On Some Banach Space Properties Sufficient for Normal Structure

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Abstract. In this paper, we present some sufficient conditions for which a Banach space has normal structure and therefore the fixed point property for nonexpansive mappings in terms of the generalized James, von Neumann-Jordan, Zbăganu constants, the Ptolemy constant and the Domínguez-Benavides coefficient. Our main results extend and improve some known results in the recent literature.

1. Introduction

It is well known that the concepts of normal structure and weak normal structure play an important role in metric fixed point theory for nonexpansive mappings (see [14]). For many years metric fixed point theory just studied some extensions of Banach Theorem relaxing the contractiveness condition, and the extension of this result for multivalued mappings. In 1965, metric fixed point theory received a strong boost when Kirk [20] proved that every reflexive Banach space X with normal structure has the fixed point property for nonexpansive mappings. In other words, if C is any nonempty, bounded, closed and convex subset of X, then every nonexpansive (i.e., 1-Lipschitz) self-mapping of C has a fixed point.

The result obtained by Kirk is, in some sense, surprising because it uses geometric properties of Banach spaces (commonly used in Linear Functional Analysis, but rarely considered in Nonlinear Analysis until then). Thus, it is the starting point for a new mathematical field: the application of the Geometric Theory of Banach Spaces to Fixed Point Theory. From that moment on, many researchers have tried to exploit this connection.

Recently, many geometric constants for a Banach space have been investigated and a good deal of investigations have focused on finding the sufficient conditions with various geometrical constants for a Banach space to have normal structure. Whether or not a Banach space has normal structure depends on the geometry of its unit ball or its unit sphere. Many mathematicians have established that, under various geometric properties of a Banach space often measured by different geometric constants, normal structure of the space is guaranteed. The James and von Neumann-Jordan constants are two most widely studied constants, due to their connections with various geometric structure

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of Banach spaces. The interested reader is referred to [2–10, 12, 13, 17, 19, 22, 24] for a widespread discussion and recent results in this direction.

The aim of this work is to investigate some sufficient conditions for normal structure of Banach spaces in terms of the generalized James, von Neumann-Jordan, Zbăganu constants, the Ptolemy constant and the Domínguez-Benavides coefficient. The obtained results extend and improve several recent results on this subject. Furthermore, we give different examples which show that these results are strictly more general than other results previously known to imply normal structure.

2. Preliminaries

We shall begin by recalling the definition of some geometric parameters with which we will be concerned throughout the paper.

Let X be a Banach space with the closed unit ball $B_X = \{x \in X \ : \ \|x\| \leq 1\}$ and the unit sphere $S_X = \{x \in X \ : \ \|x\| = 1\}$. $x_n \xrightarrow{w} x$ stands for weak convergence of a sequence $\{x_n\}$ in X to a point x in X. The following two constants of a Banach space X,

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \ : \ x, y \in X, \ \|x\| + \|y\| > 0\right\},$$

$$J(X) = \sup\left\{\min\{\|x+y\|, \|x-y\|\} \ : \ x, y \in S_X\right\},$$

are called the von Neumann-Jordan [3] and James constants [12], respectively. As stated in the introduction, these constants are widely studied by many authors.

Recently, both constants are generalized in the following ways (see [5, 6]): for $0 \leq a \leq 2$,

$$C_{NJ}(a, X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \ : \ x, y, z \in X, \ \|x\| + \|y\| + \|z\| > 0, \ \text{and} \ \|y-z\| \leq a\|x\|\right\},$$

$$J(a, X) = \sup\left\{\min\{\|x+y\|, \|x-z\|\} \ : \ x, y, z \in B_X, \ \text{and} \ \|y-z\| \leq a\|x\|\right\}.$$

It is clear that $C_{NJ}(0, X) = C_{NJ}(X)$ and $J(0, X) = J(X)$.

