \left( \frac{t}{(m-1)^n} \right)
\]

for all \( x_1, x_2, \cdots, x_m \in X, t > 0 \) and \( n \geq 1 \) and so, from (36), it follows that

\[
\Phi_{x_1, x_2, \cdots, x_m} \left( \frac{t}{(m-1)^n} \right) \geq \Phi_{x_1, x_2, \cdots, x_m} \left( \frac{t}{(m-1)^n} \alpha^n \right)
\]
Since $\lim_{n\to\infty} \Phi_{x_1,x_2,\ldots,x_m}(t) = 1$ for all $x_1, x_2, \ldots, x_m \in X$ and $t > 0$, we have
\[
\mu \sum_{i_1 < \cdots < i_m} A\left(\sum_{i_1}^{n_i} x_{i_1} + \sum_{i_1 < \cdots < i_m} x_{i_m}\right) \to 1 \mbox{ for all } x_1, x_2, \ldots, x_m \in X \mbox{ and } t > 0.
\]

for all $x_1, x_2, \ldots, x_m \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ satisfies (2).

On the other hand
\[
A((m-1)x) - (m-1)A(x) = (m-1)\left[\lim_{n \to \infty} (m-1)^{-1} f\left(\frac{x}{(m-1)^{n-1}}\right) - \lim_{n \to \infty} (m-1)^n f\left(\frac{x}{(m-1)^n}\right)\right] = 0.
\]

This completes the proof. \(\square\)

**Corollary 4.2.** Let $X$ be a real normed space, $\theta \geq 0$ and $r$ be a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying
\[
\mu \sum_{i_1 < \cdots < i_m} A\left(\sum_{i_1}^{n_i} x_{i_1} + \sum_{i_1 < \cdots < i_m} x_{i_m}\right) \to \frac{t}{t + \theta\left(\sum_{i=1}^{n_m} \|x_i\|^r\right)} \mbox{ for all } x_1, x_2, \ldots, x_m \in X \mbox{ and } t > 0.
\]

for all $x_1, x_2, \ldots, x_m \in X$. It follows from (37) that
\[
\Phi_{x_1}^{s_1}, \ldots, x_m^{s_m}, \mu \sum_{i_1 < \cdots < i_m} A\left(\sum_{i_1}^{n_i} x_{i_1} + \sum_{i_1 < \cdots < i_m} x_{i_m}\right) \to \frac{t}{t + \theta\left(\sum_{i=1}^{n_m} \|x_i\|^r\right)} \mbox{ for all } x_1, x_2, \ldots, x_m \in X \mbox{ and } t > 0.
\]

for all $x_1, x_2, \ldots, x_m \in X$ and $t > 0$. It follows from (44) that
\[
\frac{\mu}{\sum_{i_1 < \cdots < i_m} f(x)} \geq \Phi_{x_1}^{s_1}, \ldots, x_m^{s_m} \left(\sum_{i=1}^{n_m} \|x_i\|^r\right) \mbox{ for all } x_1 = \cdots = x_m = x \mbox{ in (37), we have}
\]
\[
\mu \geq \Phi_{x_1}^{s_1}, \ldots, x_m^{s_m} \left(\sum_{i=1}^{n_m} \|x_i\|^r\right) \mbox{ for all } x_1 = \cdots = x_m = x \mbox{ in (37), we have}
\]
\[
\mu \geq \Phi_{x_1}^{s_1}, \ldots, x_m^{s_m} \left(\sum_{i=1}^{n_m} \|x_i\|^r\right) \mbox{ for all } x_1 = \cdots = x_m = x \mbox{ in (37), we have}
\]
\[
\mu \geq \Phi_{x_1}^{s_1}, \ldots, x_m^{s_m} \left(\sum_{i=1}^{n_m} \|x_i\|^r\right) \mbox{ for all } x_1 = \cdots = x_m = x \mbox{ in (37), we have}
\]
\[
\mu \geq \Phi_{x_1}^{s_1}, \ldots, x_m^{s_m} \left(\sum_{i=1}^{n_m} \|x_i\|^r\right) \mbox{ for all } x_1 = \cdots = x_m = x \mbox{ in (37), we have}
\]
Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:

1. $A$ is a fixed point of $f$, that is,
   \[
   A((m - 1)x) = (m - 1)A(x)
   \]
   for all $x \in X$. The mapping $A$ is a unique fixed point of $f$ in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that $A$ is a unique mapping satisfying (45) such that there exists $u \in (0, \infty)$ satisfying $\mu_{(x) - A(x)}(ut) \geq \Phi_{x,x,-}^r(t)$ for all $x \in X$ and $t > 0$.

