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Complete Convergence and Complete Moment Convergence for Extended Negatively Dependent Random Variables

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Abstract. In this paper, we provide some probability and moment inequalities (especially the Marcinkiewicz-Zygmund type inequality) for extended negatively dependent (END, in short) random variables. By using the Marcinkiewicz-Zygmund type inequality and the truncation method, we investigate the complete convergence for sums and weighted sums of arrays of rowwise END random variables. In addition, the complete moment convergence for END random variables is obtained. Our results generalize and improve the corresponding ones of Wang et al. [18] and Baek and Park [2].

1. Introduction

It is well known that complete convergence plays an important role in probability limit theory and mathematical statistics, especially in establishing the convergence rate for sums and weighted sums of random variables. Recently, Kruglov et al. [6] obtained the following complete convergence theorem for arrays of rowwise independent random variables $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$, where $\{k_n, n \ge 1\}$ is a sequence of positive integers.

Theorem 1.1. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent random variables with $EX_{ni} = 0$ for all $1 \le i \le k_n, n \ge 1$ and $\{b_n, n \ge 1\}$ be a sequence of nonnegative constants. Suppose that the following conditions hold:*

(i) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$ for all $\varepsilon > 0$;

(ii) there exists $J \ge 1$ such that

$$\sum_{n=1}^{\infty} b_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 \right)^J < \infty.$$

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Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} b_n P\left(\max_{1\le m\le k_n}\left|\sum_{i=1}^{m} X_{ni}\right| > \varepsilon\right) < \infty.$$

Qiu et al. [10] generalized the result of Kruglov et al. [6] for independent random variables to the case of negatively dependent random variables and obtained the following result.

Theorem 1.2. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise negatively dependent random variables with $EX_{ni} = 0$ for all $1 \le i \le k_n, n \ge 1$ and $\{b_n, n \ge 1\}$ be a sequence of nonnegative constants. Suppose that condition (i) of Theorem 1.1 is satisfied and there exist constants $J \ge 1$ and $0 < p \le 2$ such that*

$$\sum_{n=1}^{\infty} b_n \left(\sum_{i=1}^{k_n} E\left|X_{ni}\right|^p\right)^J < \infty. \tag{1.1}$$

Then

$$\sum_{n=1}^{\infty} b_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.2}$$

However, Sung [12] pointed out that the proof of Theorem 1.2 is not correct. For more details about the complete convergence result, one can refer to Wu [23], Sung [13] and Wang and Hu [15].

The main purpose of the paper is to provide the correct proof of Theorem 1.2 and generalize the result of Theorem 1.2 for negatively dependent random variables to the case of extended negatively dependent (END, in short) random variables. In addition, we will provide the complete moment convergence for arrays of rowwise END random variables.

Now, let us recall the concept of END random variables.

Definition 1.3. *A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be extended negatively dependent (END, in short) if there exists a positive constant M independent of n such that both*

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \le M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n) \le M \prod_{i=1}^{n} P(X_i \le x_i)$$

hold for each $n \ge 1$ and all real numbers $x_1, x_2, \cdots, x_n$. An infinite sequence $\{X_n, n \ge 1\}$ is said to be END if every finite subcollection is END.

An array of random variables $\{X_{ni}, i \ge 1, n \ge 1\}$ is called rowwise END random variables if for every $n \ge 1$, $\{X_{ni}, i \ge 1\}$ is a sequence of END random variables.

The concept of END sequence was introduced by Liu [8]. In the case $M = 1$, the notion of END random variables reduces to the well-known notion of so-called negatively dependent (ND, in short) random variables which was introduced by Lehmann [7] (cf. also Joag-Dev and Proschan [5]). Not looking that the notion of END seems to be a straightforward generalization of the notion of negative dependence, the extended negative dependence structure is substantially more comprehensive. As it is mentioned in Liu [8], the END structure can reflect not only a negative dependence structure but also a positive one (inequalities from the definition of ND random variables hold both in reverse direction), to some extend. We refer the interested reader to Example 4.1 in Liu [8] where END random variables can be taken as negatively or positively dependent. Also, Joag-Dev and Proschan [5] pointed out that negatively associated (NA, in short) random variables are ND and thus NA random variables are END.

Some applications for END sequence have been found. See for example, Liu [8] obtained the precise large deviations for dependent random variables with heavy tails; Liu [9] studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails; Chen et al. [3] established the strong law of large numbers for extend negatively dependent random variables and showed applications to risk theory and renewal theory; Chen et al. [4] obtained the precise large deviations of random sums in presence of negative dependence and consistent variation; Shen [11] presented some probability inequalities for END sequences and gave some applications; Wang and Wang [14] studied the precise large deviations for random sums of END real-valued random variables with consistent variation; Wang et al. ([18], [19]) obtained some convergence results for weighted sums of END random variables; Wang et al. [21] established the complete consistency for the estimator of nonparametric regression models based on END error, and so forth. In this paper, our emphasis will be focused on the complete convergence for weighted sums of arrays of rowwise END random variables. In addition, the complete moment convergence for arrays of rowwise END random variables will also be considered.

The concept of stochastic domination below will be used throughout the paper.

Definition 1.4. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leq CP(|X| > x)$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

Our main results are as follows.

