A Class of Non-Holonomic Projective Connections on Sub-Riemannian Manifolds

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Abstract. The authors define a semi-symmetric non-holonomic (SSNH)-projective connection on sub-Riemannian manifolds and find an invariant of the SSNH-projective transformation. The authors further derive that a sub-Riemannian manifold is of projective flat if and only if the Schouten curvature tensor of a special SSNH-connection is zero.

1. Introduction

Since A. Friedmann and J. A. Schouten [8], in the early days of 1924, firstly introduced the concept of semi-symmetric linear connections, the research related to the semi-symmetric connection was unusually brilliant, and made a series of fruitful research results.

K. Yano [21] introduced and studied the semi-symmetric metric connection of Riemannian manifolds. N. S. Agashe and M. R. Chafle [1] introduced a semi-symmetric non-metric connection on a Riemannian manifold and this was further studied by U. C. De etc. [4–7], T. Imai [15], P. B. Zhao and H. Z. Song [27], and so on.


H. B. Yilmaz, F. O. Zengin and S. Aynur Uysal [22] considered a manifold equipped with a semi-symmetric metric connection whose torsion tensor satisfied a special condition and proved that if a manifold...
mentioned as above was conformally flat, then it was a mixed quasi-Einstein manifold. F. Y. Fu, X. P. Yang and P. B. Zhao [9] considered the geometric and physical properties of conformal mappings for the semi-Riemannian manifolds.


However, to the author’s knowledge, the study of geometric and analysis in sub-Riemannian manifolds on view of the semi-symmetric metric connection in sub-Riemannian manifolds is still a gap.

In this paper, we will, based on the setting of [14], investigate a class of semi-symmetric non-holonomic (SSNH) projective transformation following the work in [14].

2. The SSNH-Projective Transformation

Let \((M, \Delta, g_\Delta)\) be a \(n\)-dimensional sub-Riemannian manifold, where \(\Delta\) is a \(\ell\)-dimensional sub-bundle of tangent bundles, and is called a horizontal bundle, and is a Riemannian metric defined on \(\Delta\). In particular, when \(\Delta = TM\), \((M, \Delta, g_\Delta)\) will be degenerated into a Riemannian manifold. Without loss of generality, we assume \(\Delta \neq TM\). In this subsection, we will define a semi-symmetric non-holonomic (SSNH) metric connection and discuss the SSNH-projective transformation following the work in [14].

We use unless otherwise noted the following ranges for indices: \(i, j, k, h, \cdots \in \{1, \cdots, \ell\}\). The repeated indices with one upper index and one lower index indicates the summation over their range. The projection of \(X\) on the horizontal bundle is denoted by \(X_h\).

**Definition 2.1.** A non-holonomic connection on sub-bundle \(Q \subset TM\) is a mapping \(\nabla : \Gamma(Q) \times \Gamma(Q) \rightarrow \Gamma(Q)\) satisfying the following

\[
\nabla_{X_h}(Y_h + Z_h) = \nabla_{X_h}Y_h + \nabla_{X_h}Z_h, \quad \nabla_{X_h}(f Y_h) = X_h(f)Y_h + f\nabla_{X_h}Y_h, \quad \nabla_{f X_h + g Y_h}Z_h = f\nabla_{X_h}Z_h + g\nabla_{Y_h}Z_h,
\]

where \(X, Y, Z \in \Gamma(TM)\), \(f, g \in C^\infty(M)\).

**Definition 2.2.** A non-holonomic connection is said to be metric and symmetric if it satisfies respectively,

\[
(\nabla_{X_h}g_\Delta)(X_h, Y_h) = Z_h(g_\Delta(X_h, Y_h)) - g_\Delta(\nabla_{Z_h}X_h, Y_h) - g_\Delta(X_h, \nabla_{Z_h}Y_h) = 0,
\]

\[
T(X_h, Y_h) = \nabla_{X_h}Y_h - \nabla_{Y_h}X_h - [X_h, Y_h]_h = 0.
\]

**Definition 2.3.** A non-holonomic connection is said to be a sub-Riemannian connection if it is both metric and symmetric.

