Filomat 31:5 (2017), 1251–1264
DOI 10.2298/FIL1705251B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Wold-Type Decompositions for n-Tuples of Commuting Isometric Semigroups

Tudor Bînzar[a], Cristian Lăzureanu[a]

[a] *Department of Mathematics, Politehnica University of Timişoara, Piaţa Victoriei nr. 2, 300006, Timişoara, România*

Abstract. In this paper the n-tuples of commuting isometric semigroups on a Hilbert space and the product semigroup generated by them are considered. Properties of the right defect spaces and characterizations of the semigroups of type "s" are given. Also, the Wold-type decompositions with 3^n summands for n-tuples of commuting isometric semigroups are introduced. The existence and uniqueness of such decompositions are analysed and several connections with the Wold decompositions of each semigroup and their product semigroup are presented.

1. Introduction

In the work on the behaviour of stationary time series [30], H. Wold obtained an important mathematical principle of decomposition of a stationary stochastic process into a random part and its non-random part. In the operator theory the well known Wold decomposition theorem states that every isometry on a Hilbert space can be decomposed into the orthogonal sum between a unitary operator and a shift [13, 27].

In 1980, M. Słociński proposed a Wold-type decomposition of a pair of commuting isometries on a Hilbert space [25]. His idea has been exploited by many mathematicians in different frameworks. We mention a partial list of references [1–5, 10, 11, 20, 22, 23, 28].

In [14], Helson and Lowdenslager considered a Wold-type decomposition with three summands for the continuous stationary processes. A Wold-type decomposition with three summands also occurs for a semigroup of isometries on a Hilbert space. Such a decomposition was given by I. Suciu in the commutative case [26] and by G. Popescu in the noncommutative case [19]. In the case of semigroups of isometries, the Wold-type decompositions were considered by many researchers, see for example [8, 9, 12, 17, 18].

The Wold-type decompositions in various versions have numerous applications, such as: stochastic processes, spectral analysis, prediction theory, audio signals, textured images [6, 15, 16, 24].

The present work is organized as follows:

In section 2, definitions, notions and properties we need in the following sections, are given. In section 3, some results about the right defect spaces and characterizations of semigroups of type "s" are studied. In section 4, Wold-type decompositions are presented.

2010 *Mathematics Subject Classification*. Primary 47D03; Secondary 47A15
Keywords. commuting isometric semigroups, Wold decomposition
Received: 03 February 2015; Accepted: 03 October 2015
Communicated by Dragan S. Djordjević
Email addresses: tudor.binzar@upt.ro (Tudor Bînzar), cristian.lazureanu@upt.ro (Cristian Lăzureanu)

2. Preliminaries

In this part of the paper, we recall some results about semigroups of isometries acting on Hilbert spaces [7],[25],[26] and we present some new results used in the following sections. Also, we introduce the frame of our work.

In the sequel, $\mathcal{H}$ is a complex Hilbert space with inner product $< x, y >$, $x, y \in \mathcal{H}$. By $L(\mathcal{H})$ we denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in L(\mathcal{H})$, T^* is the adjoint of T.

Let $(S, \cdot)$ be an abelian semigroup having 1_S as identity element. A semigroup of operators $\{V(\sigma)\}_{\sigma\in S}$ on $\mathcal{H}$ is a mapping $\sigma \to V(\sigma)$ from S to $L(\mathcal{H})$ such that $V(1_S) = I_{\mathcal{H}}$ and $V(\sigma \cdot \tau) = V(\sigma)V(\tau)$ for all $\sigma, \tau \in S$.

A closed subspace $\mathcal{K}$ of $\mathcal{H}$ is invariant for the semigroup $\{V(\sigma)\}_{\sigma\in S}$ if $V(\sigma)\mathcal{K} \subseteq \mathcal{K}$ for each $\sigma \in S$. We say that $\mathcal{K}$ reduces $\{V(\sigma)\}_{\sigma\in S}$ if $V(\sigma)\mathcal{K} \subseteq \mathcal{K}$ and $V(\sigma)^*\mathcal{K} \subseteq \mathcal{K}$ for each $\sigma \in S$.

A closed subspace $\mathcal{L} \subseteq \mathcal{H}$ is called wandering for $\{V(\sigma)\}_{\sigma\in S}$ if for any $\sigma, \tau \in S$, $\sigma \neq \tau$, $V(\sigma)\mathcal{L} \perp V(\tau)\mathcal{L}$.

A semigroup of operators $\{V(\sigma)\}_{\sigma\in S}$ on $\mathcal{H}$ is called an isometric (a unitary) semigroup if $V(\sigma)$ is an isometry (a unitary operator) on $\mathcal{H}$ for any $\sigma \in S$.

A semigroup $\{V(\sigma)\}_{\sigma\in S}$ on $\mathcal{H}$ is called completely non-unitary (of type "c") if there is no reducing subspace $\mathcal{M} \subseteq \mathcal{H}$, $\mathcal{M} \neq \{0\}$, for $\{V(\sigma)\}_{\sigma\in S}$ such that $\{V(\sigma)|_{\mathcal{M}}\}_{\sigma\in S}$ is unitary.

According to I. Suciu [26], let $(G, \cdot)$ be an abelian group and let S be a unital sub-semigroup of G such that $S \cap S^{-1} = \{1_S\}$ and $G = SS^{-1}$, where $S^{-1} = \{\sigma^{-1} | \sigma \in S\}$. If $\{V(\sigma)\}_{\sigma\in S}$ is a semigroup of isometries on $\mathcal{H}$, then $\mathcal{H}$ decomposes into an orthogonal sum

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c \tag{1}$$

such that $\mathcal{H}_u$ and $\mathcal{H}_c$ reduce $\{V(\sigma)\}_{\sigma\in S}$, $\{V(\sigma)|_{\mathcal{H}_u}\}_{\sigma\in S}$ is unitary and $\{V(\sigma)|_{\mathcal{H}_c}\}_{\sigma\in S}$ is completely non-unitary. The decomposition is unique and the unitary part $\mathcal{H}_u$ of $\{V(\sigma)\}_{\sigma\in S}$ is given by

$$\mathcal{H}_u = \{h \in \mathcal{H} \mid \|V(\sigma)^*h\| = \|h\| \text{ for all } \sigma \in S\}. \tag{2}$$

We remark that $\mathcal{H}_u$ is the maximal subspace of $\mathcal{H}$ reducing the semigroup $\{V(\sigma)\}_{\sigma\in S}$ to a unitary semigroup [9].

Taking into account the structure of $\mathcal{H}_u$, $\mathcal{H}_u = \bigcap_{\sigma\in S} V(\sigma)\mathcal{H}$, it easily results the following:

Proposition 2.1. *Let $\{V(\sigma)\}_{\sigma\in S}$ be a semigroup of isometries on a Hilbert space $\mathcal{H}$ and let $X \in L(\mathcal{H})$. If $XV(\sigma) = V(\sigma)X$ for all $\sigma \in S$, then $X\mathcal{H}_u \subseteq \mathcal{H}_u$.*

I. Suciu [26] gave a more precise structure of the completely non-unitary part. In order to mention this decomposition, we remind that a semigroup of isometries $\{V(\sigma)\}_{\sigma\in S}$ on $\mathcal{H}$ is called of type "e" if

$$\mathcal{H} = \bigvee_{\substack{\sigma^{-1}\cdot\tau\notin S^{-1} \\ (\sigma,\tau)\in S\times S}} V(\sigma)^*V(\tau)\mathcal{H}$$

and there is no reducing subspace $\mathcal{M} \subseteq \mathcal{H}$, $\mathcal{M} \neq \{0\}$, for $\{V(\sigma)\}_{\sigma\in S}$ such that $\{V(\sigma)|_{\mathcal{M}}\}_{\sigma\in S}$ is unitary.

The semigroup of isometries $\{V(\sigma)\}_{\sigma\in S}$ on $\mathcal{H}$ is called of type "s" if there is a wandering subspace $\mathcal{R} \subseteq \mathcal{H}$ for $\{V(\sigma)\}_{\sigma\in S}$ such that

$$\mathcal{H} = \bigoplus_{\sigma\in S} V(\sigma)\mathcal{R}.$$

It was proved that the restriction of an isometric semigroup to a reducing subspace is of the same type as the semigroup is (see [26]).

I. Suciu's decomposition for an isometric semigroup $\{V(\sigma)\}_{\sigma\in S}$ on $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_e \oplus \mathcal{H}_s,$$

such that the subspaces $\mathcal{H}_u, \mathcal{H}_e, \mathcal{H}_s$ reduce $\{V(\sigma)\}_{\sigma\in S}$, and $\{V(\sigma)|_{\mathcal{H}_u}\}_{\sigma\in S}$ is unitary (of type "u"), $\{V(\sigma)|_{\mathcal{H}_e}\}_{\sigma\in S}$ is of type "e", $\{V(\sigma)|_{\mathcal{H}_s}\}_{\sigma\in S}$ is of type "s". Moreover, the decomposition is unique and

$$\mathcal{H}_s = \bigoplus_{\sigma\in S} V(\sigma)\mathcal{R},$$

where

$$\mathcal{R} = \mathcal{H} \ominus \bigvee_{\substack{\sigma^{-1}\cdot\tau\notin S^{-1}\\ (\sigma,\tau)\in S\times S}} V(\sigma)^*V(\tau)\mathcal{H} = \bigcap_{\substack{\sigma^{-1}\cdot\tau\notin S^{-1}\\ (\sigma,\tau)\in S\times S}} \ker V(\tau)^*V(\sigma) \tag{3}$$

is the right defect space of $\{V(\sigma)\}_{\sigma\in S}$ (see [7],[17],[19]).

