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Iterative Algorithm for a Split Equilibrium Problem and Fixed Problem for Finite Asymptotically Nonexpansive Mappings in Hilbert Space

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Abstract. In this paper, we propose an iterative algorithm for finding the common element of solution set of a split equilibrium problem and common fixed point set of a finite family of asymptotically nonexpansive mappings in Hilbert space. The strong convergence of this algorithm is proved.

1. Introduction

Throughout this paper, let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{N}$ denote the set of all positive integer numbers, H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping. If there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C,$$

we call T an asymptotically nonexpansive mapping. If $k_n \equiv 1$, then T is said to be a nonexpansive mapping. The set of fixed points of T is denoted by $Fix(T)$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to find $z \in C$ such that

$$F(z, y) \ge 0, \quad \forall y \in C. \tag{1.1}$$

The set of all solutions of (1.1) is denoted by $EP(F)$, i.e.,

$$EP(F) = \{z \in C : F(z, y) \ge 0, \forall y \in C\}.$$

Many problems in physics, optimization, and economics can be reduced to find the solution of (1.1); see [1–4]. In 1997, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the solution of (1.1) under the assumption that $EP(F)$ is non-empty. Later on, many iterative algorithms are considered to find the element of $Fix(S) \cap EP(F)$; see [6–8].

Recently, some new problems called split variational inequality problems are considered by some authors. Censor et al. [9] initially studied this class of split variation inequality problems. Let H_1 and H_2 be two real Hilbert spaces. Given the operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, bounded linear operator

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$A : H_1 \to H_2$, and nonempty closed convex subsets $C \subset H_1$ and $Q \subset H_2$, the split variational inequality problem is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0, \ \forall y \in Q.$$

After investigating the algorithm of Censor et al., Moudafi [10] introduced a new iterative scheme to solve the following split monotone variational inclusion:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*)$$

and such that

$$y^* = Ax^* \in H_2 \text{ sovles } 0 \in g(y^*) + B_2(y^*),$$

where $B_1 : H_i \to 2^{H_i}$ is a set-valued mappings for $i = 1, 2$.

In 2013, Kazmi and Rizvi [11] considered a new class of split problem called *split equilibrium problem*. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_2$ be a bounded linear operator. The split equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall x \in C, \tag{1.2}$$

and such that

$$y^* = Ax^* \in Q \text{ sovles } F_2(y^*, y) \geq 0, \ \ \forall y \in Q. \tag{1.3}$$

The set of all solutions of (1.2) and (1.3) is denoted by Ω, i.e., $\Omega = \{z \in C : z \in EP(F_1) \text{ such that } Az \in EP(F_2)\}$.

On split equilibrium problem, the interested author also may refer to [12, 13] in which the author gave an iterative algorithm to find the common element of sets of solutions of the split equilibrium problem and hierarchical fixed point problem.

To the knowledge of author, the split equilibrium problems and fixed point problems for asymptotically nonexpansive mappings have not been investigated in literature by far. In this paper, inspired by the results in [11–13], we propose an iterative algorithm to find the common element of solution sets of a split equilibrium problem and common fixed points set of a finite family of asymptotically nonexpansive mappings in Hilbert spaces and prove the strong convergence for the algorithm.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. It is well known that P_C is a firmly nonexpansive mapping from H onto C, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \ \ \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \ \ \forall y \in C.$$

A mapping $A : C \to H$ is called an α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in H.$$

For each $\lambda \in (0, 2\alpha]$, $I - \lambda A$ is a nonexpansive mapping of C into H; see [14].

Consider the following variational inequality on inverse strongly monotone mapping A:

$$\text{find } u \in C \text{ such that } \langle v - u, Au \rangle \geq 0, \ \forall v \in C.$$

The set of solutions of the variational inequality is denoted by $VI(C, A)$. It is know that $u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au)$ for any $\lambda > 0$.

Let $S : C \to C$ be a mapping. It is known that S is nonexpansive if and only if the complement $I - S$ is $\frac{1}{2}$-inverse strongly monotone; see [15]. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\}$. Then for any $(x, \hat{x}) \in C \times Fix(T)$, we have

$$\|T^n x - x\|^2 \leq 2\langle x - T^n x, x - \hat{x}\rangle + (k_n^2 - 1)\|x - \hat{x}\|^2, \forall n \in \mathbb{N}, \tag{2.1}$$

which is obtained directly from the following

$$\begin{aligned} k_n^2\|x - \hat{x}\|^2 &\geq \|T^n x - T^n \hat{x}\|^2 = \|T^n x - \hat{x}\|^2 = \|T^n x - x + (x - \hat{x})\|^2 \\ &= \|T^n x - x\|^2 + \|x - \hat{x}\|^2 + 2\langle T^n x - x, x - \hat{x}\rangle. \end{aligned}$$