**Definition 2.4.** A horizontal curve \(\gamma(t) : [0,1] \rightarrow M\) (i.e. \(\gamma(t) \in \Delta_{\gamma(t)}\)) is said to be a sub-Riemannian parallel (in briefly, SR-parallel) curve if it satisfies

\[
\nabla_{\dot{\gamma}}\dot{\gamma} = 0,
\]

where \(\nabla\) is the sub-Riemannian connection.

Let \(\gamma : x^i = x^i(t)\), the corresponding equation (1) is

\[
\frac{d^2x^\ell}{dt^2} + \ell_{ij}^{\ell} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0,
\]

where \(t\) is an affine parameter.
Definition 2.5. Let $D_1, D_2$ be two classes of non-holonomic connections. If a SR-parallel curve corresponding to $D_1$ coincides always with one corresponding to $D_2$, then we say that $D_1$ is a projective correspondence to $D_2$.

The second author [23] proved that a non-holonomic symmetric connection $D$ is a projective correspondence if and only if there exists a smooth horizontal 1-form $\varphi$ (i.e. a 1-form defined on $\Delta$), such that, for any two horizontal vector fields $X_h, Y_h$, there holds

$$D_{X_h}Y_h = \nabla_{X_h}Y_h + \varphi(X_h)Y_h + \varphi(Y_h)X_h.$$  \hspace{1cm} (3)

If $D$ is a non-holonomic connection with torsion, then we have the following

Proposition 2.6. A non-holonomic connection $D$ with torsion is a projective correspondence to $\mathbb{V}$ if and only if there exists 1-form $\lambda$ such that the symmetric part of tensor $A(X_h, Y_h) = D_{X_h}Y_h - \nabla_{X_h}Y_h$ is of the form

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \lambda(X_h)Y_h + \lambda(Y_h)X_h, \text{ for } X, Y \in \Gamma(TM).$$ \hspace{1cm} (4)

Proof. The necessity is obvious. We only prove the sufficiency. If (4) holds, we denote by $(D_{X_h}Y_h + D_{Y_h}X_h)/2 = \nabla_{X_h}Y_h + \nabla_{Y_h}X_h)/2 = \nabla_{X_h}Y_h$, then (4) is equivalent to $D_{X_h}Y_h - \nabla_{X_h}Y_h = \lambda(X_h)Y_h + \lambda(Y_h)X_h$. Hence $D$ and $\nabla$ have the same SR-parallel curves by (3).

On the other hand, if $\gamma(t)$ is a SR-parallel curve of $D$, then $\gamma(t)$ is also a SR-parallel curve of $\tilde{D}$ by a simple computation. Hence $\tilde{D}$ and $D$ also have the same SR-parallel curves, so do $\tilde{V}$ and $\mathbb{V}$. Therefore, $D$ and $\mathbb{V}$ have the same SR-parallel curves, namely, $D$ is a projective correspondence of $\mathbb{V}$. \hfill \square

Definition 2.7. If $\tilde{V}$ is a projective correspondence to $\mathbb{V}$ with torsion,

$$\tilde{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h,$$ \hspace{1cm} (5)

where $\pi$ is a given 1-form, then we say that $\tilde{V}$ is a semi-symmetric non-holonomic projective connection, in briefly, a SSNH-projective connection.

Theorem 2.8. $\tilde{V}$ is a SSNH-projective connection if and only if there exist two 1-form $p, q$ such that

$$\nabla_{X_h}Y_h = \nabla_{X_h}Y_h + p(X_h)Y_h + q(Y_h)X_h,$$ \hspace{1cm} (6)

for any $X, Y \in \Gamma(TM)$.

Proof. Let $A(X_h, Y_h) = \nabla_{X_h}Y_h - \nabla_{X_h}Y_h$. Since $\nabla$ is a SSNH-projective connection, from Proposition 2.6, there exists a smooth 1-form $\varphi$ such that

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \varphi(X_h)Y_h + \varphi(Y_h)X_h, \text{ for } X, Y \in TM$$ \hspace{1cm} (7)

and 1-form $\pi$ such that the torsion of $\tilde{V}$ is of the form $\tilde{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h$, we can deduce from the above equation

$$A(X_h, Y_h) - A(Y_h, X_h) = \pi(Y_h)X_h - \pi(X_h)Y_h.$$ \hspace{1cm} (8)

By (6) and (8), we arrive at $A(X_h, Y_h) = (p - \pi/2)(X_h)Y_h + (p + \pi/2)(Y_h)X_h$ for $p = \pi - \pi/2, q = \pi + \pi/2$.