We remark that the subspace $\mathcal{H}_\alpha$, $\alpha \in \{u, e, s\}$, in the I. Suciu's decomposition is the largest subspace that reduces $\{V(\sigma)\}_{\sigma\in S}$ to a semigroup of type "α" [9], [26].

Now, let us introduce the general framework of the present paper. Throughout this paper n is a natural number, $n \geq 2$ and I_n stands for the set $\{1, 2, \ldots, n\}$.

We consider n unital sub-semigroups $S_1, S_2, \ldots, S_n$ of multiplicative abelian groups $G_1, G_2, \ldots, G_n$ respectively, such that $S_i \cap S_i^{-1} = \{1_{S_i}\}$ and $G_i = S_iS_i^{-1}$, $i \in I_n$. Also, let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be commuting isometric semigroups on a Hilbert space $\mathcal{H}$, i.e.

$$V_i(\sigma_i)V_j(\sigma_j) = V_j(\sigma_j)V_i(\sigma_i) \text{ for all } \sigma_i \in S_i,\ \sigma_j \in S_j,\ i, j \in I_n.$$

We say that the semigroups $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, are doubly commuting on a Hilbert space $\mathcal{H}$ if they commute and

$$V_i(\sigma_i)^*V_j(\sigma_j) = V_j(\sigma_j)V_i(\sigma_i)^* \text{ for all } \sigma_i \in S_i,\ \sigma_j \in S_j,\ i, j \in I_n, i \neq j.$$

Let $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ be the product semigroup generated by the commuting semigroups of isometries $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, defined by $V(\overline{\sigma}) = V_1(\sigma_1)V_2(\sigma_2)\ldots V_n(\sigma_n)$, $\overline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_1 \times S_2 \times \ldots \times S_n = \overline{S}$. It is clear that $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ is an isometric semigroup.

At the end of this section, we give a description of the unitary part of $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$.

Proposition 2.2. *Let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be commuting semigroups of isometries on a Hilbert space $\mathcal{H}$ and let $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ be the corresponding product semigroup. If $\mathcal{H}_u^i$ is the unitary part of $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, and $\mathcal{H}_u$ is the unitary part of $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$, then $\mathcal{H}_u \subseteq \bigcap_{i=1}^{n} \mathcal{H}_u^i$. Moreover, if the semigroups $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, are doubly commuting on $\mathcal{H}$, then $\mathcal{H}_u = \bigcap_{i=1}^{n} \mathcal{H}_u^i$.*

Proof. Let $i \in I_n$. Using (2) for $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ and taking $\sigma_j = 1_{S_j}$ for all $j \in I_n$, $j \neq i$ one gets $\mathcal{H}_u \subseteq \mathcal{H}_u^i$, whence $\mathcal{H}_u \subseteq \bigcap_{i=1}^{n} \mathcal{H}_u^i$.

Now, let us assume that the semigroups $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, are doubly commuting on $\mathcal{H}$. It only remains to prove that $\bigcap_{i=1}^{n} \mathcal{H}_u^i \subseteq \mathcal{H}_u$. Using Proposition 2.1, it results

$$\begin{aligned} V_1(\sigma_1)V_2(\sigma_2)\ldots V_n(\sigma_n)\left(\bigcap_{i=1}^{n}\mathcal{H}_u^i\right) &\subseteq V_1(\sigma_1)V_2(\sigma_2)\ldots V_n(\sigma_n)\mathcal{H}_u^n \\ &\subseteq V_1(\sigma_1)V_2(\sigma_2)\ldots V_{n-1}(\sigma_{n-1})\mathcal{H}_u^n \subseteq \mathcal{H}_u^n. \end{aligned}$$

Similarly, one obtains $V_1(\sigma_1)V_2(\sigma_2)\ldots V_n(\sigma_n)\left(\bigcap_{i=1}^{n}\mathcal{H}_u^i\right) \subseteq \mathcal{H}_u^j$ for all $j \in I_n$. Therefore $\bigcap_{i=1}^{n} \mathcal{H}_u^i$ is invariant for the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$.

Analogously, using our assumption it follows $\bigcap_{i=1}^{n} \mathcal{H}_u^i$ is invariant for the semigroup $\{V(\overline{\sigma})^*\}_{\overline{\sigma}\in\overline{S}}$. Thus, $\bigcap_{i=1}^{n} \mathcal{H}_u^i$ is a reducing subspace for the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$. It is obvious that the semigroup $\{V(\overline{\sigma})|_{\bigcap_{i=1}^{n} \mathcal{H}_u^i}\}_{\overline{\sigma}\in\overline{S}}$ is unitary, whence $\bigcap_{i=1}^{n} \mathcal{H}_u^i \subseteq \mathcal{H}_u$. □

3. Right Defect Spaces

Let us consider an n-tuple of commuting semigroups of isometries $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, on a Hilbert space $\mathcal{H}$ and the corresponding product semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$. In this section, connections between the right defect space $\mathcal{R}$ of $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ and the right defect spaces $\mathcal{R}_i$ of $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, are given.

The first result establishes some inclusions between the aforementioned right defect spaces.

Theorem 3.1. *Let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be commuting isometric semigroups on a Hilbert space $\mathcal{H}$ and let $\mathcal{R}_i$, $i \in I_n$, be their corresponding right defect spaces. If $\mathcal{R}$ is the right defect space of the product semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$, then the following relations*

a) $\mathcal{R} \subseteq \bigcap_{i=1}^{n} \mathcal{R}_i$;

b) $\bigoplus_{\check{\overline{\sigma}}_i\in\check{\overline{S}}_i} V_1(\sigma_1)\ldots V_{i-1}(\sigma_{i-1})V_{i+1}(\sigma_{i+1})\ldots V_n(\sigma_n)\mathcal{R} \subseteq \mathcal{R}_i$, $i \in I_n$, *where*

$$\check{\overline{\sigma}}_i = (\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) \in S_1\times\ldots\times S_{i-1}\times S_{i+1}\times\ldots\times S_n = \check{\overline{S}}_i;$$

c) $\bigoplus_{\sigma_j\in S_j} V_j(\sigma_j)\mathcal{R} \subseteq \bigcap_{\substack{i=1\\ i\neq j}}^{n} \mathcal{R}_i$, $j \in I_n$,

hold.

Also, if $\mathcal{R}'$ is a closed subspace of $\mathcal{H}$ such that

$$\mathcal{R}' \subseteq \mathcal{R}_1,\ \bigoplus_{\sigma_1\in S_1} V_1(\sigma_1)\mathcal{R}' \subseteq \mathcal{R}_2,\ldots,\bigoplus_{\check{\overline{\sigma}}_n\in\check{\overline{S}}_n} V_1(\sigma_1)\ldots V_{n-1}(\sigma_{n-1})\mathcal{R}' \subseteq \mathcal{R}_n,$$

then $\mathcal{R}'$ is a wandering subspace for the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$.

Proof. a) By the definition of $\mathcal{R}$ it follows that

$$\mathcal{R} = \left[\bigvee_{\substack{\sigma_1^{-1}\cdot\tau_1\notin S_1^{-1}\\ (\overline{\sigma},\overline{\tau})\in\overline{S}\times\overline{S}}} V_1(\sigma_1)^*\ldots V_n(\sigma_n)^*V_1(\tau_1)\ldots V_n(\tau_n)\mathcal{H}\vee\ldots\vee \bigvee_{\substack{\sigma_n^{-1}\cdot\tau_n\notin S_n^{-1}\\ (\overline{\sigma},\overline{\tau})\in\overline{S}\times\overline{S}}} V_1(\sigma_1)^*\ldots V_n(\sigma_n)^*V_1(\tau_1)\ldots V_n(\tau_n)\mathcal{H}\right]^{\perp} \tag{4}$$

Consequently, if $x \in \mathcal{R}$, then for every $i \in I_n$ we have the following:

$$x\perp V_1(\sigma_1)^*\ldots V_i(\sigma_i)^*\ldots V_n(\sigma_n)^*V_1(\tau_1)\ldots V_i(\tau_i)\ldots V_n(\tau_n)\mathcal{H} \tag{5}$$

for all $(\overline{\sigma},\overline{\tau}) \in \overline{S}\times\overline{S}$ with the properties $\sigma_i^{-1}\cdot\tau_i \notin S_i^{-1}$.

Let $i \in I_n$. If we put in (5) $\sigma_j = \tau_j = 1_{S_j}$, $j \neq i$ one gets $x \perp V_i(\sigma_i)^* V_i(\tau_i)\mathcal{H}$ for all $\sigma_i, \tau_i \in S_i$ with the property $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$, whence $x \in \mathcal{R}_i$. Therefore $\mathcal{R} \subseteq \bigcap_{i=1}^{n} \mathcal{R}_i$.

b) Let $i \in I_n$, fixed. Taking $\tau_j = 1_{S_j}$, $j \neq i$, $j \in I_n$, by (5) we deduce

$$x \perp V_1(\sigma_1)^* \dots V_{i-1}(\sigma_{i-1})^* V_{i+1}(\sigma_{i+1})^* \dots V_n(\sigma_n)^* V_i(\sigma_i)^* V_i(\tau_i)\mathcal{H}$$

for every $x \in \mathcal{R}, \sigma_j \in S_j$ and $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$. It results:

$$V_1(\sigma_1) \dots V_{i-1}(\sigma_{i-1}) V_{i+1}(\sigma_{i+1}) \dots V_n(\sigma_n) x \perp V_i(\sigma_i)^* V_i(\tau_i)\mathcal{H}$$

for every $x \in \mathcal{R}, \sigma_j \in S_j$ and $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$. Using relation (3), the conclusion follows.

c) It results by b).