Let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.1 [16] *Let C be a nonempty closed convex subset of a Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies the conditions* (A1)-(A4). *For $x \in H$ and $r > 0$, define a mapping $T_r^F : H \to C$ by*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}. \tag{2.2}$$

Then T_r is well defined and the following hold:

(1) *T_r^F is single-valued;*

(2) *T_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(3) *$Fix(T_r^F) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 2.2 [17] *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-A(4). *Let T_r^F and T_s^F be defined as in Lemma 2.1 with $r, s > 0$. Then, for any $x, y \in H$, one has*

$$\|T_r^F x - T_s^F y\| \leq |x - y| + \left|1 - \frac{s}{r}\right| \|T_r^F x - x\|.$$

Lemma 2.3 [8] *Let $F : C \times C \to \mathbb{R}$ be a functions satisfying the conditions* (A1)-(A4) *and let T_s^F and T_t^F be defined as in Lemma 2.1 with $s, t > 0$. Then the following holds:*

$$\|T_s^F x - T_t^F x\|^2 \leq \frac{s - t}{s}\langle T_s x - T_t x, T_s x - x\rangle$$

for all $x \in H$.

Lemma 2.4 [18] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in* $[0, 1]$ *with* $0 < \liminf_n \beta_n \limsup_n \beta_n < 1$. *Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)y_n$ for all integers $n \geq 0$ and $\limsup_n(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_n \|y_n - x_n\| = 0$.*

Lemma 2.5 [19] *Let T be an asymptotically nonexpansive mapping on a closed and convex subset C of a real Hilbert space H. Then $I - T$ is demiclosed at any point $y \in H$. That is, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to y \in H$, then $x - Tx = y$.*

Lemma 2.6 [20] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative numbers such that*

$$\alpha_{n+1} \le (1 - a_n)\alpha_n + a_n t_n, \ \ n \ge 0,$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{t_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} a_n = \infty$;
(2) *either* $\limsup_{n\to\infty} t_n \le 0$ *or* $\sum_{n=0}^{\infty} |a_n t_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main Results

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed and convex subsets. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying* (A1-A4) *and assume that G is upper semicontinuous in the first argument. Let $f : C \to C$ be ρ-contraction and $\{T_i\}_{i=1}^{l} : C \to C$ be l asymptotically nonexpansive mappings with the same sequence $\{k_n\}$ satisfying the condition that*

$$\lim_{n\to\infty} \sup_{x\in K} \|T_i^{n+1}x - T_i^n x\| = 0 \tag{Γ}$$

for any bounded subset K of C and each $i = 1, \cdots, l$. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $Fix(T) \cap \Omega \ne \emptyset$, where $Fix(T) = \cap_{i=1}^{l} Fix(T_i)$ and $\Omega = \{v \in C : v \in EP(F)$ such that $Av \in EP(G)\}$. Let $\{\alpha_n\} \subset (0, 1)$ be a sequence. Define the sequence $\{x_n\}$ by the following manner: $x_0 \in C$ and

$$\begin{cases} u_n = T_{r_n}^F (I - \gamma A^*(I - T_{s_n}^G)A)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \dfrac{(1-\alpha_n)}{l} \displaystyle\sum_{i=1}^{l} T_i^n u_n, \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$, $\gamma \subset (0, 1/L^2]$, L is the spectral radius radius of the operator A^*A and A^* is the adjoint of A. If the control sequences $\{\alpha_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{k_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |s_n - s_{n-1}| < \infty$;
(iii) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$,

then $\{x_n\}$ generated by (3.1) *strongly converges to $z = P_{Fix(T)\cap\Omega} f(z)$.*

Remark 3.1. For each $n \in \mathbb{N}$, $A^*(I - T_{r_n}^G)A$ is a $\frac{1}{2L^2}$-inverse strongly monotone mapping. In fact, since $T_{s_n}^G$ is (firmly) nonexpansive and $I - T_{s_n}^G$ is $\frac{1}{2}$-inverse strongly monotone, we have

$$\begin{aligned} \|A^*(I - T_{s_n}^G)Ax - A^*(I - T_{s_n}^G)Ay\|^2 &= \langle A^*(I - T_{s_n}^G)(Ax - Ay), A^*(I - T_{s_n}^G)(Ax - Ay)\rangle \\ &= \langle (I - T_{s_n}^G)(Ax - Ay), AA^*(I - T_{s_n}^G)(Ax - Ay)\rangle \\ &\le L^2 \langle (I - T_{s_n}^G)(Ax - Ay), (I - T_{s_n}^G)(Ax - Ay)\rangle \\ &= L^2 \|(I - T_{s_n}^G)(Ax - Ay)\|^2 \\ &\le 2L^2 \langle Ax - Ay, (I - T_{s_n}^G)(Ax - Ay)\rangle \\ &= 2L^2 \langle x - y, A^*(I - T_{s_n}^G)(Ax - Ay)\rangle, \end{aligned}$$

for all $x, y \in H$, which implies that $A^*(I - T_{s_n}^G)A$ is a $\frac{1}{2L^2}$-inverse strongly monotone mapping. Note that $\gamma \in (0, \frac{1}{L^2}]$. Thus $I - \gamma A^*(I - T_{s_n}^G)A$ is nonexpansive.

