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# Existence and Uniqueness Results for a Class of Fractional Differential Equations with an Integral Fractional Boundary Condition

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**Abstract.** The aim of this work is to study a class of boundary value problem including a fractional order differential equation. Sufficient and necessary conditions will be presented for the existence and uniqueness of solution of this fractional boundary value problem.

## 1. Introduction

In this paper, the existence and uniqueness of solution for a class of nonlinear fractional boundary value problem will be discussed. this problem includes a nonlinear fractional differential equation of order $\alpha \in (0,1]$ and a fractional integral boundary conditions. In fact the following boundary value problem of fractional differential equation is considered.

$$\begin{array}{ll} {}^cD^\alpha y(t) = f(t, y(t)), & 0 < \alpha < 1, \quad t \in J = [0,1], \\ y(0) = \eta I^\beta y(\tau), & 0 < \tau < 1. \end{array} \tag{1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha$, $f : J \times \mathbb{R} \to \mathbb{R}$ that will be specified later, $\eta \in \mathbb{R}$ is such that $\eta \neq \Gamma(\beta+1)/\tau^\beta$, $\Gamma$ is the Euler Gamma function and $I^\beta$, $0 < \beta < 1$, is the Riemman-Liouville fractional integral of order $\beta$.

Recently, factional differential equations have been proved to be valuable tools in modelling of many phenomena in various fields of engineering, physics, chemistry and economics. We can find numerous applications in viscoelasticity, electrochemistry, control and electromagnetic. For some recent development on the topic see [1–4]. There has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al.[5], Miller Ross [9], Lakshmikantham et al.[6], Podlubny[8].

The existence of solution for the BVP (1) has been studied by Ntouyas [7]. Let us mention, however, the assumption on $f$ are strong (f is continuous and satisfies uniformly Lipschitz condition or uniformly bounded). In this paper the new existence and uniqueness results will be presented for the boundary value problem (1) by virtue of fractional calculus and fixed point method under some weak conditions.

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Compared with the results appeared in [7], there are some differences. The most important of them is that the assumptions on $f$ are more general and easy to check, but here the $f$ is not continuous necessarily.

The rest of this paper is organized as follows. In section 2, some notations will be given also some concepts and preparing results will be recalled. In section 3, a generalized singular type Granwall inequality which can be used to establish the estimate of fixed point set $\{y = \lambda Fy, \lambda \in (0, 1)\}$ will be given. In section 4, two main results will be presented, the first result based on Banach contraction principle, the second result based on Shaefer's fixed point theorem and in section 5 one example will be given that satisfies in hypothesis of our results.

## 2. Preliminaries and Linear Problem

In this section, notations, definitions and preliminary facts which are used throughout this paper are introduced. At first, we use the notation of $C$ as a Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm $\|y\|_\infty := \sup\{|y(t)| : t \in J\}$. For measurable function $m : J \to \mathbb{R}$, define the norm

$$\|m\|_{L^p(J,\mathbb{R})} := \begin{cases} \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \{\sup\limits_{t \in J - \bar{J}} |m(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure of $\bar{J}$. Let $L^p(J, \mathbb{R})$ be the Banach space of all Lebesgue measurabel functions $m : J \to \mathbb{R}$ with $\|m\|_{L^p(J,\mathbb{R})} < \infty$.
We need some basic definitions and properties of fractional calculous theory which are used in this paper. For more details, see [5, 8].

**Definition 2.1.** *The Riemann-Liouville fractional integral of order* $\alpha>0$ *of a Lebesgue-measurable function* $f : \mathbb{R}^+ \to \mathbb{R}$ *is defined by (the Abel-integral operator)*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds \tag{2}$$

*provided that the integral exist.*

**Definition 2.2.** *The fractional derivative (in the sense of Caputo) of order* $0 < \alpha < 1$ *of a function* $f : \mathbb{R}^+ \to \mathbb{R}$ *is defined as the left inverse of the fractional integral of* $f$

$$^cD^\alpha f(t) = I^{1-\alpha} \frac{d}{dt} f(t) \tag{3}$$

*That is*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f'(s) ds, \tag{4}$$

*provided that the right side exists.*

For proving the existence and uniqueness solution of the problem (1), we need some fixed point theorems.