Recently, Gao and Saejung in [13] defined a new constant as follows: for $a \geq 0$,

$$C_Z(a, X) = \sup\left\{\frac{2\|x+y\|\|x-z\|}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \ : \ x, y, z \in X, \ \|x\| + \|y\| + \|z\| > 0, \ \text{and} \ \|y-z\| \leq a\|x\|\right\},$$

which is inspired by Zbăganu paper [26]. It is clear that

$$C_Z(0, X) = C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-z\|}{\|x\|^2 + \|y\|^2} \ : \ x, y \in X, \ \|x\| + \|y\| > 0\right\}.$$

Recall also that the real number

$$C_p(X) = \sup\left\{\frac{\|x-y\|\,\|z\|}{\|x-z\|\,\|y\| + \|z-y\|\,\|x\|} \ : \ x, y, z \in X \backslash \{0\}, \ x \neq y \neq z \neq x\right\}$$

is called the Ptolemy constant. The notion of the Ptolemy constant of Banach spaces was introduced in [23] and recently it has been studied by Llorens-Fuster et al. in [21].

As regards the above constants, the following inequalities do hold (see [21]):

$$\frac{1}{2}\big(J(X)\big)^2 \le C_Z(X) \le \min\big\{C_{NJ}(X), C_p(X)\big\}. \tag{1}$$

The coefficient $M(X) \in [1, 2]$ of X, introduced by Domínguez-Benavides [10], is defined by

$$M(X) = \sup\left\{\frac{1+\alpha}{R(\alpha, X)} \; : \; \alpha \ge 0\right\},$$

with

$$R(\alpha, X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le \alpha$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D(\{x_n\}) = \limsup_{n\to\infty}\left(\limsup_{m\to\infty} \|x_n - x_m\|\right) \le 1.$$

Obviously, $1 \le R(1, X) \le 2$.

Recall that a Banach space X is said to have normal structure (weak normal structure, respectively) [1] if for every bounded closed (weakly compact, respectively) convex subset K in X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\big\{\|x_0 - y\| \; : \; y \in K\big\} < \sup\big\{\|x - y\| \; : \; x, y \in K\big\}.$$

It is clear that for a reflexive Banach space, normal structure and weak normal structure coincide. It is well known (see [14]) that if X fails to have weak normal structure, then there exist a weakly compact convex subset $C \subset X$ and a sequence $\{x_n\} \subset C$ such that $\text{dist}\big(x_{n+1}, \text{co}\{x_k\}_{k=1}^n\big) \to \text{diam}(C) = 1$.

A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\big\{\|x_0 - y\| \; : \; y \in K\big\} < c\,\sup\big\{\|x - y\| \; : \; x, y \in K\big\}.$$

Recall also that a Banach space X is uniformly nonsquare [15] if and only if there exists $\delta > 0$ such that $\|x + y\| \le 2 - \delta$ or $\|x - y\| \le 2 - \delta$ for all $x, y \in B_X$. It is known that every uniformly nonsquare space is super-reflexive (see [15]). It is worth mentioning that if $J(a, X) < 2$ or $C_{NJ}(a, X) < 2$ for some $0 \le a < 2$, then X is uniformly nonsquare (see [5, 6]).

In the sequel, we recall some basic facts about ultrapowers which are the main ingredient of our results. Ultrapowers are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting.

Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{U}$, denoted by $\lim_{\mathcal{U}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} \; : \; x_i \in U\} \in \mathcal{U}$. Let $\ell_\infty(X)$ denotes the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty,$$

and let

$$N_{\mathcal{U}} = \Big\{(x_n) \in \ell_\infty(X) \; : \; \lim_{\mathcal{U}} \|x_n\| = 0\Big\}.$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $\frac{\ell_\infty(X)}{N\mathcal{U}}$ equipped with the quotient norm. Write $(x_n)_\mathcal{U}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\left\|(x_n)_\mathcal{U}\right\| = \lim_{\mathcal{U}} \|x_n\|.$$

Note that X can be embedded into $\widetilde{X}$ isometrically. For more detailed discussion on the Banach space ultrapower construction, the reader is directed to [14, 18, 25]. We also note that if X is super-reflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [18]).

3. Main Results

We first start with a lemma. The following lemma is our main tool for proving our results. The idea of the proof is same as the proof of Lemma 5 in [24].