Proof. We first show that $\{x_n\}$ is bounded. Let $p \in Fix(S) \cap \Omega$. Then $p = T^F_{r_n} p$ and $(I - \gamma A^*(I - T^G_{s_n})A)p = p$. Thus we have

$$\begin{aligned} \|u_n - p\| &= \|T^F_{r_n}(I - \gamma A^*(I - T^G_{s_n})A)x_n - T^F_{r_n}(I - \gamma A^*(I - T^G_{s_n})A)p\| \\ &\le \|(I - \gamma A^*(I - T^G_{s_n})A)x_n - (I - \gamma A^*(I - T^G_{s_n})A)p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{3.2}$$

Take $\epsilon \in (0, 1-\rho)$. Since $(k_n - 1)/\alpha_n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $(k_n - 1) < \epsilon\alpha_n$. Let $T^n = \frac{1}{l}\sum_{i=1}^{l} T_i^n$ for each $n \in \mathbb{N}$. It is easy to see that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. From (3.1) and (3.2) it follows that, for all $n > N$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - f(p)) + \alpha_n(f(p) - p) + (1-\alpha_n)(T^n u_n - p)\| \\ &\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|T^n u_n - p\| \\ &\le \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)k_n\|u_n - p\| \\ &= (1 - \alpha_n(1-\rho))\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)(k_n - 1)\|u_n - p\| \\ &\le (1 - \alpha_n(1-\rho))\|x_n - p\| + \alpha_n\|f(p) - p\| + \alpha_n\epsilon\|x_n - p\| \\ &= (1 - \alpha_n(1-\rho-\epsilon))\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\le \max\{\|x_n - p\|, \frac{1}{1-\rho-\epsilon}\|f(p) - p\|\}. \end{aligned} \tag{3.3}$$

By induction, we see that, for all $n > N$,

$$\|x_n - p\| \le \max\left\{\|x_N - p\|, \frac{1}{1-\rho-\epsilon}\|f(p) - p\|\right\}.$$

It follows that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Next we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $(I - \gamma A^*(I - T^G_{s_n})A)$ is nonexpansive, by Lemma 2.2 we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|T^F_{r_{n+1}}(I - \gamma A^*(I - T^G_{s_{n+1}})A)x_{n+1} - T^F_{r_n}(I - \gamma A^*(I - T^G_{s_n})A)x_n\| \\ &\le \|(I - \gamma A^*(I - T^G_{s_{n+1}})A)x_{n+1} - (I - \gamma A^*(I - T^G_{s_n})A)x_n\| \\ &\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|T^F_{r_{n+1}}(I - \gamma A^*(I - T^G_{s_{n+1}})A)x_{n+1} - (I - \gamma A^*(I - T^G_{s_{n+1}})A)x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \|(I - \gamma A^*(I - T^G_{s_{n+1}})A)x_n - (I - \gamma A^*(I - T^G_{s_n})A)x_n\| \\ &\quad + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1} \\ &= \|x_{n+1} - x_n\| + \gamma\|A^*(T^G_{s_n} - T^G_{s_{n+1}})Ax_n\| + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1}, \end{aligned}$$

where $\sigma_n = \sup\{n \in \mathbb{N}\}\|T^F_{r_n}(I - \gamma A^*(I - T^G_{s_n})A)x_n - (I - \gamma A^*(I - T^G_{s_n})A)x_n\|$. Further, by Lemma 2.3 we get

$$\begin{aligned} \|u_{n+1} - u_n\| &\le \|x_{n+1} - x_n\| + \gamma\|A\|\|T^G_{s_n}Ax_n - T^G_{s_{n+1}}Ax_n\| + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1} \\ &\le \|x_{n+1} - x_n\| + \gamma\|A\|\Big(\frac{|s_n - s_{n+1}|}{s}|\langle T^G_{s_n}Ax_n - T^G_{s_{n+1}}Ax_n, T^G_{s_n}Ax_n - Ax_n\rangle|\Big)^{\frac{1}{2}} \\ &\quad + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1} \\ &\le \|x_{n+1} - x_n\| + \gamma\|A\|\Big(\frac{|s_n - s_{n+1}|}{s}\eta_n\Big)^{\frac{1}{2}} + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1}, \end{aligned} \tag{3.4}$$

where $\eta_n = \sup_{n\in\mathbb{N}}|\langle T^G_{s_n}Ax_n - T^G_{s_{n+1}}Ax_n, T^G_{s_n}Ax_n - Ax_n\rangle|$.