Theorem 1.5. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise END random variables with $EX_{ni} = 0$ for $1 \leq i \leq k_n, n \geq 1$ and $\{b_n, n \geq 1\}$ be a sequence of nonnegative constants. Suppose that the condition (i) of Theorem A is satisfied and there exist constants $J \geq 1$ and $0 < p \leq 2$ such that (1.1) satisfies. Then (1.2) holds.*

Applying Theorem 1.5, we can get the following complete convergence result for arrays of rowwise END random variables by using the Marcinkiewicz-Zygmund type inequality of END random variables.

Theorem 1.6. *Suppose that $\beta \geq -1$. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise END random variables with mean zero, which is stochastically dominated by a random variable X satisfying $E|X|^p < \infty$ for some $p \geq 1$. Let $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of constants satisfying*

$$\max_{1\leq i\leq k_n} |a_{ni}| = O\left(n^{-\gamma}\right) \text{ for some } \gamma > 0 \tag{1.3}$$

and

$$\sum_{i=1}^{k_n} |a_{ni}|^q = O\left(n^{-1-\beta+\gamma(p-q)}\right) \text{ for some } q < p. \tag{1.4}$$

Further assume that

$$\sum_{i=1}^{k_n} |a_{ni}|^t = O\left(n^{-\alpha}\right) \text{ for some } 0 < t \leq 2 \text{ and some } \alpha > 0 \tag{1.5}$$

if $p \geq 2$. Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni} X_{ni}\right| > \varepsilon\right) < \infty. \tag{1.6}$$

If $k_n = n$ and $a_{ni} \equiv n^{-\gamma}$ for $1 \leq i \leq n$ and $n \geq 1$, then we can get the following complete convergence result for END random variables.

Theorem 1.7. *Suppose that $\gamma p \geq 1$, $p \geq 1$ and $\gamma > \frac{1}{2}$. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise END random variables with mean zero, which is stochastically dominated by a random variable X satisfying $E|X|^p < \infty$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\gamma p-2} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon n^{\gamma}\right) < \infty. \tag{1.7}$$

By using Theorem 1.7, we can get the following complete moment convergence for END random variables.

Theorem 1.8. *Let the conditions of Theorem 1.7 hold and $p > 1$. Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} E\left(\left|\sum_{i=1}^{n} X_{ni}\right| - \varepsilon n^{\gamma}\right)^{+} < \infty. \tag{1.8}$$

Remark 1.9. Baek and Park [2] established the following result on complete convergence for arrays of rowwise ND random variables (see Theorem 3.1 of Baek and Park [2]).

Theorem 1.10. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise pairwise ND random variables with mean zero, which is stochastically dominated by a random variable X. Suppose that $\beta \geq -1$ and that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants satisfying*

$$\sup_{i \geq 1} |a_{ni}| = O\left(n^{-\gamma}\right) \ \textit{for some} \ \gamma > 0 \tag{1.9}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O\left(n^{\mu}\right) \ \textit{for some} \ \mu \in [0, \gamma). \tag{1.10}$$

If If $1 + \mu + \beta > 0$ and there exists some $\delta > 0$ such that $\mu/\gamma + 1 < \delta \leq 2$, $s = \max(1 + (1 + \mu + \beta)/\gamma, \delta)$, and $E|X|^s < \infty$, then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon\right) < \infty. \tag{1.11}$$

We point out that Theorem 1.10 can be obtained by Theorem 1.6 immediately. In fact, applying Theorem 1.6 with $k_n = \infty$, $p = 1 + (1 + \mu + \beta)/\gamma$, $q = 1$ and $\alpha = \gamma - \mu$, we can get that (1.4) holds by (1.9) and (1.10), and $E|X|^p < \infty$ is weaker than $E|X|^s < \infty$. Furthermore,

$$\sum_{i=1}^{\infty} |a_{ni}|^2 \leq \sup_{i \geq 1} |a_{ni}| \sum_{i=1}^{\infty} |a_{ni}| = O\left(n^{-\alpha}\right),$$

which implies that (1.5) holds for $t = 2$. Hence, (1.11) follows from Theorem 1.6 immediately.

Remark 1.11. Compared Theorem 1.7 with Corollary 3.1 of Wang et al. [18], we can see that the range of p and γ in Theorem 1.7 are wider than those in Corollary 3.1 of Wang et al. [18]. For example, γp can take value 1 in Theorem 1.7, while it can't take value 1 in Corollary 3.1 of Wang et al. [18]. In addition, the condition $E|X|\log|X| < \infty$ in Corollary 3.1 of Wang et al. [18] can be weakened by $E|X| < \infty$ when $p = 1$. So, our results of Theorem 1.7 generalize and improve the corresponding ones of Corollary 3.1 of Wang et al. [18].

In order to prove the main results of the paper, we need the Marcinkiewicz-Zygmund type inequality of END random variables, which will be presented in Section 2. The proofs of the main results will be given in Section 3.

Throughout the paper, all random variables are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Let $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of constants and $\{k_n, n \ge 1\}$ be a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$. Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise END random variables with the same constant $M > 0$ in each row. C and M denote positive constants not depending on n, which may be different in various places. We should note that all the results of this article remain true in the case $k_n = \infty$ for some/all $n \ge 1$, provided the series $\sum_{i=1}^{\infty} X_{ni}$ or $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ converges almost surely. Of course, we should consider sup instead of max in the case of infinite sums. For an event $A \in \mathcal{F}$, we denote by $I(A)$ the indicator function. Denote $x^+ = xI(x \ge 0)$ and $x^- = -xI(x < 0)$.