Lemma 3.1. *Let X be a Banach space for which B_{X^*} is w^*-sequentially compact (for example, X is reflexive or separable, or has an equivalent smooth norm). Suppose that X fails to have weak normal structure. Then, for any $\varepsilon > 0$, there exist $z_1, z_2 \in S_X$ and $g_1, g_2 \in S_{X^*}$ such that the following conditions are satisfied:*

(a) $\left| \|z_1 - z_2\| - 1 \right| < \varepsilon$ *and* $\left| g_i(z_j) \right| < \varepsilon$ *for all $i \neq j$;*

(b) $g_i(z_i) = 1$ *for $i = 1, 2$;*

(c) $\|z_1 + z_2\| \leq R(1, X) + \varepsilon$.

Proof. By the assumptions, there exists a sequence $\{x_n\} \subset X$ such that

(1) $x_n \xrightarrow{w} 0$,

(2) $\mathrm{diam}\left(\{x_n\}_{n=1}^\infty\right) = 1 = \lim_{n \to \infty} \|x_n - x\|$ for all $x \in \overline{\mathrm{co}}\,\{x_n\}_{n=1}^\infty$.

Since B_{X^*} is w^*-sequentially compact, passing to a suitable subsequence, we may assume that there exist a sequence $(f_n) \subset S_{X^*}$ such that

(3) $f_n(x_n) = \|x_n\|$ for all $n \in \mathbb{N}$, and

(4) $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$.

Observe that 0 is in the weakly closed convex hull of $\{x_n\}_{n=1}^\infty$ which equals the norm closed convex hull $\overline{\mathrm{co}}\,\{x_n\}_{n=1}^\infty$. This implies that $\lim_{n \to \infty} \|x_n\| = 1$.

Let us suppose that $\varepsilon \in (0, 1)$. Pick $\eta = \frac{\varepsilon}{3}$. By the properties of the sequence $\{x_n\}$, we first choose a natural number n_1 so that

$$\left| f(x_{n_1}) \right| < \frac{\eta}{2} \qquad \text{and} \qquad 1 - \eta \leq \|x_{n_1}\| \leq 1.$$

Note that $\{x_n\}$ is a weakly null sequence and verifies $D(\{x_n\}) = 1$. It follows from the definition of Domínguez-Benavides' coefficient that

$$\liminf_{n \to \infty} \|x_n + x_{n_1}\| \leq R(1, X).$$

Next, we choose $n_2 > n_1$ so that

$$\|x_{n_2} + x_{n_1}\| \leq R(1, X) + \eta,$$

$$1 - \eta \leq \|x_{n_2}\| \leq 1, \qquad 1 - \eta \leq \|x_{n_2} - x_{n_1}\| \leq 1,$$

$$\left|f_{n_1}(x_{n_2})\right| < \eta \qquad \text{and} \qquad \left|(f_{n_2} - f)(x_{n_1})\right| < \frac{\eta}{2}.$$

This implies that

$$\left|f_{n_2}(x_{n_1})\right| \leq \left|(f_{n_2} - f)(x_{n_1})\right| + \left|f(x_{n_1})\right| < \eta.$$

Put

$$z_1 := \frac{x_{n_1}}{\|x_{n_1}\|}, \qquad z_2 := \frac{x_{n_2}}{\|x_{n_2}\|}, \qquad g_1 := f_{n_1}, \qquad g_2 := f_{n_2}.$$

We now prove that (a), (b) and (c) are satisfied. Clearly, (b) holds. Moreover, for $i \neq j$,

$$\left|g_i(z_j)\right| = \frac{\left|f_{n_i}(x_{n_j})\right|}{\|x_{n_j}\|} < \frac{\eta}{1 - \eta} < 2\eta < \varepsilon.$$