**Theorem 2.3.** *(Schaefer's fixed point theorem) Let* $J = [t_0, t_0 + T]$ *and* $F : C(J, \mathbb{R}) :\to C(J, \mathbb{R})$ *be a completely continuous operator. If the set*

$$E(F) = \{x \in C(J, \mathbb{R}) : x = \lambda Fx \text{ for some} \lambda \in [0, 1]\}$$

*is bounded, then F has at least a fixed point.*

**Lemma 2.4.** *[7] Let $\beta \neq \frac{\Gamma(\beta+1)}{\tau^{\beta}}$. Then for $g \in C$, the solution of the fractional differential equation*

$$^{c}D^{\alpha}y(t) = g(t), \quad 0 < \alpha \leq 1 \tag{5}$$

*subject to the boundary condition*

$$y(0) = \eta I^{\beta} y(\eta), \tag{6}$$

*is given by*

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1) - \eta\tau^{\alpha}} \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s)ds, \quad t \in [0,1]. \tag{7}$$

As a consequence of Lemma 2.4, we have the following result which is useful in what follows.

**Lemma 2.5.** *A function $y \in C$ is a solution of the fractional integral equation*

$$y(t) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1) - \eta\tau^{\alpha}} \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, y(s))ds, \quad t \in [0,1], \tag{8}$$

*if and only if $y$ is a solution of the fractional BVP* (1).

## 3. A Special Singular Type Gronwall Inequality

In order to apply the schaefer fixed point theorem to show the existence of solutions, we need a new generalized singular type Granwall inequality with mixed type singular integral operator. It may play an essential role in the study of fractional boundary value problem.

**Lemma 3.1.** *Let $y \in C$ satisfy the following inequality*

$$|y(t)| \leq a + b\int_0^t (t-s)^{\alpha-1}|y(s)|^{\lambda}ds + c\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|y(s)|^{\lambda}ds, \tag{9}$$

*where $\alpha, \beta \in (0,1)$, $\lambda \in [0, 1-\frac{1}{p}]$ for some $1 < p < \frac{1}{1-\alpha-\beta}$, $a, b, c \geq 0$ are constants. Then there exists a constant $M^{*} > 0$ such that*

$$|y(t)| \leq M^{*}.$$

*Proof.* Let $M = (b+c)\left[\frac{1}{p(\alpha+\beta-1)+1}\right]^{\frac{1}{p}}$ and

$$x(t) = \begin{cases} 1, & |y(t)| \leq 1 \\ y(t), & |y(t)| \geq 1. \end{cases}$$

$$\begin{aligned}
|y(t)| &\leq |x(t)| \leq a + 1 + b\int_0^t (t-s)^{\alpha-1}|x(s)|^{\lambda}ds + c\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|x(s)|^{\lambda}ds \\
&\leq (a+1) + b\int_0^t (t-s)^{\alpha+\beta-1}|x(s)|^{\lambda}ds + c\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|x(s)|^{\lambda}ds \\
&\leq (a+1) + b\left(\int_0^t (t-s)^{p(\alpha+\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t |x(s)|^{\frac{\lambda p}{p-1}}ds\right)^{\frac{p-1}{p}} + c\left(\int_0^{\tau}(\tau-s)^{p(\alpha+\beta-1)}ds\right)^{\frac{1}{p}}\left(\int_0^{\tau}|x(s)|^{\frac{\lambda p}{p-1}}ds\right)^{\frac{p-1}{p}} \\
&\leq (a+1) + b\left[\frac{1}{p(\alpha+\beta-1)+1}\right]^{\frac{1}{p}}\int_0^t |x(s)|^{\frac{\lambda p}{p-1}}ds + c\left[\frac{1}{p(\alpha+\beta-1)+1}\right]^{\frac{1}{p}}\int_0^{\tau}|x(s)|^{\frac{\lambda p}{p-1}}ds \\
&\leq (s+1) + (b+c)\left[\frac{1}{p(\alpha+\beta-1)+1}\right]^{\frac{1}{p}}\int_0^{\tau}|x(s)|^{\frac{\lambda p}{p-1}}ds \\
&= (a+1) + M\int_0^{\tau}|x(s)|^{\frac{\lambda p}{p-1}}ds \leq (a+1) + M\int_0^{1}|x(s)|^{\frac{\lambda p}{p-1}}ds \leq
\end{aligned}$$