2. Preliminaries

In this section, we will present some important lemmas which will be used to prove the main results of the paper. Let $\{X_n, n \ge 1\}$ be a sequence of END random variables. Denote $S_n = \sum_{i=1}^{n} X_i$ and $M_{t,n} = \sum_{i=1}^{n} E|X_i|^t$ for some $t > 0$ and each $n \ge 1$.

The first one is the basic property for END random variables, which was given by Liu [9].

Lemma 2.1. *Let random variables $X_1, X_2, \cdots, X_n$ be END with some concrete constant $M > 0$.*

(i) If $f_1, f_2, \cdots, f_n$ are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ are END.

(ii) For each $n \ge 1$,

$$E\left(\prod_{j=1}^{n} X_j^+\right) \le M \prod_{j=1}^{n} EX_j^+.$$

The next one is the probability inequality for END random variables which will play an essential role to prove the main results of the paper. The proof can be found in Shen [11].

Lemma 2.2. *Let $0 < t \le 1$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with some concrete constant $M > 0$. Then for all $n \ge 1$, $x > 0$ and $y > 0$,*

$$P(|S_n| \ge x) \le \sum_{i=1}^{n} P(|X_i| \ge y) + 2M \exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy^{t-1}}{M_{t,n}}\right)\right\}. \tag{2.12}$$

For $1 \le t \le 2$, we have the following result.

Lemma 2.3. *Let $1 \le t \le 2$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with some concrete constant $M > 0$, and $EX_n = 0$ for each $n \ge 1$. Then for all $n \ge 1$, $x > 0$ and $y > 0$, (2.12) holds.*

Proof. The proof is similar to that of Lemma 3.2 and Theorem 2.2 in Asadian et al. [1]. So we omit the details. □

Remark 2.4. Combining Lemma 2.2 and Lemma 2.3, we can see that (2.12) holds for $0 < t \le 2$, provided that $EX_n = 0$ when $1 \le t \le 2$.

By using Lemma 2.3 and similar to the proof of Theorem 2.2 in Asadian et al. [1], we can get the following result. Here we omit the details of the proof.

Lemma 2.5. *Let $1 \le t \le 2$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with some concrete constant $M > 0$, and $EX_n = 0$ for each $n \ge 1$. Let $g(x)$ be a nonnegative even function and nondecreasing on the half-line $[0, \infty)$. Assume that $g(0) = 0$ and $Eg(X_i) < \infty$ for each $i \ge 1$, then for all $n \ge 1$ and $r > 0$,*

$$Eg(S_n) \le \sum_{i=1}^{n} Eg(rX_i) + 2Me^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x).$$

By taking $g(x) = |x|^p, p \ge t$ in Lemma 2.5, we can get the following moment inequality for END random variables.

Corollary 2.6. *Let $1 \le t \le 2$, $p \ge t$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with some concrete constant $M > 0$. Assume further that $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \ge 1$. Then there exists a positive constant $C(M, p, t)$ depending only on M, p and t such that*

$$E|S_n|^p \le C(M, p, t)\left(M_{p,n} + M_{t,n}^{p/t}\right). \tag{2.13}$$

Proof. Taking $g(x) = |x|^p, p \ge t$ in Lemma 2.5, we can get that for every $r > 0$,

$$E|S_n|^p \le r^p \sum_{i=1}^{n} E|X_i|^p + 2pMe^r \int_0^\infty x^{p-1}\left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx. \tag{2.14}$$

It is easy to check that

$$K := \int_0^\infty x^{p-1}\left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx = \int_0^\infty x^{p-1}\left(1 - \frac{x^t}{r^{t-1}M_{t,n} + x^t}\right)^{r} dx.$$

If we set $y = \frac{x^t}{r^{t-1}M_{t,n}+x^t}$ in the last equality above, then we have for $r > p/t$ that

$$K = \frac{r^{p-p/t}M_{t,n}^{p/t}}{t}\int_0^1 y^{\frac{p}{t}-1}(1-y)^{r-\frac{p}{t}-1}dy = \frac{r^{p-p/t}M_{t,n}^{p/t}}{t}B\left(\frac{p}{t}, r-\frac{p}{t}\right),$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx, \quad \alpha, \beta > 0$$

is the Beta function. Substitute I to (2.14) and choose

$$C(M, p, t) = \max\left\{r^p, 2pMe^r \frac{B\left(\frac{p}{t}, r-\frac{p}{t}\right) r^{p-p/t}}{t}\right\},$$

we can obtain the desired result (2.13) immediately. The proof is completed. □

If we set $p = t$ in Corollary 2.6, then we can get the following Marcinkiewicz-Zygmund type inequality for END random variables.