Next, we observe that

$$\begin{aligned}\|z_1 - z_2\| &= \left\|\frac{x_{n_1}}{\|x_{n_1}\|} - \frac{x_{n_2}}{\|x_{n_2}\|}\right\| \\ &\leq \left\|\frac{x_{n_1}}{\|x_{n_1}\|} - x_{n_1}\right\| + \|x_{n_1} - x_{n_2}\| + \left\|x_{n_2} - \frac{x_{n_2}}{\|x_{n_2}\|}\right\| \\ &= \left|1 - \|x_{n_1}\|\right| + \|x_{n_1} - x_{n_2}\| + \left|1 - \|x_{n_2}\|\right| \\ &< 1 + 2\eta < 1 + \varepsilon,\end{aligned}$$

and

$$\|z_1 - z_2\| \geq g_1(z_1 - z_2) = g_1(z_1) - g_1(z_2) \geq 1 - \eta > 1 - \varepsilon,$$

that is (a) is satisfied. Moreover, (c) is satisfied, since

$$\begin{aligned}\|z_1 + z_2\| &= \left\|\frac{x_{n_1}}{\|x_{n_1}\|} + \frac{x_{n_2}}{\|x_{n_2}\|}\right\| \\ &\leq \left\|\frac{x_{n_1}}{\|x_{n_1}\|} - x_{n_1}\right\| + \|x_{n_1} + x_{n_2}\| + \left\|x_{n_2} - \frac{x_{n_2}}{\|x_{n_2}\|}\right\| \\ &= \left|1 - \|x_{n_1}\|\right| + \|x_{n_1} + x_{n_2}\| + \left|1 - \|x_{n_2}\|\right| \\ &< 1 + 3\eta < 1 + \varepsilon.\end{aligned}$$

This completes the proof. $\square$

Lemma 3.1 can be rewritten in ultrapower language as follows.

Lemma 3.2. *If a super-reflexive Banach space X fails to have normal structure, then there are $\widetilde{x}_1, \widetilde{x}_2 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2 \in S_{(\widetilde{X})^*}$ such that*

(a) $\|\widetilde{x}_1 - \widetilde{x}_2\| = 1$ and $\widetilde{f}_i(\widetilde{x}_j) = 0$ for all $i \neq j$;

(b) $\widetilde{f}_i(\widetilde{x}_i) = 1$ for $i = 1, 2$;

(c) $\|\widetilde{x}_1 + \widetilde{x}_2\| \leq R(1, X)$.

Now, we are ready to state and prove our main results.

Theorem 3.3. *Let X be a Banach space such that*

$$J(a, X) < 1 + a + \frac{1 - a}{R(1, X)}$$

for some $0 \leq a \leq 1$. Then X has normal structure.

Proof. It is proved in [5] that $J(1, X) < 2$ implies X has normal structure. We now suppose that X fails to have normal structure and there exists $0 \leq a < 1$ such that the inequality above holds. It is clear that X is uniformly nonsquare and hence super-reflexive. Then there are elements $\widetilde{x}_1, \widetilde{x}_2 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2 \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 3.2 We now use these elements to estimate the generalized James constant of the Banach space ultrapower $\widetilde{X}$ which is the same as that of the space X itself. We put

$$\widetilde{x} = \widetilde{x}_2 - \widetilde{x}_1, \qquad \widetilde{y} = a\widetilde{x}_2 + (1 - a)\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}, \qquad \text{and} \qquad \widetilde{z} = a\widetilde{x}_1 + (1 - a)\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}.$$

We noted here that $\|\widetilde{x}_2 + \widetilde{x}_1\| \geq \widetilde{f}_2(\widetilde{x}_2 + \widetilde{x}_1) = 1$. It is easy to see that

$$\|\widetilde{y}\| \leq 1, \qquad \|\widetilde{z}\| \leq 1, \qquad \text{and} \qquad \widetilde{y} - \widetilde{z} = a\widetilde{x}.$$