Now, suppose that $x, y \in \mathcal{R}'$. We will prove that $V(\overline{\sigma})x \perp V(\overline{\tau})y$ for every $\overline{\sigma}, \overline{\tau} \in \overline{S}$ with $\overline{\sigma} \neq \overline{\tau}$ or equivalently

$$V(\sigma_1) \dots V_n(\sigma_n) x \perp V_1(\tau_1) \dots V_n(\tau_n) y \tag{6}$$

for every $\sigma_i, \tau_i \in S_i, i \in I_n$ with $\sigma_j \neq \tau_j$ for some j, $j \in I_n$.

If $\sigma_n \neq \tau_n$, taking into account that $\mathcal{R}_n$ is wandering for $\{V_n(\sigma_n)\}_{\sigma_n) \in S_n}$, one gets

$$\begin{aligned} &< V_1(\sigma_1) \dots V_n(\sigma_n) x, V_1(\tau_1) \dots V_n(\tau_n) y >= \\ &=< V_n(\sigma_n) V_1(\sigma_1) \dots V_{n-1}(\sigma_{n-1}) x, V_n(\tau_n) V_1(\tau_1) \dots V_{n-1}(\tau_{n-1}) y > \\ &= 0, \end{aligned}$$

i.e. relation (6) is proved.

If $\sigma_n = \tau_n$, relation (6) is reduced to

$$V_1(\sigma_1) \dots V_{n-1}(\sigma_{n-1}) x \perp V_1(\tau_1) \dots V_{n-1}(\tau_{n-1}) y$$

with $\sigma_k \neq \tau_k$ for some k, $k \in \{1, 2, \dots, n-1\}$. Performing the above steps, the conclusion follows. □

The next theorem furnishes sufficient conditions for the inclusions in Theorem 3.1 to become equalities.

Theorem 3.2. *Suppose that the isometric semigroups $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, are doubly commuting on a Hilbert space $\mathcal{H}$. Then:*

a) The right defect space $\mathcal{R}_i$ of the corresponding semigroup $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, is a reducing subspace of the semigroup $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in I_n$, $j \neq i$;

b) $\mathcal{R} = \bigcap_{i=1}^{n} \mathcal{R}_i$;

c) Let $i, j \in I_n, i \neq j$. If the semigroup $\{V_i(\sigma_i)|_{\mathcal{R}_j}\}_{\sigma_i \in S_i}$ is of type "s", then

$$\bigoplus_{\sigma_i \in S_i} V_i(\sigma_i) \left(\mathcal{R}_i \cap \mathcal{R}_j\right) = \mathcal{R}_j;$$

d) Let $i \in I_n$, fixed. If the semigroup $\{V_i(\sigma_i)|_{\bigcap_{j=1, j\neq i}^{n} \mathcal{R}_j}\}_{\sigma_i \in S_i}$ is of type "s", then

$$\bigoplus_{\sigma_i \in S_i} V_i(\sigma_i) \mathcal{R} = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} \mathcal{R}_j;$$

e) If the semigroup $\{V_1(\sigma_1) \dots V_{n-1}(\sigma_{n-1})|_{\mathcal{R}_n}\}_{\check{\overline{\sigma}}_n \in \check{\overline{S}}_n}$ is of type "s", then

$$\bigoplus_{\check{\overline{\sigma}}_n \in \check{\overline{S}}_n} V_1(\sigma_1) \dots V_{n-1}(\sigma_{n-1}) \mathcal{R} = \mathcal{R}_n.$$

Proof. a) Let $i, j \in I_n, i \neq j$ and let $x \in \mathcal{R}_i = \mathcal{H} \ominus \bigvee_{\substack{\sigma_i^{-1}\cdot\tau_i \notin S_i^{-1} \\ (\sigma_i,\tau_i)\in S_i\times S_i}} V_i(\sigma_i)^* V_i(\tau_i)\mathcal{H}$. Then $x \perp V_i(\sigma_i)^* V_i(\tau_i) y$ for every $y \in \mathcal{H}$ and for all $\sigma_i, \tau_i \in S_i$ such that $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$, hence $x \perp V_i(\sigma_i)^* V_i(\tau_i) V_j(\sigma_j)^* z$ for every $z \in \mathcal{H}$, for all $\sigma_j \in S_j$ and for all $\sigma_i, \tau_i \in S_i$ such that $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$.

Since $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$ and $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$ are doubly commuting it follows that

$$< V_j(\sigma_j)x, V_i(\sigma_i)^* V_i(\tau_i) z >=< x, V_i(\sigma_i)^* V_i(\tau_i) V_j(\sigma_j)^* z >= 0$$

for every $z \in \mathcal{H}$, for all $\sigma_j \in S_j$ and for all $\sigma_i, \tau_i \in S_i$ such that $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$. Therefore $V_j(\sigma_j)x \in \mathcal{R}_i$ for each $\sigma_j \in S_j$, that is $\mathcal{R}_i$ is an invariant space of $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$. Analogously one shows that $\mathcal{R}_i$ is an invariant space of $\{V_j(\sigma_j)^*\}_{\sigma_j \in S_j}$, hence $\mathcal{R}_i$ is a reducing subspace of $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \neq i$.

b) By Theorem 3.1 side a), the inclusion $\mathcal{R} \subseteq \bigcap_{i=1}^{n} \mathcal{R}_i$ holds. It only remains to prove that $\bigcap_{i=1}^{n} \mathcal{R}_i \subseteq \mathcal{R}$.

Let $x \in \bigcap_{i=1}^{n} \mathcal{R}_i$. Then

$$x \perp \bigvee_{\substack{\sigma_i^{-1}\cdot\tau_i \notin S_i^{-1} \\ (\sigma_i,\tau_i)\in S_i\times S_i}} V_i(\sigma_i)^* V_i(\tau_i)\mathcal{H}$$

for each $i \in I_n$.

Let $i \in I_n$ and let $y \in \mathcal{H}$. Since the semigroups $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in I_n$, are doubly commuting, one obtains

$$< x, V_1(\sigma_1)^* \dots V_n(\sigma_n)^* V_1(\tau_1) \dots V_n(\tau_n) y >=< x, V_i(\sigma_i)^* V_i(\tau_i) z >= 0$$

for all $(\overline{\sigma}, \overline{\tau}) \in \overline{S} \times \overline{S}$ with the property $\sigma_i^{-1} \cdot \tau_i \notin S_i^{-1}$, where $z = V_1(\sigma_1)^* \dots V_{i-1}(\sigma_{i-1})^* V_{i+1}(\sigma_{i+1})^* \dots V_n(\sigma_n)^* V_1(\tau_1) \dots V_{i-1}(\tau_{i-1}) V_{i+1}(\tau_{i+1}) \dots V_n(\tau_n) y$. Taking into account the relation (4) it results $x \in \mathcal{R}$. Therefore $\bigcap_{i=1}^{n} \mathcal{R}_i \subseteq \mathcal{R}$.

c) We denote by $\mathcal{R}_i'$ the right defect space of the semigroup $\{V_i(\sigma_i)|_{\mathcal{R}_j}\}_{\sigma_i \in S_i}$. For $x \in \mathcal{R}_i'$ we have $x \in \mathcal{R}_j$ and $x \perp V_i(\sigma_i)^* V_i(\tau_i) \mathcal{R}_j$ for every $\sigma_i, \tau_i \in S_i$ with $\sigma_i^{-1}\tau_i \notin S_i^{-1}$. We deduce that $V_i(\tau_i)^* V_i(\sigma_i) x = 0$ for every $\sigma_i, \tau_i \in S_i$ with $\sigma_i^{-1}\tau_i \notin S_i^{-1}$ and consequently $x \in \mathcal{R}_i$.Thus $\mathcal{R}_i' \subseteq \mathcal{R}_i \cap \mathcal{R}_j$, $j \neq i$. Since $\{V_i(\sigma_i)|_{\mathcal{R}_j}\}_{\sigma_i \in S_i}$ is of type "s", it easily follows the conclusion.

d) Let $\mathcal{R}'$ be the right defect space of the semigroup $\{V_i(\sigma_i)|_{\bigcap_{j=1, j\neq i}^{n} \mathcal{R}_j}\}_{\sigma_i \in S_i}$. As before one gets $\mathcal{R}' \subseteq \bigcap_{i=1}^{n} \mathcal{R}_i = \mathcal{R}$.

Now, using Theorem 3.1 c), we obtain

$$\bigcap_{\substack{j=1 \\ j\neq i}}^{n} \mathcal{R}_j = \bigoplus_{\sigma_i \in S_i} V_i(\sigma_i)\mathcal{R}' \subseteq \bigoplus_{\sigma_i \in S_i} V_i(\sigma_i)\mathcal{R} \subseteq \bigcap_{\substack{j=1 \\ j\neq i}}^{n} \mathcal{R}_j.$$

e) It immediately follows from c). □

In the last theorem of this section, a characterization for the semigroup $V(\overline{\sigma})\}_{\overline{\sigma} \in \overline{S}}$ to be of type "s" is given.