Corollary 2.7. *Let $1 \le p \le 2$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with some concrete constant $M > 0$. Assume further that $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \ge 1$. Then there exists a positive constant $C(M, p)$ depending only on M and p such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le C(M, p)\sum_{i=1}^{n} E|X_i|^p. \tag{2.15}$$

Remark 2.8. Assume that (2.15) holds for each $n \geq 1$ and $\sum_{i=1}^{\infty} X_i$ converges almost surely. Then

$$E\left|\sum_{i=1}^{\infty} X_i\right|^p \leq C(M,p)\sum_{i=1}^{\infty} E|X_i|^p \text{ for } 1 \leq p \leq 2. \tag{2.16}$$

In fact, by Fatou's Lemma and (2.15), we can get that

$$\begin{aligned} E\left|\sum_{i=1}^{\infty} X_i\right|^p &= E\left|\varliminf_{n\to\infty}\sum_{i=1}^{n} X_i\right|^p \leq E\left(\varliminf_{n\to\infty}\left|\sum_{i=1}^{n} X_i\right|^p\right) \\ &\leq \varliminf_{n\to\infty} E\left|\sum_{i=1}^{n} X_i\right|^p \leq C(M,p)\sum_{i=1}^{\infty} E|X_i|^p. \end{aligned}$$

This completes the proof of (2.16).

The following one is a fundamental inequality for stochastic domination. For the proof, one can refer to Wu [22], or Wang et al. ([16], [17]).

Lemma 2.9. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_{ni}|^{\alpha} I(|X_{ni}| \leq b) \leq C_1 \left[E|X|^{\alpha} I(|X| \leq b) + b^{\alpha} P(|X| > b)\right],$$

$$E|X_{ni}|^{\alpha} I(|X_{ni}| > b) \leq C_2 E|X|^{\alpha} I(|X| > b),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_{ni}|^{\alpha} \leq CE|X|^{\alpha}$.

The last one is the Rosenthal type inequality for END random variables, which was obtained by Shen [11].

Lemma 2.10. *'Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with some concrete constant $M > 0$. Assume further that $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exists a positive constant $C(M,p)$ depending only on M and p such that*

$$E|S_n|^p \leq C(M,p)\left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{p/2}\right].$$

3. Proofs of the Main Results

Proof of Theorem 1.5. Applying Remark 2.4 with $x = \varepsilon$, $y = \varepsilon/J$ and $t = p$, we can get that

$$\begin{aligned} P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon/J) + 2Me^J\left(1 + \frac{\varepsilon^p/J^{p-1}}{\sum_{i=1}^{k_n} E|X_{ni}|^p}\right)^{-J} \\ &\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon/J) + 2Me^J J^{J(p-1)}\varepsilon^{-Jp}\left(\sum_{i=1}^{k_n} E|X_{ni}|^p\right)^J. \end{aligned}$$

Hence, the desired result (1.2) follows from conditions (i) of Theorem 1.1, (1.1) and the inequality above immediately. The proof is complete. □

Proof of Theorem 1.6. Without loss of generality, we assume that $a_{ni} \ge 0$ for all $1 \le i \le k_n$ and $n \ge 1$ (Otherwise, we use a_{ni}^+ and a_{ni}^- instead of a_{ni}, respectively and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). For $1 \le i \le k_n$ and $n \ge 1$, define

$$\begin{aligned} X'_{ni} &= -n^\gamma I(X_{ni} < -n^\gamma) + X_{ni} I(|X_{ni}| \le n^\gamma) + n^\gamma I(X_{ni} > n^\gamma), \\ X''_{ni} &= X_{ni} - X'_{ni} = (X_{ni} - n^\gamma) I(X_{ni} > n^\gamma) + (X_{ni} + n^\gamma) I(X_{ni} < -n^\gamma). \end{aligned}$$

By Lemma 2.1 (i), we can see that $\{X'_{ni}, 1 \le i \le k_n, n \ge 1\}$ and $\{X''_{ni}, 1 \le i \le k_n, n \ge 1\}$ are still arrays of rowwise END random variables, which implies that $\{a_{ni}X'_{ni}, 1 \le i \le k_n, n \ge 1\}$ and $\{a_{ni}X''_{ni}, 1 \le i \le k_n, n \ge 1\}$ are also arrays of rowwise END random variables. It is easily seen that

$$\begin{aligned} \left|X'_{ni}\right| &= |X_{ni}| I(|X_{ni}| \le n^\gamma) + n^\gamma I(|X_{ni}| > n^\gamma) \le |X_{ni}|, \\ \left|X''_{ni}\right| &= (X_{ni} - n^\gamma) I(X_{ni} > n^\gamma) - (X_{ni} + n^\gamma) I(X_{ni} < -n^\gamma) \\ &\le |X_{ni}| I(|X_{ni}| > n^\gamma) \le |X_{ni}|. \end{aligned}$$

Since $EX_{ni} = 0$, in order to prove (1.6), it suffices to show that for all $\varepsilon > 0$,

$$H := \sum_{n=1}^{\infty} n^\beta P\left(\left|\sum_{i=1}^{k_n} a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right| > \varepsilon\right) < \infty \tag{3.17}$$

and

$$G := \sum_{n=1}^{\infty} n^\beta P\left(\left|\sum_{i=1}^{k_n} a_{ni}\left(X''_{ni} - EX''_{ni}\right)\right| > \varepsilon\right) < \infty. \tag{3.18}$$

We will consider the following three cases.

Case 1: $p = 1$.