Hence, we have

$$
\begin{aligned}
J(a, X) = J(a, \widetilde{X}) \\
&\geq \min\left\{\|\widetilde{x} + \widetilde{y}\|, \|\widetilde{x} - \widetilde{z}\|\right\} \\
&= \min\left\{\left\|\widetilde{x}_2 - \widetilde{x}_1 + a\widetilde{x}_2 + (1 - a)\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}\right\|, \left\|\widetilde{x}_2 - \widetilde{x}_1 - a\widetilde{x}_1 - (1 - a)\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}\right\|\right\} \\
&\geq \min\left\{\widetilde{f}_2\left(\widetilde{x}_2 - \widetilde{x}_1 + a\widetilde{x}_2 + (1 - a)\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}\right), (-\widetilde{f}_1)\left(\widetilde{x}_2 - \widetilde{x}_1 - a\widetilde{x}_1 - (1 - a)\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}\right)\right\} \\
&\geq 1 + a + \frac{1 - a}{\|\widetilde{x}_2 + \widetilde{x}_1\|} \\
&\geq 1 + a + \frac{1 - a}{R(1, X)},
\end{aligned}
$$

which is a contradiction. This completes the proof. □

Corollary 3.4. ([22, Corollary 24], [5, Corollary 3.5] and [13, Theorem 12]) *A Banach space X has normal structure if*

$$J(X) < 1 + \frac{1}{R(1, X)} \qquad \text{or} \qquad J(1, X) < 2.$$

Remark 3.5. *When $a = 0$, Theorem 3.3 is sharp in the sense that there is a Banach space X such that $J(X) = 1 + \frac{1}{R(1,X)}$ and X fails to have normal structure. Consider the Bynum space $\ell_{2,\infty}$ defined as $\ell_{2,\infty} := (\ell_2, \| \cdot \|_{2,\infty})$ where $\|x\|_{2,\infty} := \max\{\|x^+\|_2, \|x^-\|_2\}$ with $x^+(i) = \max\{x(i), 0\}$ for each $i \geq 1$ and $x^- = x^+ - x$. It is known that $J(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [17]) and $R(1, \ell_{2,\infty}) = \sqrt{2}$ (see [10]). Thus, we have*

$$J(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}} = 1 + \frac{1}{R(1, \ell_{2,\infty})},$$

and $\ell_{2,\infty}$ fails to have weak normal structure.

It is noted that Theorem 3.3 is a result strictly more applicable than Theorem 2 of Gao and Saejung [13] when $a = 0$. More precisely, Proposition 26 of Mazcuñán-Navarro [22] shows that Theorem 2 of Gao and Saejung [13] implies Theorem 3.3 when $a = 0$. But in the next example, we see that the converse of this implication is not true.

Example 3.6. For $\beta \geq 1$, let E_β be the space ℓ_2 endowed with the norm

$$|x|_\beta := \max\left\{\|x\|_2, \beta\|x\|_\infty\right\}.$$

The space E_β has uniform normal structure if and only if $\beta < \sqrt{2}$. It is known that $J(E_\beta) = \min(2, \beta\sqrt{2})$ (see [19]) and $R(1, E_\beta) = \max\left(\frac{\beta}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$ (see [11]). Thus, for any $\beta \in \left[\frac{1+\sqrt{5}}{2\sqrt{2}}, \frac{1+\sqrt{2/3}}{\sqrt{2}}\right)$, we have

$$\frac{1 + \sqrt{5}}{2} \leq \beta\sqrt{2} = J(E_\beta) < 1 + \sqrt{\frac{2}{3}} = 1 + \frac{1}{R(1, E_\beta)}.$$

Therefore, E_β has normal structure by Theorem 3.3 but lies out of the scope of [13, Theorem 2].

Remark 3.7. *Theorem 3.3 is better than Theorem 12 of Gao and Saejung [13] involving the coefficient of weak orthogonality of Sims $\mu(X)$. Note that $1 \leq R(1, X) \leq 2$ and $1 \leq \mu(X) \leq 3$.*

Theorem 3.8. *Let X be a Banach space such that*

$$C_{NJ}(a, X) < \frac{\left(1 + \frac{1-a}{(R(1,X))^2} + a\right)^2}{1 + \left(\frac{1-a}{R(1,X)} + a\right)^2}$$

for some $0 \leq a \leq 1$. Then X has normal structure.