2. \(d^n f, A) \to 0\) as $n \to \infty$. This implies the equality \(\lim_{n \to \infty} f((m - 1)^n x) = A(x)\) for all $x \in X$.

3. $d(f, A) \leq \frac{d(f, f)}{1 - \frac{\alpha}{m - 1}}$ with $f \in \Omega$, which implies the inequality
   \[
   \mu_{f(x) - A(x)} \left( \frac{2t}{(m - 1)(m - 1 - \alpha)} \right) \geq \Phi_{x,x,-}^r(t)
   \]
   for all $x \in X$ and $t > 0$. This implies that the inequality (43) holds. The rest of the proof is similar to the proof of Theorem 4.1.

**Corollary 4.4.** Let $X$ be a real normed space, $\theta \geq 0$ and $r$ be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying (42). Then the limit $A(x) = \lim_{n \to \infty} f((m - 1)^n x)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that
   \[
   \mu_{f(x) - A(x)}(t) \geq \frac{(m - 1)^{r+1} - 1}{(m - 1)^{r+1} - 1 + 2(m - 1)^{r-1}t\|x\|^r}
   \]
   for all $x \in X$ and $t > 0$.

**Proof.** The proof follows from Theorem 4.1 if we take $\Phi_{x_1, x_2, \cdots, x_m}(t) = \frac{t}{t + \theta \left( \sum \|x_i\| \right)}$ for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. In fact, if we choose $\alpha = (m - 1)^{-r}$, then we get the desired result.


Throughout this section, using direct method, we prove the Hyers-Ulam stability of the functional equation (2) in random normed spaces.

**Theorem 5.1.** Let $X$ be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\varphi : X^n \to Z$ be a function such that there exists $0 < \alpha < \frac{1}{m - 1}$ such that
   \[
   \mu'_{\varphi(x_1, x_2, \cdots, x_m)}(t) \geq \mu'_{\varphi(x_1, x_2, \cdots, x_m)}(t)
   \]
   for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$ and \(\lim_{n \to \infty} \frac{t}{(m - 1)^n} = 1\) for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping such that
   \[
   \sum_{i,j \neq k} f \left( \frac{x_i^2 + x_j^2}{2} \right) \geq \frac{\|x_i\|}{2} \sum_{i} f(x_i)
   \]
   for all $x_1, x_2, \cdots, x_m \in X$, $t > 0$, then the limit $A(x) = \lim_{n \to \infty} f\left( \frac{x}{(m - 1)^n} \right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that
   \[
   \mu_{f(x) - A(x)} \left( \frac{m(m - 1)(1 - (m - 1)\alpha)t}{2\alpha} \right)
   \]
   for all $x \in X$ and $t > 0$. 

\[\]
Proof. Putting $x_1 = x_2 = \cdots = x_m = x$ in (47), we obtain
\[
\mu_{f(x)} - (m-1)f(\frac{x}{m-1}) (t) \geq \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2} \right) \tag{49}
\]
for all $x \in X$. Replacing $x$ by $\frac{x}{(m-1)^n}$ in (49) and using (46), we obtain
\[
\mu_{(m-1)^n f(\frac{x}{(m-1)^n}) - (m-1)^n f(\frac{x}{(m-1)^n})} (t) \geq \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2(m-1)^n} \right) \geq \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2(m-1)^n n_{\alpha+1}} \right).
\]
Since
\[
(m-1)^n f\left( \frac{x}{(m-1)^n} \right) - f(x) = \sum_{k=0}^{n-1} (m-1)^{k+1} f\left( \frac{x}{(m-1)^{k+1}} \right) - (m-1)^k f\left( \frac{x}{(m-1)^k} \right)
\]
so we have
\[
\mu_{(m-1)^{k+1} f(\frac{x}{(m-1)^{k+1}}) - f(x)} \left( \sum_{k=0}^{n-1} (m-1)^{k+1} t \right) = \mu_{\sum_{k=0}^{n-1} (m-1)^{k+1} f(\frac{x}{(m-1)^{k+1}}) - (m-1)^k f(\frac{x}{(m-1)^k})} \left( \sum_{k=0}^{n-1} (m-1)^{k+1} t \right) \geq T_{k=0}^{n-1} \left( \mu_{(m-1)^{k+1} f(\frac{x}{(m-1)^{k+1}}) - (m-1)^k f(\frac{x}{(m-1)^k})} \right) \left( (m-1)^k f(\frac{x}{(m-1)^k}) \right) \geq T_{k=0}^{n-1} \left( \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2} \right) \right) = \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2} \right).
\]
This implies that
\[
\mu_{(m-1)^{k+1} f(\frac{x}{(m-1)^{k+1}}) - f(x)} (t) \geq \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2 \sum_{k=0}^{n-1} (m-1)^k \alpha^{k+1}} \right) \tag{50}
\]
Replacing $x$ by $\frac{x}{(m-1)^n}$ in (50), we obtain
\[
\mu_{(m-1)^n f(\frac{x}{(m-1)^n}) - (m-1)^n f(\frac{x}{(m-1)^n})} (t) \geq \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2 \sum_{k=p}^{n-1} (m-1)^k \alpha^{k+1}} \right) \tag{51}
\]
Since
\[
\lim_{p,n \to \infty} \mu'_{\psi(x,x,\cdots,x)} \left( \frac{m(m-1)t}{2 \sum_{k=p}^{n-1} (m-1)^k \alpha^{k+1}} \right) = 1,
\]
it follows that \( \left\{ (m-1)^n f \left( \frac{x}{(m-1)^n} \right) \right\}_{n=1}^{\infty} \) is a Cauchy sequence in a complete RN-space (\( Y, \mu, \min \)) and so there exists a point \( A(x) \in Y \) such that \( \lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right) = A(x) \). Fix \( x \in X \) and put \( p = 0 \) in (51). Then we obtain