For H, we have by Markov's inequality, Remark 2.8, Lemma 2.9, (1.3) and (1.4) that

$$\begin{aligned} H &\ll \sum_{n=1}^{\infty} n^\beta E\left|\sum_{i=1}^{k_n} a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right|^2 \ll \sum_{n=1}^{\infty} n^\beta \sum_{i=1}^{k_n} a_{ni}^2 E\left|X'_{ni}\right|^2 \\ &\ll \sum_{n=1}^{\infty} n^\beta \sum_{i=1}^{k_n} a_{ni}^2 \left[EX^2 I(|X| \le n^\gamma) + n^{2\gamma} P(|X| > n^\gamma)\right] \\ &\ll \sum_{n=1}^{\infty} n^\beta \max_{1\le i\le k_n} |a_{ni}|^{2-q} \sum_{i=1}^{k_n} |a_{ni}|^q \left[EX^2 I(|X| \le n^\gamma) + n^{2\gamma} P(|X| > n^\gamma)\right] \\ &\ll \sum_{n=1}^{\infty} n^\beta n^{-\gamma(2-q)} n^{-1-\beta+\gamma(1-q)} \left[EX^2 I(|X| \le n^\gamma) + n^{2\gamma} P(|X| > n^\gamma)\right] \\ &= \sum_{n=1}^{\infty} n^{-1-\gamma} EX^2 I(|X| \le n^\gamma) + C\sum_{n=1}^{\infty} n^{-1+\gamma} P(|X| > n^\gamma) \\ &= \sum_{n=1}^{\infty} n^{-1-\gamma} \sum_{i=1}^{n} EX^2 I((i-1)^\gamma < |X| \le i^\gamma) + \sum_{n=1}^{\infty} n^{-1+\gamma} \sum_{i=n}^{\infty} P(i^\gamma < |X| \le (i+1)^\gamma) \\ &\ll \sum_{i=1}^{\infty} EX^2 I((i-1)^\gamma < |X| \le i^\gamma) i^{-\gamma} + \sum_{i=1}^{\infty} P(i^\gamma < |X| \le (i+1)^\gamma) i^\gamma \\ &\ll E|X| < \infty. \end{aligned}$$

For G, we first prove that

$$\sum_{i=1}^{k_n} |a_{ni}| E\left|X''_{ni}\right| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

By Lemma 2.9, (1.3) and (1.4) again, we can get that

$$\begin{aligned}\sum_{i=1}^{k_n}|a_{ni}|\,E\left|X''_{ni}\right| &\ll \sum_{i=1}^{k_n}|a_{ni}|\,E\,|X|\,I\,(|X|>n^{\gamma})\\ &\ll \max_{1\le i\le k_n}|a_{ni}|^{1-q}\sum_{i=1}^{k_n}|a_{ni}|^{q}\,E\,|X|\,I\,(|X|>n^{\gamma})\\ &\ll n^{-1-\beta}E\,|X|\,I\,(|X|>n^{\gamma})\to 0\ \text{ as }\ n\to\infty,\end{aligned}$$

which implies (3.19).

Hence, in order to prove (3.18), it suffices to prove that

$$G^{*}:=\sum_{n=1}^{\infty}n^{\beta}P\left(\left|\sum_{i=1}^{k_n}a_{ni}X''_{ni}\right|>\varepsilon\right)<\infty. \tag{3.20}$$

Take $0<\delta<1$ such that $1-\delta=p-\delta>q$. Thus, we have by Markov's inequality, Lemma 2.9, (1.3) and (1.4) that

$$\begin{aligned}G^{*} &\ll \sum_{n=1}^{\infty}n^{\beta}E\left|\sum_{i=1}^{k_n}a_{ni}X''_{ni}\right|^{1-\delta} \ll \sum_{n=1}^{\infty}n^{\beta}\sum_{i=1}^{k_n}|a_{ni}|^{1-\delta}\,E\left|X''_{ni}\right|^{1-\delta}\\ &\ll \sum_{n=1}^{\infty}n^{\beta}\sum_{i=1}^{k_n}|a_{ni}|^{1-\delta}\,E\,|X_{ni}|^{1-\delta}\,I\,(|X_{ni}|>n^{\gamma})\\ &\ll \sum_{n=1}^{\infty}n^{\beta}\max_{1\le i\le k_n}|a_{ni}|^{1-\delta-q}\sum_{i=1}^{k_n}|a_{ni}|^{q}\,E\,|X|^{1-\delta}\,I\,(|X|>n^{\gamma})\\ &\ll \sum_{n=1}^{\infty}n^{\beta}n^{-\gamma(1-\delta-q)}n^{-1-\beta+\gamma(1-q)}E\,|X|^{1-\delta}\,I\,(|X|>n^{\gamma})\\ &= \sum_{n=1}^{\infty}n^{-1+\gamma\delta}E\,|X|^{1-\delta}\,I\,(|X|>n^{\gamma})\\ &= \sum_{n=1}^{\infty}n^{-1+\gamma\delta}\sum_{i=n}^{\infty}E\,|X|^{1-\delta}\,I\,(i^{\gamma}<|X|\le(i+1)^{\gamma})\\ &= \sum_{i=1}^{\infty}E\,|X|^{1-\delta}\,I\,(i^{\gamma}<|X|\le(i+1)^{\gamma})\sum_{n=1}^{i}n^{-1+\gamma\delta}\\ &\ll \sum_{i=1}^{\infty}E\,|X|^{1-\delta}\,I\,(i^{\gamma}<|X|\le(i+1)^{\gamma})\,i^{\gamma\delta}\ \ll\ E|X|\ <\ \infty,\end{aligned} \tag{3.21}$$

which implies (3.20)

Case 2: $1<p<2$**.**

In this case, we can get that $H\ll E|X|^{p}<\infty$ by the similar method as that in Case 1.