$$\leq \quad (a+1) + M\int_0^1 |x(s)|ds.$$

Therefor by the standard Granwall inequality, we have

$$|y(t)| \leq |x(t)| \leq (a+1)e^M := M^*.$$

□

## 4. Main Results

In this section we shall present and prove our main result. First, consider the following hypotheses:

**(H1)** The function $f : J \times \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable with respect to $t$ on $J$.

**(H2)** There exists a constant $p \in [0, \alpha)$ and real valued function $h(t) \in L^{\frac{1}{p}}(J, \mathbb{R}_+)$ such that $|f(t, y)| \leq h(t)$ for each $t \in J$ and all $y \in \mathbb{R}$.

**(H3)** There exists a constant $q \in [0, \alpha)$ and real valued function $m(t) \in L^{\frac{1}{q}}(J, \mathbb{R}_+)$ such that

$$|f(t, u_1) - f(t, u_2)| \leq m(t)|u_1 - u_2|$$

for each $t \in J$ and all $u_1, u_2 \in \mathbb{R}$.

**(H4)** There exists constant $\lambda \in [0, 1 - \frac{1}{p}]$ for some $1 < p < \frac{1}{1-\alpha}$ and $N > 0$ such that

$$|f(t, u)| \leq N(1 + |u|^\lambda) \quad \text{for each } t \in J \text{ and all } u \in \mathbb{R}.$$

For convenience, let $M = \|m\|_{L^{\frac{1}{p}}(J,\mathbb{R})}$, $H = \|h\|_{L^{\frac{1}{q}}(J,\mathbb{R})}$.
Our first result is based on the Banach fixed point theorem.

**Theorem 4.1.** *Assume that (H1)-(H3) hold. If*

$$\Omega_{\alpha,q}(t) = \frac{M}{\Gamma(\alpha)(\frac{\alpha-q}{1-q})^{1-q}} + \frac{M|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1) - \eta\tau^\beta|\Gamma(\alpha+\beta)(\frac{\alpha+\beta-q}{1-q})^{1-q}} \leq \rho < 1 \tag{10}$$

*then the system* (1) *has a unique solution.*

*Proof.* For each $t \in J$, we have

$$\begin{aligned}
\int_0^t |(t-s)^{\alpha-1} f(s, y(s))|ds &\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}} ds\right)^{1-p} \left(\int_0^t (h(s))^{\frac{1}{p}} ds\right)^p \\
&\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}} ds\right)^{1-p} \left(\int_0^1 (h(s))^{\frac{1}{p}} ds\right)^p \\
&\leq \frac{H}{(\frac{\alpha-p}{1-p})^{1-p}}
\end{aligned}$$

Thus $|(t-s)^{\alpha-1} f(s, y(s))|$ is a Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $y \in C$. Then $(t-s)^{\alpha-1} f(s, y(s))$ is a Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$.
Hence the fractional BVP (1) is equivalent to the following fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1) - \eta\tau^\beta}\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, y(s))ds$$

Let

$$r \geq \frac{H}{\Gamma(\alpha)(\frac{\alpha-p}{1-p})^{1-p}} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{H}{\Gamma(\alpha+\beta)(\frac{\alpha+\beta-p}{1-p})^{1-p}}$$

Now we define the operator $F$ on $B_r := \{y \in C : \|y\|_\infty \leq r\}$ as follows

$$(Fy)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\beta}}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,y(s))ds. \tag{11}$$

Therefore the existence of a solution of the fractional BVP (1) is equivalent to that the operator $F$ has a fixed point in $B_r$. The Banach contraction principle will be used to prove that $F$ has a fixed point. The proof will be presented in two steps.