Let $y_n = \frac{x_{n+1}-\alpha_n f(x_n)}{1-\alpha_n}$. Then from (3.1) and (3.4) it follows that

$$\begin{aligned}
&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
&= \|T^{n+1}u_{n+1} - T^n u_n\| - \|x_{n+1} - x_n\| \\
&\le \|T^{n+1}u_{n+1} - T^{n+1}u_n\| - \|x_{n+1} - x_n\| + \|T^{n+1}u_n - T^n u_n\| \\
&\le (k_{n+1} - 1)\|x_{n+1} - x_n\| + k_n\Big[\gamma\|A\|\Big(\frac{|s_n - s_{n+1}|}{s}\eta_n\Big)^{\frac{1}{2}} + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1}\Big] \\
&\quad + \|T^{n+1}u_n - T^n u_n\| \\
&\le (k_{n+1} - 1)(\|x_{n+1}\| + \|x_n\|) + k_n\Big[\gamma\|A\|\Big(\frac{s_n - s_{n+1}}{s}\eta_n\Big)^{\frac{1}{2}} + \frac{|r_{n+1} - r_n|}{r}\sigma_{n+1}\Big] \\
&\quad + \frac{1}{l}\|\sum_{i=1}^{l} T_i^{n+1}u_n - \sum_{i=1}^{l} T_i^n u_n\|.
\end{aligned}$$

Since the mappings $\{T_i\}_{i=1}^l$ satisfy the condition (Γ), by the condition (ii) we get

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) = 0.$$

Hence, by Lemma 2.4 we conclude that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Further, by (3.4) we get

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.6}$$

From (3.1) and (3.5) it follows that

$$\lim_{n\to\infty} \|T^n u_n - x_n\| = 0. \tag{3.7}$$

Now we prove that $\lim_{n\to\infty} \|T_i x_n - x_n\| \to 0$ for each $i \in \{1, \cdots, l\}$. To show this, we first prove that $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Since $A^*(I - T_{s_n}^G)A$ is $\frac{1}{2L^2}$-inverse strongly monotone, by (3.1) we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^F(I - \gamma A^*(I - T_{s_n}^G)A)x_n - T_{r_n}^F(I - \gamma A^*(I - T_{s_n}^G)A)p\|^2 \\
&\le \|(I - \gamma A^*(I - T_{s_n}^G)A)x_n - (I - \gamma A^*(I - T_{s_n}^G)A)p\|^2 \\
&= \|(x_n - p) - \gamma(A^*(I - T_{s_n}^G)Ax_n - A^*(I - T_{s_n}^G)Ap)\|^2 \\
&= \|x_n - p\|^2 - 2\gamma\langle x_n - p, A^*(I - T_{s_n}^G)Ax_n - A^*(I - T_{s_n}^G)Ap\rangle \\
&\quad\quad + \gamma^2\|A^*(I - T_{s_n}^G)Ax_n - A^*(I - T_{s_n}^G)Ap\|^2 \\
&\le \|x_n - p\|^2 - \frac{\gamma}{L^2}\|A^*(I - T_{s_n}^G)Ax_n - A^*(I - T_{s_n}^G)Ap\|^2 \\
&\quad\quad + \gamma^2\|A^*(I - T_{s_n}^G)Ax_n - A^*(I - T_{s_n}^G)Ap\|^2 \\
&= \|x_n - p\|^2 + \gamma(\gamma - \frac{1}{L^2})\|A^*(I - T_{s_n}^G)Ax_n - A^*(I - T_{s_n}^G)Ap\|^2 \\
&= \|x_n - p\|^2 + \gamma(\gamma - \frac{1}{L^2})\|A^*(I - T_{s_n}^G)Ax_n\|^2
\end{aligned}$$