For G, we take $\delta>0$ such that $p-\delta>\max\{1,q\}$. Similar to the proof of (3.21), we have by Corollary 2.7, Remark 2.8 and C_r inequality that

$$\begin{aligned}
G &\ll \sum_{n=1}^{\infty} n^{\beta} E\left|\sum_{i=1}^{k_n} a_{ni}\left(X''_{ni} - EX''_{ni}\right)\right|^{p-\delta} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} |a_{ni}|^{p-\delta} E\left|X''_{ni}\right|^{p-\delta} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \max_{1\le i\le k_n} |a_{ni}|^{p-\delta-q} \sum_{i=1}^{k_n} |a_{ni}|^{q} E|X|^{p-\delta} I(|X| > n^{\gamma}) \\
&\ll \sum_{n=1}^{\infty} n^{-1+\gamma\delta} E|X|^{p-\delta} I(|X| > n^{\gamma}) \\
&= \sum_{n=1}^{\infty} n^{-1+\gamma\delta} \sum_{i=n}^{\infty} E|X|^{p-\delta} I(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\ll \sum_{i=1}^{\infty} E|X|^{p-\delta} I(i^{\gamma} < |X| \le (i+1)^{\gamma})\, i^{\gamma\delta} \ll E|X|^{p} < \infty. \qquad (3.22)
\end{aligned}$$

Case 3: $p \ge 2$**.**

In this case, we will prove (3.17) and (3.18) by using Theorem 1.5. To prove (3.17), we take $\delta > 0$. Hence, we have by Markov's inequality, C_r's inequality, Lemma 2.9, (1.3) and (1.4) that for all $\varepsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} P\left(\left|a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right| > \varepsilon\right) &\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} E\left|a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right|^{p+\delta} \ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} |a_{ni}|^{p+\delta} E\left|X'_{ni}\right|^{p+\delta} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \max_{1\le i\le k_n} |a_{ni}|^{p+\delta-q} \sum_{i=1}^{k_n} |a_{ni}|^{q} \left[E|X|^{p+\delta} I(|X| \le n^{\gamma}) + n^{\gamma(p+\delta)} P(|X| > n^{\gamma})\right] \\
&\ll \sum_{n=1}^{\infty} n^{-1-\gamma\delta} E|X|^{p+\delta} I(|X| \le n^{\gamma}) + \sum_{n=1}^{\infty} n^{-1+\gamma p} P(|X| > n^{\gamma}) \\
&= \sum_{n=1}^{\infty} n^{-1-\gamma\delta} \sum_{i=1}^{n} E|X|^{p+\delta} I((i-1)^{\gamma} < |X| \le i^{\gamma}) \\
&\quad + \sum_{n=1}^{\infty} n^{-1+\gamma p} \sum_{i=n}^{\infty} P(i^{\gamma} < |X| \le (i+1)^{\gamma}) \\
&\ll \sum_{i=1}^{\infty} E|X|^{p+\delta} I((i-1)^{\gamma} < |X| \le i^{\gamma})\, i^{-\gamma\delta} + \sum_{i=1}^{\infty} P(i^{\gamma} < |X| \le (i+1)^{\gamma})\, i^{\gamma p} \\
&\ll E|X|^{p} < \infty. \qquad (3.23)
\end{aligned}$$

Take $J \ge 1$ such that $\alpha J - \beta > 1$. We have by (1.5) and $E|X|^{t} < \infty$ (since $t \le p$) that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{k_n} E\left|a_{ni}\left(X'_{ni} - EX'_{ni}\right)\right|^{t}\right)^{J} &\ll \sum_{n=1}^{\infty} n^{\beta} \left[\sum_{i=1}^{k_n} |a_{ni}|^{t} \left(E\left|X'_{ni}\right|^{t} + \left(E\left|X'_{ni}\right|\right)^{t}\right)\right]^{J} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \left[\sum_{i=1}^{k_n} |a_{ni}|^{t} \left(E|X|^{t} + (E|X|)^{t}\right)\right]^{J} \\
&\ll \sum_{n=1}^{\infty} n^{\beta-\alpha J} < \infty. \qquad (3.24)
\end{aligned}$$

Therefore, (3.17) follows from Theorem 1.5 and the statements above immediately.