Conversely, we assume $\nabla_{X_h}Y_h = \nabla_{X_h}Y_h + p(X_h)X_h + q(X_h)Y_h$, then

$$(A(X_h, Y_h) + A(Y_h, X_h))/2 = \frac{p + q}{2}(X_h)Y_h,$$

$$A(X_h, Y_h) - A(Y_h, X_h) = (p - q)(X_h)Y_h - (p - q)(X_h)Y_h.$$

By virtue of Proposition 2.6 again, we know $\tilde{V}$ is a projective correspondence to $\mathbb{V}$, and we get

$$\tilde{T}(X_h, Y_h) = \nabla_{X_h}Y_h - \nabla_{X_h}Y_h - [X_h, Y_h]_h = (p - q)(Y_h)X_h - (p - q)(X_h)Y_h.$$  

Let $\pi = p - q$, then $\tilde{T}(X_h, Y_h) = \pi(Y_h)X_h - \pi(X_h)Y_h$.

This completes the proof of Theorem 2.8. \hfill \square
In a basis \( \{ e_i \} \), (6) can be rewritten as
\[
\mathbf{F}_ij = (t^i_j + p_i \delta^i_j + q_i \delta^i_j) = (t^i_j) + q_j \delta^i_j + p_i \delta^i_j - \rho_i \delta^i_j,
\]
where \( \rho_i = \alpha_i/2, p_i = \varphi_i - \rho_i, q_i = \varphi_i + \rho_i \). The Schouten curvature tensor, Ricci tensor and sub-Weyl projective curvature tensor are given, respectively, as
\[
\begin{align*}
\hat{\mathbf{R}}_{ijk}^h &= \mathbf{R}_{ijk}^h + \beta_{ij} \delta^h_j + \alpha_{jk} \delta^h_i - \delta^h_i \delta^h_j \mathbf{R}_{jk}^p - \Omega_{ijk}^h p_k, \\
\mathbf{R}_{jk}^i &= \mathbf{R}_{jk}^i + \beta_{kj} - (\ell - 1) \alpha_{jk} - \Omega_{jk}^h p_k - \Omega_{jk}^p p_k, \\
W_{ijk}^h &= \mathbf{R}_{ijk}^h + \frac{1}{\ell - 1} (\delta^h_i \mathbf{R}_{jk}^p - \delta^h_i \mathbf{R}_{jk}^p),
\end{align*}
\]
where
\[
\begin{align*}
\beta_{ij} &= (\nabla_i p_j) - (\nabla_j p_i) = \varphi_{ij} - \varphi_{ji} + \rho_j - \rho_i, \\
\alpha_{ij} &= (\nabla_i q_j) - q_e(q_j) = \varphi_{ij} + \rho_i - \varphi_{ji} - \varphi_{ij}, \\
\varphi_{ij} &= e_\ell(q_j) - \Gamma_{ij}^p q_k - \varphi_i \delta^i_j = \nabla_i \varphi_j - \varphi_i \varphi_j, \\
\rho_{ij} &= e_\ell(p_j) - \Gamma_{ij}^p e_\ell - p_i \varphi_j = \nabla_i p_j - \varphi_i \varphi_j, \\
\mathbf{R}_{jk}^i &= \mathbf{R}_{jk}^i + \Omega_{jk}^h p_k.
\end{align*}
\]

**Theorem 2.9.** The tensor \( \mathbf{S}_{ijk}^h \) is an invariant under a SSNH-projective transformation, where
\[
\mathbf{S}_{ijk}^h = \mathbf{R}_{ijk}^h + \frac{1}{\ell - 1} (\delta^h_i \mathbf{R}_{jk}^p - \delta^h_i \mathbf{R}_{jk}^p)
\]
\[
+ \frac{1}{\ell^2 - \ell - 2} \left( \delta^h_i \mathbf{R}_{jk}^p - \delta^h_i \mathbf{R}_{jk}^p - \delta^h_i \mathbf{R}_{jk}^p - (\ell - 1) \delta^h_i \mathbf{A}_{ijk} \right),
\]
and \( \mathbf{A}_{ij} = \mathbf{R}_{ijk}^i \).