Thus we have

$$\begin{aligned}
||x_{n+1}-p||^2 &\le \alpha_n||f(x_n)-p||^2+(1-\alpha_n)k_n^2||u_n-p||^2\\
&\le \alpha_n||f(x_n)-p||^2+(1-\alpha_n)k_n^2[||x_n-p||^2\\
&\quad +\gamma(\gamma-\frac{1}{L^2})||A^*(I-T^G_{s_n})Ax_n||^2]\\
&= \alpha_n||f(x_n)-p||^2+(1-\alpha_n)(1+2\theta_n+\theta_n^2)||x_n-p||^2\\
&\quad +\gamma(1-\alpha_n)k_n^2(\gamma-\frac{1}{L^2})||A^*(I-T^G_{s_n})Ax_n||^2\\
&\le \alpha_n||f(x_n)-p||^2+||x_n-p||^2+(1-\alpha_n)(2\theta_n+\theta_n^2)||x_n-p||^2\\
&\quad +\gamma(1-\alpha_n)k_n^2(\gamma-\frac{1}{L^2})||A^*(I-T^G_{s_n})Ax_n||^2,
\end{aligned}$$

where $\theta_n = k_n - 1$. Therefore,

$$\begin{aligned}
\gamma(1-\alpha_n)k_n^2(\frac{1}{L^2}-\gamma)||A^*(I-T^G_{s_n})Ax_n||^2 &\le \alpha_n||f(x_n)-p||^2\\
+||x_n-x_{n+1}||(||x_n-p||+||x_{n+1}-p||)&+(1-\alpha_n)(2\theta_n+\theta_n^2)||x_n-p||^2.
\end{aligned}$$

Since $\alpha_n \to 0$, $k_n \to 1$ and both $\{f(x_n)\}$ and $\{x_n\}$ are bounded, by (3.5) we have

$$\lim_{n\to\infty}||A^*(I-T^G_{s_n})Ax_n|| = 0, \tag{3.8}$$

which implies that

$$\lim_{n\to\infty}||(I-T^G_{s_n})Ax_n|| = 0. \tag{3.9}$$

Since $T^F_{r_n}$ is firmly nonexpansive and $(I-\gamma A^*(T^G_{s_n}-I)A)$ is nonexpansive, by (3.1) we have

$$\begin{aligned}
\left\|u_n-p\right\|^2 &= \left\|T^F_{r_n}\Big(x_n+\gamma A^*(T^G_{s_n}-I)Ax_n\Big)-T^F_{r_n}(p)\right\|^2\\
&\le \Big\langle u_n-p, x_n+\gamma A^*(T^G_{s_n}-I)Ax_n-p\Big\rangle\\
&= \frac{1}{2}\Big\{\left\|u_n-p\right\|^2+\left\|x_n+\gamma A^*(T^G_{s_n}-I)Ax_n-p\right\|^2\\
&\quad -\left\|u_n-p-[x_n+\gamma A^*(T^G_{s_n}-I)Ax_n-p]\right\|^2\Big\}\\
&= \frac{1}{2}\Big\{\left\|u_n-p\right\|^2+\left\|(I-\gamma A^*(T^G_{s_n}-I)A)x_n-(I-\gamma A^*(T^G_{s_n}-I)A)p\right\|^2\\
&\quad -\left\|u_n-x_n-\gamma A^*(T^G_{s_n}-I)Ax_n\right\|^2\Big\}\\
&\le \frac{1}{2}\Big\{\left\|u_n-p\right\|^2+\left\|x_n-p\right\|^2-\left\|u_n-x_n-\gamma A^*(T^G_{s_n}-I)Ax_n\right\|^2\Big\}\\
&= \frac{1}{2}\Big\{\left\|u_n-p\right\|^2+\left\|x_n-p\right\|^2-\Big[\left\|u_n-x_n\right\|^2+\gamma^2\left\|A^*(T^G_{s_n}-I)Ax_n\right\|^2\\
&\quad -2\gamma\langle u_n-x_n, A^*(T^G_{s_n}-I)Ax_n\rangle\Big]\Big\},
\end{aligned}$$

which implies that

$$||u_n-p||^2 \le ||x_n-p||^2-||u_n-x_n||^2+2\gamma||u_n-x_n||||A^*(T^G_{s_n}-I)Ax_n||. \tag{3.10}$$

Now, from (3.1) and (3.10) we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|f(x_n)-p\|^2 + (1-\alpha_n)\|T^n u_n - p\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2 + (1-\alpha_n)k_n^2\|u_n-p\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2 + (1-\alpha_n)k_n^2[\|x_n-p\|^2 - \|u_n-x_n\|^2\\
&\qquad + 2\gamma\|u_n-x_n\|\|A^*(T^G_{s_n}-I)Ax_n\|]\\
&= \alpha_n\|f(x_n)-p\|^2 + (1-\alpha_n)(1+2\theta_n+\theta_n^2)\|x_n-p\|^2\\
&\qquad - (1-\alpha_n)k_n^2\|u_n-x_n\|^2 + 2(1-\alpha_n)k_n^2\gamma\|u_n-x_n\|\|A^*(T^G_{s_n}-I)Ax_n\|]\\
&\le \alpha_n\|f(x_n)-p\|^2 + (2\theta_n+\theta_n^2)\|x_n-p\|^2 - (1-\alpha_n)k_n^2\|u_n-x_n\|^2 + \|x_n-p\|^2\\
&\qquad + 2k_n^2\gamma\|u_n-x_n\|\|A^*(T^G_{s_n}-I)Ax_n\|]
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-\alpha_n)k_n^2\|u_n-x_n\|^2 &\le \alpha_n\|f(x_n)-p\|^2 + \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\qquad + (2\theta_n+\theta_n^2)\|x_n-p\|^2 + 2k_n^2\gamma(\|u_n\|+\|x_n\|)\|A^*T^G_{s_n}-I)Ax_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $k_n \to 1$ and $\{x_n\}$ and $\{u_n\}$ are bounded, by (3.5) and (3.10) we have