Theorem 3.3. *If $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, are commuting semigroups of isometries on a Hilbert space $\mathcal{H}$ with the corresponding right defect spaces $\mathcal{R}_i$, then the following conditions are equivalent:*
a) The semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma} \in \overline{S}}$, $V(\overline{\sigma}) = V_1(\sigma_1)V_2(\sigma_2)\dots V_n(\sigma_n)$ is of type "s";
b) For every $i \in I_n$, $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$ and $\{V_1(\sigma_1)\dots V_{i-1}(\sigma_{i-1})V_{i+1}(\sigma_{i+1})\dots V_n(\sigma_n)\}_{\check{\overline{\sigma}}_i \in \check{\overline{S}}_i}$ are doubly commuting isometric semigroups of type "s";

c) $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$*,* $i\in I_n$*, are doubly commuting isometric semigroups of type "s";*

d) $\{V_1(\sigma_1)\dots V_{n-1}(\sigma_{n-1})\}_{\check{\check{\sigma}}_n\in\check{\check{S}}_n}$ *is of type "s", its right defect space is* $\bigcap\limits_{i=1}^{n-1}\mathcal{R}_i$ *and*

$$\bigoplus_{\sigma_n\in S_n} V_n(\sigma_n)\left(\bigcap_{i=1}^{n}\mathcal{R}_i\right)=\bigcap_{i=1}^{n-1}\mathcal{R}_i;$$

e) $\{V_n(\sigma_n)\}_{\sigma_n\in S_n}$ *is of type "s", the subspace* $\bigcap\limits_{i=1}^{n}\mathcal{R}_i$ *is wandering for the semigroup* $\{V_1(\sigma_1)\dots V_{n-1}(\sigma_{n-1})\}_{\check{\check{\sigma}}_n\in\check{\check{S}}_n}$ *and*

$$\bigoplus_{\check{\check{\sigma}}_n\in\check{\check{S}}_n} V_1(\sigma_1)\dots V_{n-1}(\sigma_{n-1})\left(\bigcap_{i=1}^{n}\mathcal{R}_i\right)=\mathcal{R}_n;$$

f) $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ *is of type "s" and* $\bigoplus\limits_{\sigma_j\in S_j} V_j(\sigma_j)\left(\bigcap\limits_{i=1}^{j}\mathcal{R}_i\right)=\bigcap\limits_{i=1}^{j-1}\mathcal{R}_i$*, for every* $j\in I_n\setminus\{1\}$*;*

g) $\bigcap\limits_{i=1}^{n}\mathcal{R}_i$ *is a wandering subspace for the semigroup* $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ *and*

$$\bigoplus_{\overline{\sigma}\in\overline{S}} V(\overline{\sigma})\left(\bigcap_{i=1}^{n}\mathcal{R}_i\right)=\mathcal{H}.$$

Proof. "a) ⇒ b)" Let $\mathcal{R}$ be a wandering subspace for the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$ such that $\mathcal{H}=\bigoplus\limits_{\overline{\sigma}\in\overline{S}}V(\overline{\sigma})\mathcal{R}=\bigoplus\limits_{\sigma_1\in S_1}V_1(\sigma_1)\mathcal{R}_1'$, where $\mathcal{R}_1'=\bigoplus\limits_{\check{\check{\sigma}}_1\in\check{\check{S}}_1}V_2(\sigma_2)\dots V_n(\sigma_n)\mathcal{R}$. It follows that $\mathcal{R}_1'$ is a wandering subspace for $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$, whence $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ is of type "s" and $\mathcal{R}_1'=\mathcal{R}_1$.

Analogously, the semigroups $\{V_1(\sigma_1)\dots V_{i-1}(\sigma_{i-1})V_{i+1}(\sigma_{i+1})\dots V_n(\sigma_n)\}_{\check{\check{\sigma}}_i\in\check{\check{S}}_i}$ and $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$ are of type "s".

In the sequel we prove that $V_1(\sigma_1)^*$ commutes with $V_2(\sigma_2)\dots V_n(\sigma_n)$ for all $(\sigma_1,\dots,\sigma_n)\in\overline{S}$.

Let $x\in\mathcal{H}$. Then $x=\sum\limits_{\sigma_1'\in S_1}V_1(\sigma_1')x_{\sigma_1'}$, $x_{\sigma_1'}\in\mathcal{R}_1$. We have

$$\begin{aligned}V_1(\sigma_1)^*V_2(\sigma_2)\dots V_n(\sigma_n)x &= \sum_{\sigma_1'\in S_1}V_1(\sigma_1)^*V_1(\sigma_1')V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1'}\\ &= \sum_{\substack{\sigma_1'\in S_1\\ (\sigma_1')^{-1}\cdot\sigma_1\notin S_1^{-1}}}V_1(\sigma_1)^*V_1(\sigma_1')V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1'}+\\ &+\sum_{\substack{\sigma_1'\in S_1\\ (\sigma_1')^{-1}\cdot\sigma_1\in S_1^{-1}}}V_1(\sigma_1)^*V_1(\sigma_1')V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1'}\\ &= \sum_{\substack{\sigma_1'\in S_1\\ (\sigma_1')^{-1}\cdot\sigma_1\in S_1^{-1}}}V_1(\sigma_1)^*V_1(\sigma_1')V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1'}\end{aligned}$$

since $x_{\sigma_1'}\in\mathcal{R}_1$ implies $V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1'}\in\mathcal{R}_1$ for every $\sigma_i\in S_i$, $i\in\{2,3,\dots,n\}$ and $\mathcal{R}_1$ is the right defect space of $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$.

If $(\sigma_1')^{-1}\cdot\sigma_1\in S_1^{-1}$, there exists $\tau_1\in S_1$ such that $(\sigma_1')^{-1}\cdot\sigma_1=\tau_1^{-1}$, hence $\sigma_1'=\sigma_1\cdot\tau_1$. Consequently,

$$V_1(\sigma_1)^*V_2(\sigma_2)\dots V_n(\sigma_n)x=\sum_{\tau_1\in S_1}V_1(\tau_1)V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1\tau_1}.$$

On the other hand,

$$\begin{aligned} V_2(\sigma_2)\dots V_n(\sigma_n)V_1(\sigma_1)^*x &= \sum_{\substack{\sigma_1{}'\in S_1\\ (\sigma_1{}')^{-1}\cdot\sigma_1\notin S_1^{-1}}} V_2(\sigma_2)\dots V_n(\sigma_n)V_1(\sigma_1)^*V_1(\sigma_1{}')x_{\sigma_1{}'} \\ &+ \sum_{\substack{\sigma_1{}'\in S_1\\ (\sigma_1{}')^{-1}\cdot\sigma_1\in S_1^{-1}}} V_2(\sigma_2)\dots V_n(\sigma_n)V_1(\sigma_1)^*V_1(\sigma_1{}')x_{\sigma_1{}'} \\ &= \sum_{\substack{\sigma_1{}'\in S_1\\ (\sigma_1{}')^{-1}\cdot\sigma_1\in S_1^{-1}}} V_2(\sigma_2)\dots V_n(\sigma_n)V_1(\sigma_1)^*V_1(\sigma_1{}')x_{\sigma_1{}'} \end{aligned}$$

since $x_{\sigma_1{}'}\in\mathcal{R}_1$ implies

$$< x_{\sigma_1{}'}, V_1(\sigma_1{}')^*V_1(\sigma_1)y >=< V_1(\sigma_1)^*V_1(\sigma_1{}')x_{\sigma_1{}'}, y >= 0$$

for all $y\in\mathcal{H}$ and for all $\sigma_1,\sigma_1{}'\in S_1$ such that $(\sigma_1{}')^{-1}\cdot\sigma_1\notin S_1^{-1}$.

As before, we obtain that

$$V_2(\sigma_2)\dots V_n(\sigma_n)V_1(\sigma_1)^*x = \sum_{\tau_1\in S_1} V_1(\tau_1)V_2(\sigma_2)\dots V_n(\sigma_n)x_{\sigma_1\tau_1}.$$

Thus, the assertion is proved.

"b) $\Rightarrow$ c)" We show that the semigroups $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ are doubly commuting.

By our assumption

$$V_1(\sigma_1)\left(V_2(\sigma_2)V_3(\sigma_3)\dots V_n(\sigma_n)\right)^* = \left(V_2(\sigma_2)V_3(\sigma_3)\dots V_n(\sigma_n)\right)^* V_1(\sigma_1)$$

for all $(\sigma_1,\dots,\sigma_n)\in\overline{S}$. Taking $\sigma_3 = 1_{S_3},\dots,\sigma_n = 1_{S_n}$, the conclusion follows.

Similarly, the semigroups $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$ and $\{V_j(\sigma_j)\}_{\sigma_j\in S_j}$ are doubly commuting, $i,j\in I_n, i\neq j$.

"b) $\Rightarrow$ e)" Corroborating the implication "b) $\Rightarrow$ c)", Theorem 3.2 a), Theorem 3.2 b), Theorem 3.2 e) and the fact that the restriction of an isometric semigroup of type "s" to one of its reducing subspaces is also of type "s", the conclusion follows.

"c) $\Rightarrow$ f)" By Theorem 3.2 a) and Theorem 3.2 d), for $n = 2$, $n = 3$, ..., it results the conclusion.

"e) $\Rightarrow$ g)" As like as in the end of the proof of Theorem 3.1, it results that $\bigcap_{i=1}^{n}\mathcal{R}_i$ is a wandering subspace for $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$.