**Proof.** For simplicity, we choose \( \{ e_i \} \) as a local frame field such that \( \{ e_i, e_j \} \in \mathcal{V}_M \), and hence we have \( \Omega_{ij}^h = 0 \). Then the Schouten curvature tensors and Ricci curvature tensors can be written simply as
\[
\hat{\mathbf{R}}_{ijk}^h = \mathbf{R}_{ijk}^h + \beta_{ij} \delta^h_j + \alpha_{jk} \delta^h_i - \delta^h_i \delta^h_j \mathbf{R}_{jk}^p, \\
\mathbf{R}_{jk}^i = \mathbf{R}_{jk}^i + \beta_{kj} - (\ell - 1) \alpha_{jk} \mathbf{R}_{jk}^p.
\]
Let \( k = h = i \), and denote by \( \mathbf{A}_{ij} = \mathbf{R}_{ijk}^i, \mathbf{A}_{ij} = \mathbf{R}_{ijk}^i \), one obtains
\[
\mathbf{A}_{ij} = \mathbf{A}_{ij} + \ell \beta_{ij} + \alpha_{ij} - \alpha_{ji},
\]
hence one arrives at
\[
\beta_{jk} = \frac{1}{\ell^2 - \ell - 2} \left[ (\mathbf{R}_{jk}^i - \mathbf{R}_{kj}^i) - (\mathbf{R}_{jk}^i - \mathbf{R}_{kj}^i) + (\ell - 1) (\mathbf{A}_{jk} - \mathbf{A}_{jk}) \right],
\]
\[
\alpha_{jk} = \frac{1}{\ell - 1} (\mathbf{R}_{jk}^i - \mathbf{R}_{kj}^i) - \frac{1}{\ell - 1} (\ell - 1) (\ell^2 - \ell - 2) \left[ (\mathbf{R}_{jk}^i - \mathbf{R}_{kj}^i) - (\mathbf{R}_{jk}^i - \mathbf{R}_{kj}^i) \right]
- \frac{1}{\ell - 1} (\ell^2 - \ell - 2) (\mathbf{A}_{jk} - \mathbf{A}_{jk}).
\]
moreover one has

\[ R^b_{ijk} = R^b_{ijk} + \frac{\delta^b_k}{\ell^2 - \ell - 2}[(R^i_{ij} - R^i_{ji}) - (R^i_{ij} - R^i_{ji}) + (\ell - 1)(A^i_{ij} - A^i_{ji})] \]

\[ + \frac{\delta^b_j}{(\ell - 1)(\ell^2 - \ell - 2)}((R^b_{ik} - R^b_{ki}) - (R^b_{ik} - R^b_{ki})) \]

\[ - \frac{\delta^b_i}{(\ell - 1)(\ell^2 - \ell - 2)}(\bar{A}^b_{ik} - A^b_{ki}) - \frac{\delta^b_k}{\ell - 1}((R^b_{jk} - R^b_{kj}) - (R^b_{jk} - R^b_{kj})) \]

\[ + \frac{\delta^b_j}{(\ell - 1)(\ell^2 - \ell - 2)}(\bar{A}^b_{jk} - A^b_{jk}). \]

Rewriting the above equation by

\[ R^b_{ijk} = R^b_{ijk} + \frac{\delta^b_k}{\ell^2 - \ell - 2}((R^i_{ij} - R^i_{ji}) + (\ell - 1)(A^i_{ij} - A^i_{ji})] \]

\[ + \frac{\delta^b_j}{(\ell - 1)(\ell^2 - \ell - 2)}((R^b_{ik} - R^b_{ki}) + (\ell - 1)(A^b_{ik} - A^b_{ki})) \]

\[ - \frac{\delta^b_i}{(\ell - 1)(\ell^2 - \ell - 2)}((R^b_{jk} - R^b_{kj}) + (\ell - 1)(A^b_{jk} - A^b_{kj})). \]

that is

\[ R^b_{ijk} + \frac{1}{\ell - 1}(\delta^b_k(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})) \]

\[ -(\ell - 1)\delta^b_i(R^i_{ij} - R^i_{ji}) + \frac{1}{(\ell - 1)(\ell^2 - \ell - 2)}[(R^b_{ik} - R^b_{ki}) - \delta^b_i(R^b_{jk} - R^b_{kj})] \]

\[ -\frac{1}{\ell - 1}(\delta^b_k(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})) \]

\[ -(\ell - 1)(\delta^b_i(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})). \]