$$\lim_{n\to\infty}\|u_n-x_n\| = 0. \tag{3.11}$$

Combing (3.5) and (3.11), by $\|u_n-x_{n+1}\| \le \|u_n-x_n\| + \|x_n-x_{n+1}\|$ we see that

$$\lim_{n\to\infty}\|u_n-x_{n+1}\| = 0. \tag{3.12}$$

Note that

$$\begin{aligned}
\frac{1}{l}\sum_{i=1}^{l}(T_i^n u_n - u_n) &= (T^n u_n - u_n)\\
&= \frac{1}{1-\alpha_n}\Big[x_{n+1}-u_n+\alpha_n(u_n-f(x_n))\Big].
\end{aligned}\tag{3.13}$$

By (2.1) and (3.13), for each $i = 1,\cdots,l$, we have

$$\begin{aligned}
\frac{1}{l}\|T_i^n u_n - u_n\|^2 &\le \frac{1}{l}\sum_{i=1}^{l}\|T_i^n u_n - u_n\|^2\\
&\le \frac{2}{l}\sum_{i=1}^{l}\langle T_i^n u_n - u_n, u_n-p\rangle + (k_n^2-1)\|u_n-p\|^2\\
&= \frac{2}{(1-\alpha_n)}\Big[\langle x_{n+1}-u_n, u_n-p\rangle + \alpha_n\langle u_n - f(x_n), u_n-p\rangle\Big] + (k_n^2-1)\|u_n-p\|^2.
\end{aligned}\tag{3.14}$$

Since $\alpha_n \to 0$ and $k_n \to 1$, from (3.12) and (3.14) it follows that, for each $i = 1,\cdots,l$,

$$\lim_{n\to\infty}\|T_i^n u_n - u_n\| = 0. \tag{3.15}$$

Let $k_\infty = \sup_{n\in\mathbb{N}} k_n < \infty$. Consequently, by (3.6) and (3.15) we get that, for each $i = 1,\cdots,l$,

$$\begin{aligned}
\|T_i u_n - u_n\| &\le \|T_i u_n - T_i^{n+1}u_n\| + \|T_i^{n+1}u_n - T_i^{n+1}u_{n+1}\|\\
&\qquad + \|T_i^{n+1}u_{n+1} - u_{n+1}\| + \|u_{n+1}-u_n\|\\
&\le k_\infty\|u_n - T_i^n u_n\| + \|T_i^{n+1}u_{n+1}-u_{n+1}\| + (1+k_\infty)\|u_{n+1}-u_n\|\\
&\to 0, \text{ as } n\to\infty.
\end{aligned}$$

Further, we have, for each $i = 1, \cdots, l$,

$$\begin{aligned}\|T_i x_n - x_n\| &\le \|T_i x_n - T_i u_n\| + \|T_i u_n - u_n\| + \|u_n - x_n\| \\ &\le (k_1 + 1)\|u_n - x_n\| + \|T_i u_n - u_n\| \\ &\to 0, \text{ as } n \to \infty.\end{aligned} \tag{3.16}$$

Since $P_{Fix(S)\cap\Omega} f$ is a contraction, there exists a unique $z \in Fix(S) \cap \Omega$ such that $z = P_{Fix(S)\cap\Omega} f(z)$. Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty}\langle f(z) - z, x_{n_i} - z\rangle.$$

As $\{x_{n_k}\}$ is bounded, there is a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ converging weakly to some $w \in C$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup w$.

Now we show that $w \in \cap_{i=1}^{l} Fix(T_i)$. In fact, since each $x_n - T_i x_n \to 0$ and $x_{n_k} \rightharpoonup w$, by Lemma 2.3 we obtain that $w \in Fix(T_i)$. So $w \in Fix(T) = \cap_{i=1}^{l} Fix(T_i)$.