\[
\mu_{(m-1)^n f \left( \frac{x}{(m-1)^n} \right)}(t) \geq \mu'_{\psi(x,x_{\cdots},x)} \left( \frac{m(m-1)t}{2 \sum_{k=0}^{n-1} (m-1)^k x_{\cdots, x_{k+1}}} \right)
\]

and so, for any \( \epsilon > 0 \),

\[
\mu_{A(x)-f(x)}(t + \epsilon) \geq \mu_{A(x)}(t + \epsilon) \geq \mu_{A(x) - f(x)}(t)
\]

Taking \( n \to \infty \) in (52), we get

\[
\mu_{A(x)-f(x)}(t + \epsilon) \geq \mu'_{\psi(x,x_{\cdots},x)} \left( \frac{m(m-1)(1-(m-1)\alpha)t}{2\alpha} \right).
\]

Since \( \epsilon \) is arbitrary, by taking \( \epsilon \to 0 \) in (53), we get

\[
\mu_{A(x)-f(x)}(t) \geq \mu'_{\psi(x,x_{\cdots},x)} \left( \frac{m(m-1)(1-(m-1)\alpha)t}{2\alpha} \right).
\]

Replacing \( x_1, x_2, \cdots, x_m \) by \( \frac{x_1}{(m-1)^{\alpha}}, \frac{x_2}{(m-1)^{\alpha}}, \cdots, \frac{x_m}{(m-1)^{\alpha}} \), respectively, in (47), we get

\[
\mu_{(m-1)^{\alpha}} \left[ \sum_{k=1}^{n} f \left( \frac{x_{1+2k-1}}{(m-1)^{\alpha}}, \frac{x_{1+2k}}{(m-1)^{\alpha}} \right) \right] \geq \mu'_{\psi \left( \frac{\alpha_1}{(m-1)^{\alpha}}, \frac{\alpha_2}{(m-1)^{\alpha}}, \cdots, \frac{\alpha_m}{(m-1)^{\alpha}} \right)} \left( \frac{t}{(m-1)^{\alpha}} \right)
\]

for all \( x_1, x_2, \cdots, x_m \in X \) and \( t > 0 \). Since

\[
\lim_{n \to \infty} \mu'_{\psi \left( \frac{\alpha_1}{(m-1)^{\alpha}}, \frac{\alpha_2}{(m-1)^{\alpha}}, \cdots, \frac{\alpha_m}{(m-1)^{\alpha}} \right)} \left( \frac{t}{(m-1)^{\alpha}} \right) = 1,
\]

we conclude that \( A \) satisfies (2).

On the other hand

\[
(m-1)A \left( \frac{x}{m-1} \right) - A(x) = \lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right) - \lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right) = 0.
\]

This implies that \( A : X \to Y \) is an additive mapping.

To prove the uniqueness of the additive mapping \( A \), assume that there exists another additive mapping
$L : X \to Y$ which satisfies (48). Then we have

$$
\mu_{A(x)-L(x)}(t) = \lim_{n \to \infty} \mu_{(m-1)^nA\left(\frac{t}{m-1}\right)}(\frac{t}{m-1})
$$

$$
\geq \lim \min_{n \to \infty} \left\{ \mu_{(m-1)^nA\left(\frac{t}{m-1}\right)} - m1\left(\frac{t}{m-1}\right) \right\} \left( \frac{t}{m-1} \right)
$$

$$
\geq \lim_{n \to \infty} \mu'_{\psi\left(\frac{t}{m-1}x\right)} \left( \frac{m1((m-1)^n - 1)}{4(m-1)^n} \right).
$$

Since $\lim_{n \to \infty} \frac{m1((m-1)^n - 1)}{4(m-1)^n} = \infty$, we get

$$
\lim_{n \to \infty} \lim_{t \to \infty} \mu'_{\psi\left(\frac{t}{m-1}x\right)} \left( \frac{m1((m-1)^n - 1)}{4(m-1)^n} \right) = 1.
$$

Therefore, it follows that $\mu_{A(x)-L(x)}(t) = 1$ for all $t > 0$ and so $A(x) = L(x)$. This completes the proof.