To prove (3.18), we take $\delta > 0$ such that $p - \delta > \max\{1, q\}$. Similar to the proof of (3.22) and (3.23), we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} P\left(\left|a_{ni}\left(X''_{ni} - EX''_{ni}\right)\right| > \varepsilon\right) &\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} E\left|a_{ni}\left(X''_{ni} - EX''_{ni}\right)\right|^{p-\delta} \\ &\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{k_n} |a_{ni}|^{p-\delta} E\left|X''_{ni}\right|^{p-\delta} \\ &\ll E|X|^p < \infty.\end{aligned}$$

Similar to the proof of (3.24), we still have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{k_n} E\left|a_{ni}\left(X''_{ni} - EX''_{ni}\right)\right|^t\right)^J &\ll \sum_{n=1}^{\infty} n^{\beta} \left[\sum_{i=1}^{k_n} |a_{ni}|^t \left(E\left|X''_{ni}\right|^t + \left(E\left|X''_{ni}\right|\right)^t\right)\right]^J \\ &\ll \sum_{n=1}^{\infty} n^{\beta} \left[\sum_{i=1}^{k_n} |a_{ni}|^t \left(E|X|^t + (E|X|)^t\right)\right]^J \\ &\ll \sum_{n=1}^{\infty} n^{\beta - \alpha J} < \infty,\end{aligned}$$

by taking $J \geq 1$ such that $\alpha J - \beta > 1$. Therefore, (3.18) follows from Theorem 1.5 and the statements above immediately. This completes the proof of the theorem. □

Proof of Theorem 1.7. We only need to show that the conditions of Theorem 1.6 hold. Applying Theorem 1.6 with $\beta = \gamma p - 2 \geq -1$ and $a_{ni} \equiv n^{-\gamma}$ for $1 \leq i \leq n$ and $n \geq 1$, we can see that

$$\sum_{i=1}^{n} |a_{ni}|^q = \sum_{i=1}^{n} n^{-\gamma q} = n^{1-\gamma q} = n^{-1-\beta+\gamma(p-q)}$$

and

$$\sum_{i=1}^{n} |a_{ni}|^t = \sum_{i=1}^{n} n^{-\gamma t} = n^{1-\gamma t} \text{ for } 1/\gamma < t \leq 2.$$

Hence, the conditions (1.3)–(1.5) are satisfied, which yield (1.7) by Theorem 1.6. The proof is complete. □

Proof of Theorem 1.8. For any $\varepsilon > 0$, we have by Theorem 1.7 that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} E\left(\left|\sum_{i=1}^{n} X_{ni}\right| - \varepsilon n^{\gamma}\right)^+ &= \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_0^{\infty} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| - \varepsilon n^{\gamma} > t\right) dt \\ &\leq \sum_{n=1}^{\infty} n^{\gamma p-2} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > \varepsilon n^{\gamma}\right) + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^{\gamma}}^{\infty} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > t\right) dt \\ &\ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^{\gamma}}^{\infty} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > t\right) dt.\end{aligned}$$

Hence, to prove (1.8), it suffices to show that

$$Q := \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^{\gamma}}^{\infty} P\left(\left|\sum_{i=1}^{n} X_{ni}\right| > t\right) dt < \infty. \tag{3.25}$$

For $t > 0$, denote for $1 \le i \le n$ and $n \ge 1$ that

$$\begin{aligned} Y_{nit} &= -tI(X_{ni} < -t) + X_{ni}I(|X_{ni}| \le t) + tI(X_{ni} > t), \\ Z_{nit} &= X_{ni} - Y_{nit} = (X_{ni} - t)I(X_{ni} > t) + (X_{ni} + t)I(X_{ni} < -t). \end{aligned}$$

Note that $EX_{ni} = 0$, we have

$$\begin{aligned} Q &\le \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} P\left(\left|\sum_{i=1}^{n} Z_{nit}\right| > t/3\right) dt + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} P\left(\left|\sum_{i=1}^{n} EZ_{nit}\right| > t/3\right) dt \\ &\quad + \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} P\left(\left|\sum_{i=1}^{n} (Y_{nit} - EY_{nit})\right| > t/3\right) dt \\ &:= Q_1 + Q_2 + Q_3. \end{aligned}$$

For Q_1, it follows by Markov's inequality that

$$\begin{aligned} Q_1 &\le \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} \sum_{i=1}^{n} P(|X_{ni}| > t)\, dt \ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \int_{n^\gamma}^{\infty} P(|X| > t)\, dt \\ &\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \int_{n^\gamma}^{\infty} t^{-1} E|X| I(|X| > t)\, dt \\ &\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \sum_{m=n}^{\infty} \int_{m^\gamma}^{(m+1)^\gamma} t^{-1} E|X| I(|X| > t)\, dt \\ &\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \sum_{m=n}^{\infty} m^{-1} E|X| I(|X| > m^\gamma) \\ &= \sum_{m=1}^{\infty} m^{-1} E|X| I(|X| > m^\gamma) \sum_{n=1}^{m} n^{\gamma p-1-\gamma} \\ &\ll \sum_{m=1}^{\infty} m^{\gamma p-\gamma-1} E|X| I(|X| > m^\gamma) \\ &= \sum_{m=1}^{\infty} m^{\gamma p-\gamma-1} \sum_{n=m}^{\infty} E|X| I(n^\gamma < |X| \le (n+1)^\gamma) \\ &= \sum_{n=1}^{\infty} E|X| I(n^\gamma < |X| \le (n+1)^\gamma) \sum_{m=1}^{n} m^{\gamma p-\gamma-1} \\ &\ll \sum_{n=1}^{\infty} E|X| I(n^\gamma < |X| \le (n+1)^\gamma)\, n^{\gamma(p-1)} \ll E|X|^p < \infty. \end{aligned}$$