Denote by

\[ S^b_{ijk} = R^b_{ijk} + \frac{1}{\ell - 1}(\delta^b_k(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})) \]

\[ -(\ell - 1)(\delta^b_i(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})). \]

and

\[ S^b_{ijk} = R^b_{ijk} + \frac{1}{\ell - 1}(\delta^b_k(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})) \]

\[ -(\ell - 1)(\delta^b_i(R^i_{ij} - R^i_{ji}) + \delta^b_i(R^b_{jk} - R^b_{kj})). \]
then one obtains $S^h_{ijk} = S^h_{ij,k}$. This ends the proof of Theorem 2.9. □

We now similarly define the sub-Weyl projective curvature tensor of the SSNH-projective connection by

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{\ell - 1} (\delta^h_i R_{jk} - \delta^h_j R_{ik}), \quad (12)$$

then we have

$$W^h_{ijk} = W^h_{ijk} + \delta^h_i (\beta_{ij} - \Omega^h_{ij} p_s) + \frac{1}{\ell - 1} (\delta^h_i \beta_{jk} - \delta^h_j \beta_{ik})$$

$$\quad - \frac{1}{\ell - 1} (\delta^h_i \Omega_{ki} - \delta^h_j \Omega_{kj}) p_s - [\Omega^h_{ij} - \frac{1}{\ell - 1} (\delta^h_i \Omega^h_{j} - \delta^h_j \Omega^h_{i})] p_i.$$  

We denote by

$$B^h_{ijk} = \delta^h_i (\beta_{ij} - \Omega^h_{ij} p_s) + \frac{1}{\ell - 1} (\delta^h_i \beta_{jk} - \delta^h_j \beta_{ik})$$

$$\quad - \frac{1}{\ell - 1} (\delta^h_i \Omega_{ki} - \delta^h_j \Omega_{kj}) p_s - [\Omega^h_{ij} - \frac{1}{\ell - 1} (\delta^h_i \Omega^h_{j} - \delta^h_j \Omega^h_{i})] p_i,$$

then it is obvious that $W^h_{ijk} = W^h_{ijk} + B^h_{ijk}$.

**Definition 2.10.** If the 1-form $p$ and $q$ in (6) are horizontally closed, that is,

$$dp(X_b, Y_h) = X_b(p(Y_h)) - Y_h(p(X_b)) - p([X_b, Y_h]_0) = 0,$$

$$dq(X_b, Y_h) = X_b(q(Y_h)) - Y_h(q(X_b)) - q([X_b, Y_h]_0) = 0,$$

then we call a SSNH-projective connection $\nabla$ the special SSNH-projective connection.

**Theorem 2.11.** The sub-Weyl projective curvature tensor is an invariant under a special SSNH-projective transformation.

**Proof.** If $\tilde{\nabla}$ is a special SSNH-projective connection, then the 1-form $p$ and $q$ in (9) are all horizontally closed. Therefore there holds

$$0 = dp(e_i, e_j) = e_i(p_j) - e_j(p_i) - p([e_i, e_j]_0) = \varphi_{ij} - \varphi_{ji} + \rho_{ij} - \rho_{ji},$$

$$0 = dq(e_i, e_j) = e_i(q_j) - e_j(q_i) - q([e_i, e_j]_0) = \varphi_{ij} - \varphi_{ji} + \rho_{ij} - \rho_{ji}.$$  

By adding above two equations one gets $\varphi_{ij} = \varphi_{ji}$, and $\rho_{ij} = \rho_{ji}$ by subtracting these equations. Then one obtains $\beta_{ij} = 0$ and

$$\tilde{R}^h_{ijk} = R^h_{ijk} + \alpha_k \delta^h_{ij} - \alpha_k \delta^h_{ij},$$