Since $\{V_n(\sigma_n)\}_{\sigma_n\in S_n}$ is a semigroup of type "s", we have

$$\mathcal{H} = \bigoplus_{\sigma_n\in S_n} V_n(\sigma_n)\mathcal{R}_n$$

and using the hypothesis we obtain

$$\mathcal{H} = \bigoplus_{\overline{\sigma}\in\overline{S}} V(\overline{\sigma})\left(\bigcap_{i=1}^{n}\mathcal{R}_i\right).$$

"f) $\Rightarrow$ g)" First we prove that $\bigcap_{i=1}^{n}\mathcal{R}_i$ is a wandering subspace for $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$. Let $\overline{\sigma},\overline{\tau}\in\overline{S}$ such that $\overline{\sigma}\neq\overline{\tau}$. Then there exists $k\in I_n$ such that $\sigma_k\neq\tau_k$ and $\sigma_j = \tau_j$, for any $j\in I_n$, $j<k$. Let $x,y\in\bigcap_{i=1}^{n}\mathcal{R}_i$. By hypothesis, we deduce

$$V_{k+1}(\sigma_{k+1})\dots V_n(\sigma_n)\left(\bigcap_{i=1}^{n}\mathcal{R}_i\right) = \bigcap_{i=1}^{k}\mathcal{R}_i\subseteq\mathcal{R}_k,$$

whence

$$< V_1(\sigma_1)\dots V_k(\sigma_k)V_{k+1}(\sigma_{k+1})\dots V_n(\sigma_n)x, V_1(\tau_1)\dots V_k(\tau_k)V_{k+1}(\tau_{k+1})\dots V_n(\tau_n)y >=$$

$$< V_k(\sigma_k)V_{k+1}(\sigma_{k+1})\dots V_n(\sigma_n)x, V_k(\tau_k)V_{k+1}(\tau_{k+1})\dots V_n(\tau_n)y >= 0.$$

Therefore the subspace $\bigcap_{i=1}^{n}\mathcal{R}_i$ is wandering for the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$.

It is easy to show that $\mathcal{H} = \bigoplus_{\overline{\sigma}\in S_{\overline{\sigma}}} V(\overline{\sigma})\left(\bigcap_{i=1}^{n}\mathcal{R}_i\right)$.

"g) ⇒ a)" It is obvious.
"c) ⇒ d)" The conclusion follows by the equivalence "c) ⇔ b)" and by the assertions a), b) and d) of Theorem 3.2.
"d) ⇒ g)" It immediately results.

Hence the theorem is completely proved. □

A more precisely description of the subspace $\mathcal{R}$ in Theorem 3.1 is given in the following.

Proposition 3.4. *Let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be commuting isometric semigroups on a Hilbert space $\mathcal{H}$ and let $\mathcal{R}_i$, $i \in I_n$, be their corresponding right defect spaces. If $\mathcal{R}$ is the right defect space of the product semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma}\in\overline{S}}$, then $\mathcal{R} = \mathcal{H}_0 \cap \bigcap_{i=1}^{n}\mathcal{R}_i$, where $\mathcal{H}_0$ is the maximal subspace of $\mathcal{H}$ that reduces $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, and the semigroups $\{V_i(\sigma_i)|_{\mathcal{H}_0}\}_{\sigma_i\in S_i}$, $i \in I_n$, are doubly commuting.*

Proof. Since double commutativity is a hereditary property, the Hilbert space $\mathcal{H}$ has the unique decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$, where $\mathcal{H}_0$ has the aforementioned properties (see [29]). Let $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_e \oplus \mathcal{H}_s$ be the I. Suciu decomposition of the product semigroup. Using Theorem 3.3, the semigroups $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, doubly commute on $\mathcal{H}_s$. Therefore $\mathcal{H}_u \oplus \mathcal{H}_s \subseteq \mathcal{H}_0$. Since $\mathcal{R} \subseteq \mathcal{H}_s \subseteq \mathcal{H}_0$, using Theorem 3.1 side a), it follows $\mathcal{R} \subseteq \mathcal{H}_0 \cap \bigcap_{i=1}^{n}\mathcal{R}_i$. It remains to prove $\mathcal{H}_0 \cap \bigcap_{i=1}^{n}\mathcal{R}_i \subseteq \mathcal{R}$.

Let $x \in \mathcal{H}_0 \cap \bigcap_{i=1}^{n}\mathcal{R}_i$. By (3), we have $V_i(\tau_i)^*V_i(\sigma_i)x = 0$ for any $\sigma_i, \tau_i \in S_i$, $\sigma_i^{-1}\tau_i \notin S_i^{-1}$, $i \in I_n$. Let $\overline{\mu}, \overline{\nu} \in \overline{S}$ such that $\overline{\mu}^{-1}\overline{\nu} \notin \overline{S}^{-1}$. Then there exists $j \in I_n$ with the property $\mu_j^{-1}\nu_j \notin S_j^{-1}$. Since $x \in \mathcal{H}_0$, we deduce

$$\begin{aligned} V(\overline{\nu})^*V(\overline{\mu})x &= V_n(\nu_n)^*\dots V_1(\nu_1)^*V_1(\mu_1)\dots V_n(\mu_n)x \\ &= V_n(\nu_n)^*\dots V_{j+1}(\nu_{j+1})^*V_{j-1}(\nu_{j-1})^*\dots V_1(\nu_1)^*V_1(\mu_1)\dots V_{j-1}(\mu_{j-1})V_{j+1}(\mu_{j+1})\dots V_n(\mu_n)V_j(\nu_j)^*V_j(\mu_j)x \\ &= 0. \end{aligned}$$

Therefore $x \in \mathcal{R}$. This completes the proof. □

In the case when G_i is totally ordered by S_i, $i \in I_n$, a double commuting part of an n-tuple of isometric semigroups can be identified.

Proposition 3.5. *Let G_i, $i \in I_n$, be multiplicative abelian groups totally ordered by the unital sub-semigroups S_i, $i \in I_n$ such that $S_i \cap S_i^{-1} = \{1_{S_i}\}$ and $G_i = S_iS_i^{-1}$, $i \in I_n$. Let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be an n-tuple of isometric semigroups on a Hilbert space $\mathcal{H}$. Then the subspace $\mathcal{H}_{dc}$ of $\mathcal{H}$ given by*

$$\mathcal{H}_{dc} = \{h \in \mathcal{H} \mid V_1(\sigma_1)V_1(\tau_1)^*V_2(\sigma_2)V_2(\tau_2)^*\dots V_n(\sigma_n)V_n(\tau_n)^*h = V_n(\sigma_n)V_n(\tau_n)^*\dots V_1(\sigma_1)V_1(\tau_1)^*h,$$

$$\sigma_i, \tau_i \in S_i, i \in I_n\}$$

is the maximal subspace of $\mathcal{H}$ that reduces $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, and the semigroups $\{V_i(\sigma_i)|_{\mathcal{H}_{dc}}\}_{\sigma_i\in S_i}$, $i \in I_n$, are doubly commuting.

Proof. The subspace $\mathcal{H}_{dc}$ is a closed subspace, being an intersection of bounded operator kernels. It is obvious that the semigroups $\{V_i(\sigma_i)|_{\mathcal{H}_{dc}}\}_{\sigma_i\in S_i}$, $i \in I_n$, are doubly commuting.

For any $h \in \mathcal{H}_{dc}$, $\sigma_i, \tau_i \in S_i$, $i \in I_n$ and $\mu_j \in S_j$, $j \in I_n$, we have

$$V_1(\sigma_1)V_1(\tau_1)^* \dots V_j(\sigma_j)V_j(\tau_j)^* \dots V_n(\sigma_n)V_n(\tau_n)^* V_j(\mu_j)h =$$

$$V_1(\sigma_1)V_1(\tau_1)^* \dots V_j(\sigma_j)V_j(\tau_j)^* V_j(\mu_j) \dots V_n(\sigma_n)V_n(\tau_n)^* h.$$

By hypothesis, there exists $s_j \in S_j$ such that $\mu_j = \tau_j s_j$ or $\tau_j = \mu_j s_j$. Let us consider, for example, $\mu_j = \tau_j s_j$. Then

$$\begin{aligned}
V_1(\sigma_1)V_1(\tau_1)^* \dots V_j(\sigma_j)V_j(\tau_j)^* \dots V_n(\sigma_n)V_n(\tau_n)^* V_j(\mu_j)h &= \\
&= V_1(\sigma_1)V_1(\tau_1)^* \dots V_j(\sigma_j)V_j(\tau_j)^* V_j(\mu_j) \dots V_n(\sigma_n)V_n(\tau_n)^* h \\
&= V_1(\sigma_1)V_1(\tau_1)^* \dots V_j(\sigma_j s_j) \dots V_n(\sigma_n)V_n(\tau_n)^* h \\
&= V_n(\sigma_n)V_n(\tau_n)^* \dots V_j(\sigma_j s_j) \dots V_1(\sigma_1)V_1(\tau_1)^* h \\
&= V_n(\sigma_n)V_n(\tau_n)^* \dots V_j(\sigma_j)V_j(\tau_j)^* V_j(\tau_j)V_j(s_j) \dots V_1(\sigma_1)V_1(\tau_1)^* h \\
&= V_n(\sigma_n)V_n(\tau_n)^* \dots V_1(\sigma_1)V_1(\tau_1)^* V_j(\mu_j)h.
\end{aligned}$$

Therefore $\mathcal{H}_{dc}$ is invariant for $\{V_j(\sigma_j)\}_{\sigma_j\in S_j}$, $j \in I_n$. Analogously, one proves that $\mathcal{H}_{dc}$ is invariant for $\{V_j(\sigma_j)^*\}_{\sigma_j\in S_j}$, $j \in I_n$. It is easy to see that $\mathcal{H}_{dc}$ is maximal. This completes the proof. □

4. Wold-Słociński-Suciu Decompositions

In this section, in the Słociński's spirit [25], we define a Wold-type decomposition for n-tuples of commuting isometric semigroups. The existence and the uniqueness of such decomposition is proved. Also, connection between our decomposition and the I. Suciu decomposition [26] of the product semigroup generated by these n semigroups is given.