Proof. It is proved in [5] that $C_{NJ}(1, X) < 2$ implies X has normal structure. Since $R(1, X) \geq 1$, it follows that X is uniformly nonsquare and hence the inequality above implies the super-reflexivity of X. As in the proof of Theorem 3.3, we suppose that X fails to have normal structure and the inequality holds for some $0 \leq a < 1$. We will make use of the existence of elements $\widetilde{x}_1, \widetilde{x}_2 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2 \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 3.2. We follow the proof of the first theorem but we will estimate $\beta := \|\widetilde{x}_1 + \widetilde{x}_2\|$ in terms of the generalized von Neumann-Jordan constant $C_{NJ}(a, X)$. Now, we take

$$\widetilde{x} = \widetilde{x}_2 - \widetilde{x}_1, \qquad \widetilde{y} = \frac{1-a}{\beta^2}(\widetilde{x}_1 + \widetilde{x}_2) + a\widetilde{x}_2, \qquad \text{and} \qquad \widetilde{z} = \frac{1-a}{\beta^2}(\widetilde{x}_1 + \widetilde{x}_2) + a\widetilde{x}_1.$$

It follows that

$$\|\widetilde{x}\| = 1, \qquad \|\widetilde{y}\| \le \frac{1-a}{\beta} + a, \qquad \|\widetilde{z}\| \le \frac{1-a}{\beta} + a, \qquad \text{and} \qquad \|\widetilde{y} - \widetilde{z}\| = a\|\widetilde{x}\|.$$

Therefore, we have

$$
\begin{aligned}
C_{NJ}(a, X) = C_{NJ}(a, \widetilde{X}) \\
&\ge \frac{\|\widetilde{x} + \widetilde{y}\|^2 + \|\widetilde{x} - \widetilde{z}\|^2}{2\|\widetilde{x}\|^2 + \|\widetilde{y}\|^2 + \|\widetilde{z}\|^2} \\
&= \frac{\left(1 + \frac{1-a}{\beta^2} + a\right)^2}{1 + \left(\frac{1-a}{\beta} + a\right)^2} \\
&\ge \frac{\left(1 + \frac{1-a}{(R(1,X))^2} + a\right)^2}{1 + \left(\frac{1-a}{R(1,X)} + a\right)^2},
\end{aligned}
$$

which is a contradiction. This completes the proof. □

Corollary 3.9. ([22, Corollary 18] and [6, Theorem 3.6]) *A Banach space X has normal structure if*

$$C_{NJ}(X) < 1 + \frac{\left(M(X)\right)^2}{4} \qquad or \qquad C_{NJ}(1, X) < 2.$$

Remark 3.10. *When $a = 0$, Theorem 3.8 is sharp in the sense that there is a Banach space X such that $C_{NJ}(X) = 1 + \frac{1}{(R(1,X))^2}$ and X fails to have normal structure. Consider the Bynum space $\ell_{2,\infty}$ which fails to have weak normal structure. It is known that $C_{NJ}(\ell_{2,\infty}) = \frac{3}{2}$ (see [17]) and $R(1, \ell_{2,\infty}) = \sqrt{2}$ (see [10]). Therefore, we have*

$$C_{NJ}(\ell_{2,\infty}) = \frac{3}{2} = 1 + \frac{1}{\left(R(1, \ell_{2,\infty})\right)^2}.$$

In Proposition 26 of Mazcuñán-Navarro [22] it was proved that Theorem 3.16 of Dhompongsa and Kaewkhao [4] or Theorem 2 of Saejung [24] implies Corollary 3.17 of Dhompongsa and Kaewkhao [4] which in its turn implies Theorem 3.8 when $a = 0$. These considerations let us conclude that the result assuring normal structure for Banach spaces satisfying Theorem 3.8 is strictly more applicable than Corollary 3.17 of [4] and then than Theorem 3.16 of [4] or Theorem 2 of [24].

Now, we shall present an example which shows that the scope of Theorem 3.16 of [4] and Theorem 2 of [24] is strictly more limited than the scope of Theorem 3.8 when $a = 0$.