Contracting by $i$ and $h$, one gets

$$\tilde{R}_k = R_k - (\ell - 1) \alpha_k,$$

Therefore, one obtains

$$\tilde{W}^h_{ijk} = \tilde{R}^h_{ijk} + \frac{1}{\ell - 1} (\delta^h_i \tilde{R}_{jk} - \delta^h_j \tilde{R}_{ik})$$

$$\quad = R^h_{ijk} + \alpha_k \delta^h_{ij} - \alpha_k \delta^h_{ij} + \frac{1}{\ell - 1} \delta^h_i (R_{jk} - (\ell - 1) \alpha_k) - \frac{1}{\ell - 1} \delta^h_i (R_{jk} - (\ell - 1) \alpha_k)$$

$$\quad = W^h_{ijk}.$$

The proof is finished. □
Remark 2.12. It is obvious that a projectively flat sub-Riemannian manifold is transformed to a projectively flat sub-Riemanninan manifold by a SSNH-projective transformation.

Theorem 2.13. A sub-Riemannian manifold \((M, \alpha, g)\) is projective flat if and only if the Schouten curvature tensor \(\tilde{R}\) of the special SSNH-projective connection \(\tilde{D}\) is vanished.

Proof. If \(\tilde{\nabla}\) is a special SSNH-projective connection and
\[
R_{ijk}^h = R_{ijk}^h + \beta_{ij}\delta_{ik}^h + \alpha_{ik}\delta_{jh}^i - \alpha_{kh}\delta_{ih}^j = 0, \tag{13}
\]
then by contracting (13) with \(i, j, h\), we have \(\tilde{R}_{ik} = R_{ik} + \beta_{ij} - (\ell - 1)\alpha_{ik} = 0\). Since \(\tilde{\nabla}\) is special, then the 1-form \(p\) is horizontally closed. Hence we get \(\beta_{ij} = 0\), and
\[
R_{ijk}^h = \alpha_{ik}\delta_{ih}^j - \alpha_{kh}\delta_{ih}^j, \quad R_{ik} = (\ell - 1)\alpha_{ik}. \tag{14}
\]
By substituting (14) into the following equation
\[
W_{ijk}^h = R_{ijk}^h + \frac{1}{\ell - 1}(\delta_{ij}^hR_{kh} - \delta_{ih}^jR_{kj}),
\]
we obtain \(W_{ijk}^h = 0\), that is, \(M\) is projectively flat.

Conversely, if \(M\) is projectively flat, then \(W_{ijk}^h = 0\), and \(R_{ijk}^h = \frac{1}{\ell - 1}(\delta_{ij}^hR_{kh} - \delta_{ih}^jR_{kj})\), namely, \(R_{ijk}^h = \frac{1}{\ell - 1}(g_{kh}R_{jk} - g_{jk}R_{kh})\). Since \(R_{ijk} = 0\), we get \(R_{ik} = \frac{\ell}{\ell - 1}g_{ik}\). If the 1-form \(p\) is horizontally closed, then the equation \(\tilde{R}_{ij} = R_{ij} + \beta_{ij} - (\ell - 1)\alpha_{ij} = 0\) is equivalent to
\[
(\nabla_\ell q)(e_i) - q(\alpha_{ij}) = \frac{R}{\ell(\ell - 1)}g_{ij}, \tag{15}
\]
where \((\nabla_\ell q)(e_i) - q(\alpha_{ij}) = \alpha_{ij}\).