We denote $\Lambda_{WSS} = \{u, e, s\}$. For every $n \in \mathbb{N}$, $n \geq 2$, let $F(n, \Lambda_{WSS})$ be the set of all functions from I_n to Λ_{WSS}.

Definition 4.1. *Let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be an n-tuple of commuting semigroups of isometries on a Hilbert space $\mathcal{H}$. Let $\{\mathcal{H}_f\}_{f\in F(n,\Lambda_{WSS})}$ be a set of closed subspaces of $\mathcal{H}$ such that*

$$\mathcal{H} = \bigoplus_{f\in F(n,\Lambda_{WSS})} \mathcal{H}_f.$$

Such a decomposition is called the Wold-Słociński-Suciu decomposition (WSSD) of $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, if the following conditions are satisfied:
a) The space $\mathcal{H}_f$ reduces $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, for every $f \in F(n, \Lambda_{WSS})$;
b) For $i \in I_n$ and $f \in F(n, \Lambda_{WSS})$, $\{V_i(\sigma_i)|_{\mathcal{H}_f}\}_{\sigma_i\in S_i}$ is a unitary semigroup if $f(i) = u$, a semigroup of type "e" if $f(i) = e$ and a semigroup of type "s" if $f(i) = s$.

First result regards a connection between WSSD and I. Suciu's decompositions with three summands of each semigroup.

Proposition 4.2. *Let $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, be an n-tuple of commuting isometric semigroups on a Hilbert space $\mathcal{H}$, let $\mathcal{H} = \mathcal{H}_u^i \oplus \mathcal{H}_e^i \oplus \mathcal{H}_s^i$ be the I. Suciu decomposition of $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in I_n$, and let $\mathcal{H} = \bigoplus_{f\in F(n,\Lambda_{WSS})} \mathcal{H}_f$ be a Wold-Słociński-Suciu decomposition of the given n-tuple. Then the following relations:*
a) $\mathcal{H}_u^i = \bigoplus_{\substack{f\in F(n,\Lambda_{WSS})\\ f(i)=u}} \mathcal{H}_f$, $\mathcal{H}_e^i = \bigoplus_{\substack{f\in F(n,\Lambda_{WSS})\\ f(i)=e}} \mathcal{H}_f$, $\mathcal{H}_s^i = \bigoplus_{\substack{f\in F(n,\Lambda_{WSS})\\ f(i)=s}} \mathcal{H}_f$, $i \in I_n$;
b) $\mathcal{H}_f = \bigcap_{i=1}^{n} \mathcal{H}_{f(i)}^i$, $f \in F(n, \Lambda_{WSS})$,
hold.

Proof. a) It immediately results;

b) Let $f \in F(n, \Lambda_{WSS})$. By a) we have $\mathcal{H}^i_{f(i)} = \bigoplus_{\substack{g \in F(n,\Lambda_{WSS}) \\ g(i)=f(i)}} \mathcal{H}_g \supset \mathcal{H}_f$ for every $i \in I_n$, hence $\mathcal{H}_f \subseteq \bigcap_{i=1}^{n} \mathcal{H}^i_{f(i)}$.

Conversely, let $f \in F(n, \Lambda_{WSS})$. For any $g \in F(n, \Lambda_{WSS})$, $g \neq f$, there is $j \in I_n$ such that $f(j) \neq g(j)$. Consequently $\bigcap_{i=1}^{n} \mathcal{H}^i_{f(i)} \subseteq \mathcal{H}^j_{f(j)} \perp \mathcal{H}^j_{g(j)} \supseteq \mathcal{H}_g$. Then by $\mathcal{H} = \bigoplus_{g \in F(n,\Lambda_{WSS})} \mathcal{H}_g$ we get $\bigcap_{i=1}^{n} \mathcal{H}^i_{f(i)} \perp \mathcal{H} \ominus \mathcal{H}_f$ which finishes the proof. □

Remark 4.3. *Taking into account Proposition 4.2 side b), it easily results that the subspace $\mathcal{H}_f$ is the maximal subspace of $\mathcal{H}$ reducing each semigroup $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, to a semigroup of type "$f(i)$".*

As a consequence of the above proposition we can state:

Proposition 4.4. *If a Wold-Słociński-Suciu decomposition of an n-tuple of commuting semigroups of isometries exists, then it is unique.*

A positive result about the problem of the existence of the WSSD is given in the following:

Proposition 4.5. *Let $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, be an n-tuple of commuting semigroups of isometries on a Hilbert space $\mathcal{H}$ and let $\mathcal{H} = \mathcal{H}^i_u \oplus \mathcal{H}^i_e \oplus \mathcal{H}^i_s$ be the I. Suciu decomposition of $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$. Then, there exists the Wold-Słociński-Suciu decomposition of the given n-tuple if and only if the subspaces $\mathcal{H}^i_\alpha$, $\alpha \in \{u, e, s\}$, $i \in I_n$, are reducing for the semigroups $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in I_n$, $j \neq i$.*

Proof. If the WSSD exists, then by Proposition 4.2 a), one deduces that $\mathcal{H}^i_\alpha$, $\alpha \in \{u, e, s\}$, $i \in I_n$, reduces $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in I_n$, $j \neq i$.

Conversely, let us suppose that $\mathcal{H}^i_\alpha$, $\alpha \in \{u, e, s\}$, $i \in I_n$, reduces $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in I_n$, $j \neq i$. We denote by P^i_α, $\alpha \in \{u, e, s\}$, $i \in I_n$, the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}^i_\alpha$.

Recall that a subspace $\mathcal{K} \subset \mathcal{H}$ reduces $X \in L(\mathcal{H})$ if and only if X commutes with the orthogonal projection $P_{\mathcal{K}}$ onto $\mathcal{K}$ [3].

It is clear that $P^i_\alpha V_j(\sigma_j) = V_j(\sigma_j) P^i_\alpha$ for all $\sigma_j \in S_j$, whence, by Proposition 2.1, it results $P^i_\alpha \mathcal{H}^j_u \subseteq \mathcal{H}^j_u$. It follows $P^i_\alpha P^j_u = P^j_u P^i_\alpha$, $\alpha \in \{u, e, s\}$, $i, j \in I_n$, $j \neq i$.

Now, let us prove that $P^i_\alpha \mathcal{H}^j_s \subseteq \mathcal{H}^j_s$. Using $\mathcal{H}^j_s = \bigoplus_{\sigma_j \in S_j} V_j(\sigma_j)\mathcal{R}_j$, it results that $P^i_\alpha \mathcal{H}^j_s = \bigvee_{\sigma_j \in S_j} V_j(\sigma_j) P^i_\alpha \mathcal{R}_j$. Taking $r \in \mathcal{R}_j$, one gets

$$< P^i_\alpha r, V_j(\sigma_j)^* V_j(\tau_j) h > = < r, V_j(\sigma_j)^* V_j(\tau_j) P^i_\alpha h > = 0$$

for all $h \in \mathcal{H}$ and for all $\sigma_j, \tau_j \in S_j$ with $\sigma_j^{-1} \cdot \tau_j \notin S_j^{-1}$. Therefore $P^i_\alpha \mathcal{R}_j \subseteq \mathcal{R}_j$, hence $P^i_\alpha \mathcal{H}^j_s = \bigoplus_{\sigma_j \in S_j} V_j(\sigma_j) P^i_\alpha \mathcal{R}_j \subseteq \mathcal{H}^j_s$, and consequently $P^i_\alpha P^j_s = P^j_s P^i_\alpha$, $\alpha \in \{u, e, s\}$, $i, j \in I_n$, $j \neq i$.

By $P^i_e = I_{\mathcal{H}} - P^i_u - P^i_s$, $i \in I_n$, one obtains $P^i_e P^j_e = P^j_e P^i_e$, for every $i, j \in I_n$, $j \neq i$. Hence $P^i_\alpha P^j_\beta = P^j_\beta P^i_\alpha$ for every $\alpha, \beta \in \{u, e, s\}$ and for all $i, j \in I_n$.

Thus, $P^1_{\alpha_1} P^2_{\alpha_2} \dots P^n_{\alpha_n}$, $\alpha_i \in \{u, e, s\}$, $i \in I_n$, are the orthogonal projections on $\mathcal{H}^1_{\alpha_1} \cap \mathcal{H}^2_{\alpha_2} \cap \dots \cap \mathcal{H}^n_{\alpha_n}$. Since the sum of these projections is the identity operator on $\mathcal{H}$, the conclusion follows. □

Now, it is obvious that the following result holds:

Theorem 4.6. *Every n-tuple of doubly commuting semigroups of isometries has the Wold-Słociński-Suciu decomposition.*

The existence of a multiple decomposition for an n-tuple of commuting isometric semigroups $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, may be concluded by various properties like: doubly commutativity, hyperreducivity of the Lebesgue decomposition, finite dimensional wandering spaces ([2],[3],[20],[25]). Let $k \in I_n$, $2 \leq k < n$. We denote

$\mathcal{S}_k = \{J \subseteq I_n \;:\; |J| = j,\ 2 \le j \le k\}$. For a set $J \in \mathcal{S}$, the corresponding j-tuple of isometric semigroups is $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in J$. By Proposition 4.5 it is easy to see that the existence of the WSSD of an n-tuple of commuting isometric semigropus implies the existence of WSSD of any k-tuple, $k \in I_n$, $k < n$. It raises the question if a converse property holds. A positive answer is given in the next theorem.