Note that $|Z_{nit}| \le |X_{ni}| I(|X_{ni}| > t)$, we have by Markov's inequality, Lemma 2.9 and (3.11) that

$$\begin{aligned} Q_2 &\le 3\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-1} \left|\sum_{i=1}^{n} EZ_{nit}\right| dt \\ &\le 3\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-1} \sum_{i=1}^{n} E|X_{ni}| I(|X_{ni}| > t)\, dt \\ &\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \int_{n^\gamma}^{\infty} t^{-1} E|X| I(|X| > t)\, dt \ll E|X|^p < \infty. \end{aligned}$$

For fixed $n \geq 1$ and $t > 0$, it is easily seen that $\{Y_{nit} - EY_{nit}, 1 \leq i \leq n\}$ are still END random variables with mean zero by Lemma 2.1. By Markov's inequality, Jensen's inequality, C_r-inequality and Lemma 2.10, we have that for any $q \geq 2$,

$$\begin{aligned}
Q_3 &\ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-q} E\left|\sum_{i=1}^{n}(Y_{nit} - EY_{nit})\right|^q dt \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-q} \sum_{i=1}^{n} E\left|Y_{nit}\right|^q dt + C\sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-q} \left(\sum_{i=1}^{n} EY_{nit}^2\right)^{q/2} dt \\
&:= Q_{31} + Q_{32}. \qquad (3.26)
\end{aligned}$$

We will consider the following three cases.

Case 1. $\gamma > 1/2$, $\gamma p > 1$ **and** $p \geq 2$.

Take q large enough such that $q > \max\left(p, \frac{\gamma p-1}{\gamma-\frac{1}{2}}\right)$, it follows that $\gamma p - 2 - \gamma q + q/2 < -1$. We have by Lemma 2.9 and (3.26) that

$$\begin{aligned}
Q_{31} &\ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-q} \sum_{i=1}^{n} \left[E|X_{ni}|^q I(|X_{ni}| \leq t) + t^q P\left(|X_{ni}| > t\right)\right] dt \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \int_{n^\gamma}^{\infty} t^{-q} E|X|^q I(|X| \leq t) dt + \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \int_{n^\gamma}^{\infty} P\left(|X| > t\right) dt \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \sum_{m=n}^{\infty} \int_{m^\gamma}^{(m+1)^\gamma} t^{-q} E|X|^q I(|X| \leq t) dt \\
&\ll \sum_{n=1}^{\infty} n^{\gamma p-1-\gamma} \sum_{m=n}^{\infty} m^{\gamma-1-\gamma q} E|X|^q I\left(|X| \leq (m+1)^\gamma\right) \\
&\ll \sum_{m=1}^{\infty} m^{\gamma-1-\gamma q} E|X|^q I\left(|X| \leq (m+1)^\gamma\right) \sum_{n=1}^{m} n^{\gamma p-1-\gamma} \ll \sum_{m=1}^{\infty} m^{\gamma p-1-\gamma q} E|X|^q I\left(|X| \leq (m+1)^\gamma\right) \\
&\ll \sum_{m=1}^{\infty} m^{\gamma p-1-\gamma q} E|X|^q I\left(|X| \leq m^\gamma\right) + \sum_{m=1}^{\infty} m^{\gamma p-1-\gamma q} E|X|^q I\left(m^\gamma < |X| \leq (m+1)^\gamma\right) \\
&\ll \sum_{m=1}^{\infty} m^{\gamma p-1-\gamma q} \sum_{n=1}^{m} E|X|^q I\left((n-1)^\gamma < |X| \leq n^\gamma\right) + E|X|^p \\
&\ll \sum_{n=1}^{\infty} E|X|^q I\left((n-1)^\gamma < |X| \leq n^\gamma\right) \sum_{m=n}^{\infty} m^{\gamma p-1-\gamma q} + E|X|^p \\
&\ll \sum_{n=1}^{\infty} E|X|^q I\left((n-1)^\gamma < |X| \leq n^\gamma\right) n^{\gamma p-\gamma q} + E|X|^p \\
&\ll E|X|^p < \infty. \qquad (3.27)
\end{aligned}$$

Since $p \geq 2$ and $E|X|^p < \infty$, it follows that $EY_{nit}^2 \leq EX_{ni}^2 \leq CEX^2 < \infty$. Hence,

$$\begin{aligned}
Q_{32} &\ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-q} \left(\sum_{i=1}^{n} EX^2\right)^{q/2} dt = \left(EX^2\right)^{q/2} \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma+q/2} \int_{n^\gamma}^{\infty} t^{-q} dt \\
&\ll \left(EX^2\right)^{q/2} \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma q+q/2} < \infty.
\end{aligned}$$

Case 2. $\gamma > 1/2$, $\gamma p > 1$ **and** $1 < p < 2$.

Take $q = 2$. Similar to the proof of (3.26) and (3.27), we can get that

$$Q_3 \ll \sum_{n=1}^{\infty} n^{\gamma p-2-\gamma} \int_{n^\gamma}^{\infty} t^{-2} \sum_{i=1}^{n} E\left|Y_{nit}\right|^2 dt < \infty. \tag{3.28}$$

Case 3. $\gamma > 1/2$, $\gamma p = 1$.

Note that $p = 1/\gamma < 2$. Take $q = 2$, and similar to the proof of (3.28), we still have $Q_3 < \infty$.

From the statements above, we have proved (3.25). This completes the proof of the theorem. □

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