*Step 1.* $Fy \in B_r$ for every $y \in B_r$
In fact, for every $y \in B_r$ and all $t \in J$, It is verified that $F$ is continuous on $J$, i.e., $Fy \in C$, and

$$\begin{aligned}
|(Fy)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds + \frac{|\eta\Gamma(\beta+1)|}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,y(s))|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s) + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}}ds\right)^{1-p}\left(\int_0^t h(s)^{\frac{1}{p}}ds\right)^{p} \\
&\quad + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{1}{\Gamma(\alpha+\beta)}\left(\int_0^{\tau}(\tau-s)^{\frac{\alpha+\beta-1}{1-p}}ds\right)^{1-p}\left(\int_0^{\tau}h(s)^{\frac{1}{p}}ds\right)^{p} \\
&\leq \frac{H}{\Gamma(\alpha)(\frac{\alpha-p}{1-p})^{1-p}} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{H}{\Gamma(\alpha+\beta)(\frac{\alpha+\beta-p}{1-p})^{1-p}} \leq r,
\end{aligned}$$

which implies that $\|Fy\|_\infty \leq r$. Thus, It is concluded that for all $y \in B_r, Fy \in B_r$. i.e., $F : B_r \to B_r$ is well defined.

*Step 2.* $F$ is contraction mapping on $B_r$.
For $x, y \in B_r$ and any $t \in J$ using (H3) and Hölder inequality, we can get

$$\begin{aligned}
&|(Fx)(t)-(Fy)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds + \frac{|\eta\Gamma(\beta+1)|}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))-f(s,y(s))|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)|x(s)-y(s)|ds + \frac{|\eta\Gamma(\beta+1)|}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)|x(s)-y(s)|ds \\
&\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)ds + \frac{\|x-y\|_\infty|\eta\Gamma(\beta+1)|}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)ds \\
&\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{(\frac{\alpha-1}{1-q})}ds\right)^{1-q}\left(\int_0^t m(s)^{\frac{1}{q}}ds\right)^{q} \\
&\quad + \frac{|\eta\Gamma(\beta+1)|}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{\|x-y\|_\infty}{\Gamma(\alpha+\beta)}\left(\int_0^{\tau}(\tau-s)^{\frac{\alpha+\beta-1}{1-q}}ds\right)^{1-q}\left(\int_0^{\tau}m(s)^{\frac{1}{q}}ds\right)^{q} \\
&\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)(\frac{\alpha-q}{1-q})^{1-q}}\|m\|_{L^{\frac{1}{q}}(J,\mathbb{R}_+)} + \frac{|\eta\Gamma(\beta+1)|}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{\|x-y\|_\infty}{\Gamma(\alpha+\beta)(\frac{\alpha+\beta-q}{1-q})^{1-q}}\|m\|_{L^{\frac{1}{q}}(J,\mathbb{R}_+)}
\end{aligned}$$

$$\leq \left[\frac{M}{\Gamma(\alpha)(\frac{\alpha-q}{1-q})^{1-q}} + \frac{M|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|\Gamma(\alpha+\beta)(\frac{\alpha+\beta-q}{1-q})^{1-q}}\right]\|x-y\|_{\infty}.$$

So we have

$$\|Fx-Fy\|_{\infty} \leq \Omega_{\alpha,q}(t)\|x-y\|_{\infty}.$$

By banach contraction principle, It is concluded that $F$ has an unique fixed point which is just the unique solution of fractional BVP (1). □

Our second result is based on the well-known Schaefer's fixed point theorem.

**Theorem 4.2.** *Assume that (H3) - (H4) are hold. Then boundary value problem* (1) *has at least one solution on* $[0,1]$.