Theorem 4.7. *Let $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, $n \ge 3$, be an n-tuple of commuting isometric semigroups on a Hilbert space $\mathcal{H}$, let $k \in I_n$, $2 \le k < n$ fixed and let $J_1, J_2, \ldots, J_m$ be the minimum number of subsets of $\mathcal{S}_k$ such that the number of all their distinct subsets with two elements is $\binom{n}{2}$. If for every corresponding j_l-tuple of isometric semigroups $\{V_{j_l}(\sigma_{j_l})\}_{\sigma_{j_l} \in S_{j_l}}$, $l \in \{1, 2, \ldots, m\}$, the Wold-Słociński-Suciu decomposition exists, then the given n-tuple has the Wold-Słociński-Suciu decomposition.*

Proof. Let $l \in \{1, 2, \ldots, m\}$. Applying Proposition 4.5 for the j_l-tuple $\{V_{j_l}(\sigma_{j_l})\}_{\sigma_{j_l} \in S_{j_l}}$, it results the subspace $\mathcal{H}^i_\alpha$, $\alpha \in \{u, e, s\}$, $i \in J_l$, reduces $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in J_l$, $j \ne i$, whence every pair $(\{V_i(\sigma_i)\}_{\sigma_i \in S_i}, \{V_j(\sigma_j)\}_{\sigma_j \in S_j})$, $i, j \in J_l$, $i \ne j$, has WSSD. Since l is arbitrary and the number of all pairs (i, j), $i, j \in I_n$, $i < j$ is $\binom{n}{2}$, it follows that every pair $(\{V_i(\sigma_i)\}_{\sigma_i \in S_i}, \{V_j(\sigma_j)\}_{\sigma_j \in S_j})$, $i, j \in I_n$, $i \ne j$, has WSSD. By Proposition 4.5 one gets every summand $\mathcal{H}^i_\alpha$, $\alpha \in \{u, e, s\}$, of the I. Suciu decomposition of the semigroup $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, is reducing for $\{V_j(\sigma_j)\}_{\sigma_j \in S_j}$, $j \in I_n$, $j \ne i$, which is what we set out to prove. □

Remark 4.8. *We notice that $m = \binom{n}{2}$ for $k = 2$ and $m = 3$ for $k = n - 1$. Also, if there exists a pair of semigroups has not WSSD, then the n-tuple has not WSSD.*

The next result establishes relations between some subspaces of the WSSD and the subspaces of I. Suciu's decomposition.

Theorem 4.9. *Suppose that $\mathcal{H} = \bigoplus_{f \in F(n, \Lambda_{WSS})} \mathcal{H}_f$ is the Wold-Słociński-Suciu decomposition of the n-tuple of commuting semigroups of isometries $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, on a Hilbert space $\mathcal{H}$. Let $f_u, f_s \in F(n, \Lambda_{WSS})$, $f_u(j) = u$, $j \in I_n$, $f_s(j) = s$, $j \in I_n$ and let $\mathcal{F} = \{f \in F(n, \Lambda_{WSS}) | f \ne f_u$ and there exists $k \in I_n$ such that $f(k) = u\}$. If $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_e \oplus \mathcal{H}_s$ is the I. Suciu decomposition of the product semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma} \in \overline{S}}$ of $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, then*

$$\mathcal{H}_u = \mathcal{H}_{f_u}\ ,\ \ \mathcal{H}_s \subseteq \mathcal{H}_{f_s}\ \text{ and }\ \mathcal{H}_e \supset \bigoplus_{f \in \mathcal{F}} \mathcal{H}_f\ .$$

Moreover, if $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, are doubly commuting on $\mathcal{H}$, then

$$\mathcal{H}_s = \mathcal{H}_{f_s}\ \text{ and }\ \mathcal{H}_e = \bigoplus_{f \in F(n, \Lambda_{WSS}) \setminus \{f_u, f_s\}} \mathcal{H}_f.$$

Proof. First we prove that $\mathcal{H}_{f_u} = \mathcal{H}_u$.

Since $\mathcal{H}_u$ reduces the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma} \in \overline{S}}$ to a unitary semigroup, it results in particular that $\mathcal{H}_u$ reduces the semigroups $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, to unitary semigroups. By Remark 4.3 one gets $\mathcal{H}_u \subseteq \mathcal{H}_{f_u}$.

Taking into account that $\mathcal{H}_{f_u}$ reduces the semigroups $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, to unitary semigroups, it follows that $\mathcal{H}_{f_u}$ reduces the semigroup $\{V(\overline{\sigma})\}_{\overline{\sigma} \in \overline{S}}$ to a unitary semigroup, whence $\mathcal{H}_{f_u} \subseteq \mathcal{H}_u$. Thus $\mathcal{H}_{f_u} = \mathcal{H}_u$.

Using Theorem 3.3, it follows that $\{V_i(\sigma_i)|_{\mathcal{H}_s}\}_{\sigma_i \in S_i}$, $i \in I_n$, are doubly commuting semigroups of type "s", whence $\mathcal{H}_s \subseteq \mathcal{H}_{f_s}$.

It is easy to see that $\mathcal{H}_f$, $f \in \mathcal{F}$ reduces $\{V(\overline{\sigma})\}_{\overline{\sigma} \in \overline{S}}$ and the semigroup $\{V(\overline{\sigma})|_{\mathcal{H}_f}\}_{\overline{\sigma} \in \overline{S}}$ is of type "e", whence $\bigoplus_{f \in \mathcal{F}} \mathcal{H}_f \subseteq \mathcal{H}_e$.

Now, if $\{V_i(\sigma_i)\}_{\sigma_i \in S_i}$, $i \in I_n$, doubly commute, by Theorem 3.3 it results that $\mathcal{H}_{f_s} \subseteq \mathcal{H}_s$, whence $\mathcal{H}_s = \mathcal{H}_{f_s}$. Thus $\mathcal{H}_e = \bigoplus_{f \in F(n, \Lambda_{WSS}) \setminus \{f_u, f_s\}} \mathcal{H}_f$ and the theorem is completely proved. □

In the sequel, two results about a pair of doubly commuting isometric semigroups are presented.

Remark 4.10. *In the particular case $n = 2$, we have $F(2, \Lambda_{WSS}) = \{f_1, f_2, \dots, f_9\}$, where $f_1(1) = f_1(2) = u$, $f_2(1) = u$, $f_2(2) = e$, $f_3(1) = u$, $f_3(2) = s$, $f_4(1) = e$, $f_4(2) = u$, $f_5(1) = f_5(2) = e$, $f_6(1) = e$, $f_6(2) = s$, $f_7(1) = s$, $f_7(2) = u$, $f_8(1) = s$, $f_8(2) = e$, $f_9(1) = f_9(2) = s$. The Wold-Słociński-Suciu decomposition of the pair of commuting semigroups $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ has the form $\mathcal{H} = \bigoplus_{j=1}^{9} \mathcal{H}_{f_j}$. For a function $f \in F(2, \Lambda_{WSS})$, $\mathcal{H}_f$ can be denoted by $\mathcal{H}_{f(1)f(2)}$, where $f(1)f(2)$ represents the concatenation of $f(1)$ and $f(2)$. Thus, the subspaces $\mathcal{H}_{f_j}$, $j \in \{1, 2, \dots, 9\}$ become $\mathcal{H}_{uu}$, $\mathcal{H}_{ue}$, $\mathcal{H}_{us}$, $\mathcal{H}_{eu}$, $\mathcal{H}_{ee}$, $\mathcal{H}_{es}$, $\mathcal{H}_{su}$, $\mathcal{H}_{se}$, $\mathcal{H}_{ss}$, respectively, which are in accordance with the Słociński notations [25] for the summands in the Wold decomposition of a pair of commuting isometries. Thus, the WSSD has the following form*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{ue} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{eu} \oplus \mathcal{H}_{ee} \oplus \mathcal{H}_{es} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{se} \oplus \mathcal{H}_{ss} \tag{7}$$

Proposition 4.11. *Let $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ be two commuting isometric semigroups on a Hilbert space $\mathcal{H}$ and let $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_e \oplus \mathcal{H}_s$ be the I. Suciu decomposition of the product semigroup $\{V_1(\sigma_1)V_2(\sigma_2)\}_{(\sigma_1,\sigma_2)\in S_1\times S_2}$. The following conditions are equivalent:*
(i) the pair $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in \{1, 2\}$ has the Wold-Słociński-Suciu decomposition (7), the semigroups $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ doubly commute on the subspaces $\mathcal{H}_{es}, \mathcal{H}_{se}, \mathcal{H}_{ee}$ and $\mathcal{H}_{ss} = \mathcal{H}_s$;
(ii) the semigroups $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ doubly commute on $\mathcal{H}$.

Proof. Taking into account Theorems 4.6 and 4.9, it only remains to prove the implication $(i) \Rightarrow (ii)$.

The semigroup $\{V_1(\sigma_1)V_2(\sigma_2)|_{\mathcal{H}_s}\}_{(\sigma_1,\sigma_2)\in S_1\times S_2}$ is of type "s", hence, by Theorem 3.3, the semigroups $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ are doubly commuting on $\mathcal{H}_s = \mathcal{H}_{ss}$.

Now, taking into account the Fuglede-Putnam-Rosemblum theorem [21], it results that the semigroups $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ are doubly commuting on $\mathcal{H}_{uu}, \mathcal{H}_{ue}, \mathcal{H}_{us}, \mathcal{H}_{eu}, \mathcal{H}_{su}$. Therefore $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ doubly commute on $\mathcal{H}$. □

In the last proposition of the paper, the structure of the summands in the WSSD in the case of two doubly commuting semigroups of isometries is given.