Next we show that $w \in \Omega$. By (3.1), $u_n = T_{r_n}^F (I - \gamma A^*(I - T_{s_n}^G)A)x_n$, that is

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma A^*(T_{s_n}^G - I)Ax_n\rangle \ge 0, \ \forall y \in C.$$

From the monotonicity of F it follows that

$$-\frac{1}{r_n}\langle y - u_n, \gamma A^*(T_{s_n}^G - I)Ax_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \ \forall y \in C.$$

Replacing n with n_k in the above inequality, we have

$$-\frac{1}{r_{n_k}}\langle y - u_{n_k}, \gamma A^*(T_{s_n}^G - I)Ax_{n_k}\rangle + \frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_i} - x_{n_k}\rangle \ge F(y, u_{n_k}), \ \forall y \in C.$$

Since $\|u_{n_k} - x_{n_k}\| \to 0$, $\|A^*(T_{r_{n_k}}^G - I)Ax_{n_k}\| \to 0$ and $\|x_{n_k} - w\| \to 0$ as $k \to \infty$, we have

$$F(y, w) \le 0, \ \forall y \in C.$$

For any $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1 - t)w$. Then we have $y_t \in C$. Further, we have

$$\begin{aligned}0 &= F(y_t, y_t) \\ &\le tF(y_t, y) + (1 - t)F(y_t, w) \\ &\le tF(y_t, y).\end{aligned}$$

So $F(y_t, y) \ge 0$. Let $t \to 0$, one has $F(w, y) \ge 0$, i.e., $w \in EP(F)$.

Next we show that $Aw \in EP(G)$. Since A is bounded linear operator, $Ax_{n_k} \to Aw$. Then from (3.9) it follows that $T_{s_n}^G Ax_{n_k} \to Aw$. By the definition of $T_{r_n}^G Ax_{n_k}$, we have

$$G(T_{s_n}^G Ax_{n_k}, y) + \frac{1}{s_{n_k}}\langle y - T_{s_n}^G Ax_{n_k}, T_{s_n}^G Ax_{n_k} - Aw\rangle \ge 0, \ \forall y \in C. \tag{3.12}$$

Since each G is upper semicontinuous in the first argument, taking lim sup to (3.12) as $k \to \infty$, we get

$$G(Aw, y) \ge 0, \ \ \forall y \in C,$$

which implies that $Aw \in EP(G)$. Therefore, $w \in \Omega$.

By the property on $P_{Fix(T)\cap\Omega}$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \lim_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle \\ &= \langle f(z) - z, w - z\rangle \le 0.\end{aligned} \tag{3.13}$$

Since $\alpha_n \to 0$, there exists $N_1 \in \mathbb{N}$ such that $(2-\rho)\alpha_n < 1$ for all $n \geq N_1$. Now, by (3.1) we have, for all $n > N_1$,

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)T^n u_n - p\|^2 \\
&\leq (1-\alpha_n)\|T^n u_n - p\|^2 + 2\alpha_n\langle f(x_n)-z, x_{n+1}-z\rangle \\
&\leq [(1-\alpha_n)k_n]^2\|u_n-p\|^2 + 2\alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle \\
&\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\leq [(1-\alpha_n)k_n]^2\|x_n-p\|^2 + 2\rho\alpha_n\|x_n-z\|\|x_{n+1}-z\| \\
&\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\leq [(1-\alpha_n)k_n]^2\|x_n-p\|^2 + \rho\alpha_n(\|x_n-z\|^2 + \|x_{n+1}-z\|^2) \\
&\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&= [(1-\alpha_n)(k_n-1) + (1-\alpha_n)]^2\|x_n-z\|^2 \\
&\quad + \rho\alpha_n\|x_n-z\|^2 + \rho\alpha_n\|x_{n+1}-z\|^2 + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&= [1-(2-\rho)\alpha_n + \alpha_n^2 + (1-\alpha_n)^2(k_n-1)^2 + 2(1-\alpha_n)^2(k_n-1)]\|x_n-z\|^2 \\
&\quad + \rho\alpha_n\|x_{n+1}-z\|^2 + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\leq [1-(2-\rho)\alpha_n + \alpha_n^2 + (k_n-1)^2 + 2(k_n-1)]\|x_n-z\|^2 + \rho\alpha_n\|x_{n+1}-z\|^2 \\
&\quad + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle.
\end{aligned}$$

So

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-(2-\rho)\alpha_n}{1-\rho\alpha_n}\|x_n-z\|^2 + \frac{\alpha_n^2 + (k_n-1)^2 + 2(k_n-1)}{1-\rho\alpha_n}M' \\
&\quad + \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(z)-z, x_{n+1}-z\rangle \\
&= (1 - \frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n})\|x_n-z\|^2 + \frac{\alpha_n^2 + (k_n-1)^2 + 2(k_n-1)}{1-\rho\alpha_n}M' \\
&\quad + \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(z)-z, x_{n+1}-z\rangle,
\end{aligned}$$

where $M' = \sup_{n\in\mathbb{N}}\|x_n-z\|^2$. Put

$$s_n = \frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}$$

and

$$\delta_n = \frac{\alpha_n^2 + (k_n-1)^2 + 2(k_n-1)}{2(1-\rho)\alpha_n}M + \frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle.$$