Now taking a covariant derivative of Equation (15), we get
\[
(\nabla_\ell \nabla_\ell q)(e_i) + (\nabla_\ell q)(\nabla_\ell e_i) - (\nabla_\ell q)(e_i)q(e_j) - q(\nabla_\ell q)(e_i) - q(q(e_j)q(e_i) - q(q(\nabla_\ell e_i)q(e_j)) = \frac{K}{\ell(\ell - 1)}(g(\nabla_\ell e_i, e_j) + g(\nabla_\ell e_j, e_i))
\]
\[
= (\nabla_\ell \nabla_\ell q)(e_i) - q(\nabla_\ell q)(e_j)q(e_j) - (\nabla_\ell q)(\nabla_\ell e_i)q(e_j) - q(\nabla_\ell e_i)q(e_j),
\]
where the last equality follows from Equation (15). Namely,
\[
(\nabla_\ell \nabla_\ell q)(e_i) - q(\nabla_\ell q)(e_j)q(e_j) - q(\nabla_\ell q)(\nabla_\ell q)(e_j) = (\nabla_\ell \nabla_\ell q)(e_i). \tag{16}
\]
Since the horizontal 1-form \(p\) is closed, then by (15), (16) and \(W_{ijk}^h = 0\), we obtain
\[
(\nabla_\ell \nabla_\ell q - \nabla_\ell \nabla_\ell q - \nabla_\ell \nabla_\ell q)(e_i) = -R_{ij}^h g_{ih}, \tag{17}
\]
therefore there exists a solution \(q\) to Equation (15), let
\[
\tilde{\Gamma}_{ij}^k = \frac{\ell}{\ell - 1}p_i^j + p_j^k + q_i^k, \tag{18}
\]
where \(p\) is a closed horizontal 1-form.

By Theorem 2.8, we know \(\tilde{\nabla}\) whose connection coefficients are defined by (18) is a SSNH-projective connection. On the other hand, \(\alpha_{ij}\) is proportional to \(g_{ij}\) by (15), so it is symmetric and \(dq(e_j, e_j) = \alpha_{ij} - \alpha_{ji} = 0\), which implies that the 1-form \(q\) is horizontally closed.

This completes the proof of Theorem 2.13. \(\square\)
3. Example

**Example 3.1.** (Almost contact metric sub-Riemannian manifold)

Let \((M, \Delta, g)\) be a \((2n+1)\)-dimensional sub-Riemannian manifold, an almost contact structure is denoted by \((\varphi, \xi, \eta)\), where \(\varphi\) is a horizontal \((1,1)\)-tensor field (i.e. \(\varphi(X_h) \in \Delta\)), \(\xi\) is a vector field and \(\eta\) is a 1-form such that
\[\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X_h, \varphi Y_h) = g(X_h, Y_h) - \eta(X_h)\eta(Y_h).\]

then \((M, \Delta, g, \varphi, \xi, \eta)\) is called an almost contact metric sub-Riemannian manifold. In virtue of this 1-form \(\eta\), one defines a metric connection,
\[
\tilde{\nabla}_{X_h} Y_h = \nabla_{X_h} Y_h + \eta(X_h)Y_h + \eta(Y_h)X_h,
\]
in local coordinate, that is,
\[
\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \eta_i \delta^k_j + \eta_j \delta^k_i,
\]
where \(\nabla\) is the sub-Riemannian connection, then \(\tilde{\nabla}\) is actually a SSNH-projective connection.

In fact, if \(\gamma : x^a = x^a(t)\) is a SR-parallel curve with respect to sub-Riemannian connection, then it satisfies Equations (2), substituting (20) into the above Equations, one obtains,
\[
\frac{d^2x^k}{dt^2} + \tilde{\Gamma}^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = -2\eta_i \frac{dx^i}{dt} \frac{dx^k}{dt},
\]
Now we introduce a new parameter \(s\) by the equation
\[
s = \int e^{\int -2\eta ds} dt,
\]
and obtain the following relations by straight-forward calculation,
\[
\frac{ds}{dt} = e^{\int -2\eta ds} \frac{d^2 s}{dt^2} = e^{\int -2\eta ds} \left( -2\eta \frac{dx^i}{dt} \right),
\]
\[
\frac{dx^i}{dt} = e^{\int -2\eta ds} \frac{dx^i}{ds} \frac{d^2 s}{dt^2} = e^{\int -2\eta ds} \left( \frac{d^2 x^i}{ds^2} - 2\eta \frac{dx^i}{ds} \frac{dx^j}{ds} \right).
\]
hence, we have
\[
\frac{d^2x^k}{ds^2} + \tilde{\Gamma}^k_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0.
\]
that is \(\gamma : x^a = x^a(t)\) is also a SR-parallel curve associated with the connection (19). On the other hand, one can prove the converse statement is also true by the same method. Therefore, the metric connection (19) is a SSNH-projective connection.

References