Proposition 4.12. *Let $\{V_1(\sigma_1)\}_{\sigma_1\in S_1}$ and $\{V_2(\sigma_2)\}_{\sigma_2\in S_2}$ be two doubly commuting isometric semigroups on a Hilbert space $\mathcal{H}$ and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be their corresponding right defect spaces. Then the subspaces in the Wold-Słociński-Suciu decomposition (7) have the following geometric structure:*

$$\mathcal{H}_{uu} = \bigcap_{(\sigma_1,\sigma_2)\in S_1\times S_2} V_1(\sigma_1)V_2(\sigma_2)\mathcal{H};\ \ \mathcal{H}_{us} = \bigcap_{\sigma_1\in S_1} V_1(\sigma_1)\left(\bigoplus_{\sigma_2\in S_2} V_2(\sigma_2)\mathcal{R}_2\right);$$
$$\mathcal{H}_{ue} = \left[\bigcap_{\sigma_1\in S_1} V_1(\sigma_1)\mathcal{H}\right] \ominus (\mathcal{H}_{uu} \oplus \mathcal{H}_{us});\ \ \mathcal{H}_{su} = \bigcap_{\sigma_2\in S_2} V_2(\sigma_2)\left(\bigoplus_{\sigma_1\in S_1} V_1(\sigma_1)\mathcal{R}_1\right);$$
$$\mathcal{H}_{ss} = \bigoplus_{(\sigma_1,\sigma_2)\in S_1\times S_2} V_1(\sigma_1)V_2(\sigma_2)\left(\mathcal{R}_1 \cap \mathcal{R}_2\right);\ \ \mathcal{H}_{se} = \left[\bigoplus_{\sigma_1\in S_1} V_1(\sigma_1)\mathcal{R}_1\right] \ominus (\mathcal{H}_{su} \oplus \mathcal{H}_{ss});$$
$$\mathcal{H}_{eu} = \left[\bigcap_{\sigma_2\in S_2} V_2(\sigma_2)\mathcal{H}\right] \ominus (\mathcal{H}_{uu} \oplus \mathcal{H}_{su});\ \ \mathcal{H}_{es} = \left[\bigoplus_{\sigma_2\in S_2} V_2(\sigma_2)\mathcal{R}_2\right] \ominus (\mathcal{H}_{us} \oplus \mathcal{H}_{ss});$$
$$\mathcal{H}_{ee} = \mathcal{H} \ominus (\mathcal{H}_{uu} \oplus \mathcal{H}_{ue} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{eu} \oplus \mathcal{H}_{es} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{se} \oplus \mathcal{H}_{ss})$$

Proof. By Theorem 4.5, the structures of $\mathcal{H}_{uu}$ and $\mathcal{H}_{ss}$ are obtained.

Let $\mathcal{H} = \mathcal{H}_u^i \oplus \mathcal{H}_e^i \oplus \mathcal{H}_s^i$ be the I. Suciu decomposition of $\{V_i(\sigma_i)\}_{\sigma_i\in S_i}$, $i \in \{1, 2\}$. The subspace $\mathcal{H}_{us}$ is the unitary part in the I. Suciu decomposition of $\{V_1(\sigma_1)|_{\mathcal{H}_{us}\oplus\mathcal{H}_{es}\oplus\mathcal{H}_{ss}}\}_{\sigma_1\in S_1}$. Since $\mathcal{H}_{us} \oplus \mathcal{H}_{es} \oplus \mathcal{H}_{ss} = \mathcal{H}_s^2 = \bigoplus_{\sigma_2\in S_2} V_2(\sigma_2)\mathcal{R}_2$, it follows $\mathcal{H}_{us} = \bigcap_{\sigma_1\in S_1} V_1(\sigma_1)\left(\bigoplus_{\sigma_2\in S_2} V_2(\sigma_2)\mathcal{R}_2\right)$.

The I. Suciu decomposition of $\{(V_2(\sigma_2)|_{\mathcal{H}_u^1}\}_{\sigma_2\in S_2}$ is $\mathcal{H}_u^1 = \mathcal{H}_{uu} \oplus \mathcal{H}_{ue} \oplus \mathcal{H}_{us}$. Therefore $\mathcal{H}_{ue} = \mathcal{H}_u^1 \ominus (\mathcal{H}_{uu} \oplus \mathcal{H}_{us}) = \left[\bigcap_{\sigma_1\in S_1} V_1(\sigma_1)\mathcal{H}\right] \ominus (\mathcal{H}_{uu} \oplus \mathcal{H}_{us})$.

The other geometric structures are similarly obtained. □

Acknowledgements

We would like to thank the unknown referees for their valuable comments and suggestions which helped to improve the paper.

References

[1] H. Bercovici, R. G. Douglas, C. Foiaş, Canonical models for bi-isometries, Operator Theory: Advances and Applications 218 (2012) 177–205.
[2] Z. Burdak, M. Kosiek, M. Słociński, The canonical Wold decomposition of commuting isometries with finite dimensional wandering spaces, Bull. Sci. Math. 137 (2013) 653–658.
[3] Z. Burdak, M. Kosiek, P. Pagacz, M. Słociński, Shift-type properties of commuting, completely non doubly commuting pairs of isometries, Integr. Equ. Oper. Theory 79, Issue 1 (2014) 107–122.
[4] Z. Burdak, M. Kosiek, M. Słociński, Compatible pairs of commuting isometries, Linear Algebra Appl. 479 (2015) 216–259.
[5] X. Catepillan, M. Ptak, W. Szymanski, Multiple canonical decompositions of families of operators and a model of quasinormal families, Proc. Amer. Math. Soc. 121, no. 4 (1994) 1165–1172.
[6] J. M. Francos, A. Narasimhan, W. Woods, Maximum-likehood parameter estimation of the harmonic, evanescent and purely indeterministic components of discrete homogeneous random fields, IEEE Trans. Inf. Th. 42(3) (1996) 916–930.
[7] D. Gaşpar, N. Suciu, On the structure of isometric semigroups, Operator Theory: Adv. and Appl. 14 (1984) 125–139.
[8] D. Gaşpar, N. Suciu, On Wold decompositions of isometric semigroups, in: Approximation theory and functional analysis, ISNM 65, Birkhauser Verlag, Basel, 1984, pp. 99–108.
[9] D. Gaşpar, N. Suciu, Intertwining properties of isometric semigroups and Wold type decompositions, Operator Theory: Adv. and Appl. 24 (1987) 183–193.
[10] D. Gaşpar, N. Suciu, Wold decompositions for commutative families of isometries, An. Univ. Timisoara Ser. Stiint. Mat. 27 (1989) 31–38.
[11] D. Gaşpar, P. Gaşpar, Wold decompositions and the unitary model for bi-isometries, Integral Equ. Oper. Theory 49, no. 4 (2004) 419–433.
[12] P. Găvruţă, T. Bînzar, Wold decompositions of the isometric semigroups, Proc. Nat Conf. Math. Anal. App., Timişoara (2000) 101–110.
[13] P. R. Halmos, Shifts on Hilbert spaces, J. Reine Angew. Math. 208 (1961) 102–112
[14] H. Helson, D. Lowdenslager, Prediction theory and Fourier series in several variables. II, Acta Math. 106 (1961) 175–213.
[15] G. Kallianpur, V. Mandrekar, Nondeterministic random fields and Wold and Halmos decompositions for commuting isometries, in: Prediction theory and harmonic analysis, North-Holland, Amsterdam, 1983, pp. 165-190.
[16] F. Liu, R. W. Picard, A Spectral 2-D Wold Decomposition Algorithm for Homogeneous Random Fields, ICASSP 99, vol. 4 (1999) 3501–3504.
[17] P. S. Muhly, A structure theory for isometric representations of a class of semigroups, J. Reine Angew. Math. 225 (1972) 135–154.
[18] F. Pater, L. D. Lemle, T. Bînzar, On a Wold-Słociński's type decomposition of a pair of commuting semigroups of isometries, ICNAAM (2010) 1379–1381.
[19] G. Popescu, Noncommutative Wold decompositions for semigroups of isometries, Indiana Univ. Math. J. 47 (1998) 277–296.
[20] D. Popovici, A Wold-type decomposition for commuting isometric pairs, Proc. Amer. Math. Soc. 132, no. 8 (2004) 2303–2314.
[21] W. Rudin, Functional analysis, Mc Graw-Hill, New York, 1973.
[22] J. Sarkar, Wold decomposition for doubly commuting isometries, Lin. Alg. its Application 445 (2014) 289–301.
[23] A. Skalski, J. Zacharias, Wold decomposition for representations of product systems of C^*-correspondences, Internat. J. Math. 19, no. 4 (2008) 455–479.
[24] Y. Stitou, F. Turcu, M. Najim, L. Radouane, 3-D texture model based on the Wold decomposition, Proc. Eur. Signal Process, Wien (2004) 429–432.
[25] M. Słociński, On the Wold-type decomposition of a pair of commuting isometries, Ann. Polon. Math. 37 (1980) 255–262.
[26] I. Suciu, On the semigroups of isometries, Studia Math. 30 (1968) 101–110.
[27] B. Sz.-Nagy, C. Foiaş, Harmonic Analysis of operators on Hilbert space, North-Holland Punblishing Company, Amsterdam, London, 1970.
[28] D. Timotin, Regular dilations and models for multicontractions, Indiana Univ. Math. J. 47, no. 2 (1998) 671–684.
[29] W. Szymański, Decomposition of operator-valued functions in Hilbert spaces, Studia Math., 50, no. 1 (1974) 265-280.
[30] H. Wold, A study in the analysis of stationary time series, Almqvist and Wiksell, Stockholm, 1938 (2nd ed., 1954).