Then

$$\|x_{n+1}-z\|^2 \leq (1-s_n)\|x_n-z\|^2 + s_n\delta_n.$$

Note that $s_n \to 0$, $\sum_{n=1}^{\infty} s_n = \infty$ and $\limsup_{\to\infty}\delta_n \leq 0$. By theorem 2.6 we conclude that $\lim_{n\to\infty}\|x_n-z\| = 0$. This completes the proof. □

In Theorem 3.1, if $T_i \equiv T$, then the condition (Γ) is reduced to asymptotically regular and we get the following

Corollary 3.1 *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying* (A1-A4) *and assume that G is upper semicontinuous in the first argument. Let $f : C \to C$ be ρ-contraction and $T : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\}$ satisfying the condition that*

$$\lim_{n\to\infty}\sup_{x\in K}\|T^{n+1}x - T^n x\| = 0$$

for any bounded subset K of C. Assume that T is asymptotically regular and suppose that $Fix(T) \cap \Omega \neq \emptyset$*, where* $\Omega = \{v \in C : v \in EP(F) \text{ and } Av \in EP(G)\}$*. Let* $\{\alpha_n\} \subset (0,1)$ *be a sequence. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Define the sequence* $\{x_n\}$ *by the following manner:* $x_0 \in C$ *and*

$$\begin{cases} u_n = T^F_{r_n}(I - \gamma A^*(I - T^G_{s_n})A)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n u_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (r, \infty)$ *with* $r > 0$*,* $\{s_n\} \subset [s, \infty)$ *with* $s > 0$*,* $\gamma \subset (0, 1/L^2]$*, L is the spectral radius radius of the operator* A^*A *and* A^* *is the adjoint of A. If the control sequences* $\{\alpha_n\}$ *and* $\{k_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |s_n - s_{n-1}| < \infty$;

(iii) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$,

then $\{x_n\}$ *strongly converges to* $z = P_{Fix(T)\cap EP(F)} f(z)$*.*

In Corollary 3.1, if $A \equiv 0$, then we get the following

Corollary 3.2 *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty closed convex subsets. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1-A4)*. Let* $f : C \to C$ *be ρ-contraction and* $T : C \to C$ *be an asymptotically nonexpansive mapping with the sequence* $\{k_n\}$ *satisfying the condition that*

$$\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$$

for any bounded subset K of C. Assume that T is asymptotically regular and suppose that $Fix(T) \cap EP(F) \neq \emptyset$*. Let* $\{\alpha_n\} \subset (0,1)$ *be three sequence. Define the sequence* $\{x_n\}$ *by the following manner:* $x_0 \in C$ *and*

$$\begin{cases} u_n = T^F_{r_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n u_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset [r, \infty)$ *with* $r > 0$*. If the control sequences* $\{\alpha_n\}$*,* $\{r_n\}$*,* $\{s_n\}$ *and* $\{k_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;

(iii) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$,

then $\{x_n\}$ *strongly converges to* $z = P_{Fix(T)\cap EP(F)} f(z)$*.*

In Corollary 3.2, if $F(x, y) \equiv 0$ and $s_n \equiv 1$, then $u_n = P_C x_n = x_n$ and we get the following

Corollary 3.3 *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty closed convex subsets. Let* $f : C \to C$ *be ρ-contraction and* $T : C \to C$ *be an asymptotically nonexpansive mapping with the sequence* $\{k_n\}$ *satisfying the condition that*

$$\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$$

for any bounded subset K of C. Assume that T is asymptotically regular and suppose that $Fix(T) \neq \emptyset$*. Let* $\{\alpha_n\} \subset (0,1)$ *be three sequence. Define the sequence* $\{x_n\}$ *by the following manner:* $x_0 \in C$ *and*

$$\left\{ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n x_n, \ n \in \mathbb{N}. \right.$$

If the control sequences $\{\alpha_n\}$*,* $\{r_n\}$ *and* $\{k_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n < \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(iii) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$, *then* $\{x_n\}$ *strongly converges to* $z = P_{Fix(T)} f(z)$.

Remark 3.2 In [11–13], a gap appears in the computation process of $\|u_{n+1} - u_n\|$. In this paper, we use a new method to estimate the value of $\|u_{n+1} - u_n\|$ by Lemma 2.3 and the inverse strong monotonicity of $I - \gamma A^*(I - T^G_{s_n})A$, which is simpler and avoids the gap in [11–13].

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