Example 3.11. Consider the space E_β which has uniform normal structure if and only if $\beta < \sqrt{2}$. The space E_β verifies $C_{NJ}(E_\beta) = \min(2, \beta^2)$ (see [19]) and $R(1, E_\beta) = \max\left(\frac{\beta}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$ (see [11]). Hence, for any $\beta \in \left[\frac{\sqrt{1+\sqrt{3}}}{\sqrt{2}}, \frac{1+\sqrt{2/3}}{\sqrt{2}}\right)$, we have

$$\frac{1 + \sqrt{3}}{2} \le \beta^2 = C_{NJ}(E_\beta) < \frac{5}{3} = 1 + \frac{1}{\left(R(1, E_\beta)\right)^2}.$$

Therefore, E_β verifies the hypothesis in Theorem 3.8 but lies out of the scope of [4, Theorem 3.16] or [24, Theorem 2].

In the next example, we show that the hypothesis of [4, Corollary 3.17] is more restrictive than the hypothesis of Theorem 3.8 when $a = 0$. In order to see this, we show that there are spaces satisfying Theorem 3.8 which do not satisfy Corollary 3.17 of [4].

Example 3.12. Consider the Bynum space $\ell_{2,1}$ defined as $\ell_{2,1} := (\ell_2, \|\cdot\|_{2,1})$ where $\|x\|_{2,1} := \|x^+\|_2 + \|x^-\|_2$ with $x^+(i) = \max\{x(i), 0\}$ for each $i \geq 1$ and $x^- = x^+ - x$. If we consider $x = \left(\frac{1}{2}, -\frac{1}{2}, 0, \cdots\right)$ and $y = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \cdots\right)$ in $\ell_{2,1}$, then we obtain that $J(\ell_{2,1}) \geq \frac{3}{2}$. It is also known that $C_{NJ}(\ell_{2,1}) = C_{NJ}(\ell_{2,\infty}) = \frac{3}{2}$ (see [17]) and $M(\ell_{2,1}) = \sqrt{\frac{5}{2}}$ (see [11]). Thus, we have

$$C_{NJ}(\ell_{2,1}) = \frac{3}{2} > \frac{13}{9} \geq 1 + \frac{1}{\left(J(\ell_{2,1})\right)^2},$$

while

$$C_{NJ}(\ell_{2,1}) = \frac{3}{2} < \frac{13}{8} = 1 + \frac{\left(M(\ell_{2,1})\right)^2}{4}.$$

Therefore, $\ell_{2,1}$ has normal structure by Theorem 3.8 but lies out of the scope of [4, Corollary 3.17].

Theorem 3.13. *Let X be a Banach space such that*

$$C_Z(a, X) < \frac{\left(1 + \frac{1-a}{(R(1,X))^2} + a\right)^2}{1 + \left(\frac{1-a}{R(1,X)} + a\right)^2}$$

for some $0 \leq a \leq 1$. Then X has normal structure.

Proof. The proof is the same as that of Theorem 3.8, so it is left for the reader to verify. □

According to Theorem 3.13, we get the following result.

Corollary 3.14. *If X is a Banach space such that*

$$C_Z(X) < 1 + \frac{\left(M(X)\right)^2}{4},$$

then X has normal structure.

Remark 3.15. *Since $C_Z(X) \leq C_{NJ}(X)$, it follows that Corollary 3.9 is better than [22, Corollary 18].*

Corollary 3.16. *([13, Corollary 7 and Theorem 12]) If $C_Z(1, X) < 2$, then X has normal structure.*

Remark 3.17. *When $a = 0$, Theorem 3.13 is sharp in the sense that there is a Banach space X such that $C_Z(X) = 1 + \frac{1}{(R(1,X))^2}$ and X fails to have normal structure. Consider the Bynum space $\ell_{2,\infty}$ which fails to have weak normal structure. By using the inequality, $C_Z(X) \leq C_{NJ}(X)$ (see [26]), we get that $C_Z(\ell_{2,\infty}) \leq \frac{3}{2}$. Now taking the points $x = (-1, 1, 0, \cdots)$ and $y = \left(\frac{1}{2}, \frac{1}{2}, 0, \cdots\right)$ in $\ell_{2,\infty}$. Thus, we obtain that $\|x + y\| = \|x - y\| = \frac{3}{2}$, $\|x\| = 1$ and $\|y\| = \frac{1}{\sqrt{2}}$. So $C_Z(\ell_{2,\infty}) \geq \frac{3}{2}$ and consequently, $C_Z(\ell_{2,\infty}) = \frac{3}{2}$. It is known that $R(1, \ell_{2,\infty}) = \sqrt{2}$ (see [10]). Hence, we have*

$$C_Z(\ell_{2,\infty}) = \frac{3}{2} = 1 + \frac{1}{\left(R(1, \ell_{2,\infty})\right)^2}.$$