**Corollary 5.2.** Let $X$ be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $r$ be a positive real number with $r > 1$, $z_0 \in Z$ and $f : X \to Y$ be a mapping satisfying

$$
\frac{\mu}{1} \sum_{i=1}^{m-1} \left( \frac{t}{m-1} \right) \frac{\sum_{\|x\|\geq z_0} \sum_{\|x\|\geq z_0} \|f(x)\|} \right) \geq \mu'_{\psi\left(\frac{t}{m-1}x\right)} (t)
$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n \to \infty} (m-1)^n f \left( \frac{x}{(m-1)^n} \right)$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$
\mu\left(\sum_{i=1}^{m-1} (m-1)^n \right) = \frac{1}{2(m-1)^n}
$$

for all $x \in X$ and $t > 0$.

**Proof.** Let $\alpha = (m-1)^{-r}$ and $\varphi : X^m \to Z$ be a mapping defined by $\varphi(x_1, x_2, \cdots, x_m) = \left( \sum_{i=1}^{m} \|x_i\| \right) z_0$. Then, from Theorem 5.1, the conclusion follows.

**Theorem 5.3.** Let $X$ be a real linear space, $(Z, \mu', \min)$ be an RN-space and $\varphi : X^m \to Z$ be a function such that there exists $0 < \alpha < m-1$ such that

$$
\mu'_{\varphi(x_1, x_2, \cdots, x_m)}(t) \geq \mu'_{\varphi\left(\frac{t}{m-1}x_1, \frac{t}{m-1}x_2, \cdots, \frac{t}{m-1}x_m\right)}(t)
$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$ and

$$
\lim_{n \to \infty} \mu'_{\varphi((m-1)^n x_1, (m-1)^n x_2, \cdots, (m-1)^n x_m)}((m-1)^n x) = 1
$$

for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping satisfying (47). Then the limit $A(x) = \lim_{n \to \infty} \frac{f((m-1)^n x)}{(m-1)^n}$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that

$$
\mu\left(\sum_{i=1}^{m-1} (m-1)^n \right) = \frac{1}{2(m-1)^n}
$$

for all $x \in X$ and $t > 0$.
Proof. Putting $x_1 = \cdots = x_m = x$ in (47), we have
\[
\mu \left( \frac{m-1}{m} f(x) \right)(t) \geq \mu_{\phi(x,x,\cdots,x)} \left( \frac{m(m-1)^2 t}{2} \right) \tag{56}
\]
for all $x \in X$ and $t > 0$. Replacing $x$ by $(m-1)^r x$ in (56), we obtain that
\[
\mu \left( \frac{m-1}{m} f(x) \right)(t) \geq \mu_{\phi((m-1)^r x,(m-1)x,\cdots,(m-1)^r x)} \left( \frac{m(m-1)^{r+2} t}{2} \right) 
\geq \mu_{\phi(x,x,\cdots,x)} \left( \frac{m(m-1)^{r+2} t}{2} \right).
\]

The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 5.4. Let $X$ be a real normed linear space, $(Z, \mu', \min)$ be an RN-space and $(Y, \mu, \min)$ be a complete RN-space. Let $r$ be a positive real number with $0 < r < \frac{1}{m}$, $z_0 \in Z$ and $f : X \to Y$ is a mapping satisfying
\[
\mu \left( \sum_{i=0}^{m-1} f(x_{i+1}) \right)(t) \geq \mu \left( \prod_{i=1}^{m} \|x_i\| \right) z_0 (t) \tag{57}
\]
for all $x_1, x_2, \cdots, x_m \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n \to \infty} \frac{f((m-1)^r x)}{(m-1)^r}$ exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that
\[
\mu_{f(x)-A(x)}(t) \geq \mu_{\phi^{m-1} z_0} \left( \frac{m((m-1)^{mr+2} - (m-1))t}{2(m-1)^{mr}} \right)
\]
for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = (m-1)^{-mr}$ and $\varphi : X^m \to Z$ be a mapping defined by
\[
\varphi(x_1, x_2, \cdots, x_m) = \left( \prod_{i=1}^{m} \|x_i\| \right) z_0.
\]

Then, from Theorem 5.3, the conclusion follows. □

References