Proposition 3.18. *Let X be a Banach space. The conditions*

(i) $C_Z(X) < \frac{1+\sqrt{3}}{2}$,

(ii) $C_Z(X) < 1 + \frac{1}{(J(X))^2}$,

(iii) $C_Z(X) < 1 + \frac{(M(X))^2}{4}$

satisfy the chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. [(i) $\Rightarrow$ (ii)] Since the inequality $2x(x-1) < 1$ holds if and only if $x \in \left(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2} \right)$, it follows that $2C_Z(X)\big(C_Z(X) - 1\big) < 1$. On the other hand, $C_Z(X) \geq \frac{(J(X))^2}{2}$. Thus, we have

$$\big(J(X)\big)^2\big(C_Z(X) - 1\big) \leq 2C_Z(X)\big(C_Z(X) - 1\big) < 1.$$

Therefore, $C_Z(X) < 1 + \frac{1}{(J(X))^2}$.

[(ii) $\Rightarrow$ (iii)] Because $J(X) \geq R(1, X)$ (see [4, 22]), we get $J(X) \geq \frac{2}{M(X)}$, which implies the desired inequality. $\square$

In view of Proposition 3.18, [13, Corollary 8] derives from [13, Corollary 7]. The above proposition also shows that Theorem 3.13 applies whenever Theorem 6 of Gao and Saejung [13] does so when $a = 0$.

The following example shows that Theorem 3.13 is a result strictly more applicable than Corollary 8 of Gao and Saejung [13] or Theorem 5 of Llorens-Fuster, Mazcuñán-Navarro and Reich [21] when $a = 0$.

Example 3.19. Consider the space E_β which has uniform normal structure if and only if $\beta < \sqrt{2}$. Thus, by using the inequalities (1), we conclude that $C_Z(E_\beta) = \min(2, \beta^2)$. It is easy to see that $R(1, E_\beta) = \max\left(\frac{\beta}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \right)$ (see [10]). Hence, for any $\beta \in \left[\frac{\sqrt{1+\sqrt{3}}}{\sqrt{2}}, \frac{1+\sqrt{2/3}}{\sqrt{2}} \right)$, we have

$$\frac{1 + \sqrt{3}}{2} \leq \beta^2 = C_Z(E_\beta) < \frac{5}{3} = 1 + \frac{1}{\big(R(1, E_\beta)\big)^2}.$$

Therefore, Theorem 3.13 let us deduce the normal structure of E_β, while these spaces lie out of the scope of [13, Corollary 8].

Remark 3.20. *Theorem 3.13 is better than Theorem 12 of Gao and Saejung [13] involving the coefficient of weak orthogonality of Sims* $\mu(X)$. *Note that* $1 \leq R(1, X) \leq 2$ *and* $1 \leq \mu(X) \leq 3$.

Returning to the Ptolemy constant and bearing in mind the fact that $C_Z(X) \leq C_p(X)$, we immediately obtain the following result.

Corollary 3.21. *If X is a Banach space such that*

$$C_p(X) < 1 + \frac{\big(M(X)\big)^2}{4},$$

then X has normal structure.

Remark 3.22. *Corollary 3.21 is sharp in the sense that there is a Banach space X such that $C_p(X) = 1 + \frac{(M(X))^2}{4}$ and X fails to have normal structure. Consider the Bynum space $\ell_{2,\infty}$ which fails to have weak normal structure. It is known that $C_p(\ell_{2,\infty}) = \frac{3}{2}$ (see [21]) and $M(\ell_{2,\infty}) = \sqrt{2}$ (see [10]). Hence, we have*

$$C_p(\ell_{2,\infty}) = \frac{3}{2} = 1 + \frac{\left(M(\ell_{2,\infty})\right)^2}{4}.$$

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