*Proof.* Transform the fractional boundary value problem (1) into a fixed point problem. Consider the operator $F : C \to C$ defined as (11). It is obvious that $F$ is well defined due to (H4).

*Step 1* $F$ is a continuous operator.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C$. Then for each $t \in J$, we have

$$\begin{aligned}
&|(Fy_n)(t)-(Fy)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,y_n(s))-f(s,y(s))|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)|y_n(s)-y(s)|ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)|y_n(s)-y(s)|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)\sup_{s\in J}|y_n(s)-y(s)|ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}m(s)\sup_{s\in J}|y_n(s)-y(s)|ds \\
\leq\ & \frac{\|y_n-y\|_{\infty}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{\|y_n-y\|_{\infty}}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}m(s)ds \\
\leq\ & \frac{\|y_n-y\|_{\infty}}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds\right)^{1-q}\left(\int_0^t m(s)^{\frac{1}{q}}ds\right)^{q} \\
&+\frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{\|y_n-y\|_{\infty}}{\Gamma(\alpha+\beta)}\left(\int_0^{\tau}(\tau-s)^{\frac{\alpha+\beta-1}{1-q}}ds\right)^{q}\left(\int_0^t m(s)^{\frac{1}{q}}ds\right)^{q} \\
\leq\ & \|y_n-y\|_{\infty}\left(\frac{M}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha+\beta)}\frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\right)
\end{aligned}$$

Since $y_n \to y$, we have

$$\|Fy_n-Fy\|_{\infty} \leq \|y_n-y\|_{\infty}\left(\frac{M}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha+\beta)}\frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\right) \to 0$$

as $n \to \infty$.

*Step 2.* $F$ maps bounded set into bounded set in $C$.
Indeed, it is enough to show that for any $\eta* > 0$, there exists a $l > 0$ such that for each $y \in B_{\eta^*} = \{y \in C : \|y\|_{\infty} \leq \eta^*\}$, we have $\|Fy\|_{\infty} \leq l$.

For each $t \in J$, we get

$$
\begin{aligned}
|(Fy)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,y(s))|ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+|y(s)|^{\lambda})ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}(1+|y(s)|^{\lambda})ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}ds \\
&\quad + \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)|^{\lambda}ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|y(s)|^{\lambda}ds \\
&\leq \frac{N}{\Gamma(\alpha+1)} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N}{\Gamma(\alpha+\beta+1)} \\
&\quad + \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)|^{\lambda}ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|y(s)|^{\lambda}ds \\
&\leq \frac{N}{\Gamma(\alpha+1)} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N}{\Gamma(\alpha+\beta+1)} + \frac{N(\eta*)^{\lambda}}{\Gamma(\alpha+1)} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N(\eta*)^{\lambda}}{\Gamma(\alpha+\beta+1)} \\
&= \frac{N(1+(\eta*)^{\lambda})}{\Gamma(\alpha+1)} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N(1+(\eta*)^{\lambda})}{\Gamma(\alpha+\beta+1)}
\end{aligned}
$$

which implies that

$$
\|Fy\|_{\infty} \leq \frac{N(1+(\eta*)^{\lambda})}{\Gamma(\alpha+1)} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\beta}|}\frac{N(1+(\eta*)^{\lambda})}{\Gamma(\alpha+\beta+1)} := l
$$

*Step 3.* $F$ maps bounded sets into equicontinuous sets of $C$. Let $0 \leq t_1 < t_2 \leq 1$, $y \in B_{\eta^*}$. Using (H4), we have

$$
\begin{aligned}
& |(Fy)(t_2) - (Fy)(t_1)| = \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds\right| \\
\leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,y(s))ds\right| + \left|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds\right| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]|f(s,y(s))|ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,y(s))|ds \\
\leq & \frac{N}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}](1+|y(s)|^{\lambda})ds + \frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1+|y(s)|^{\lambda})ds \\
\leq & \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds + \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
\leq & \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha)}[|t_1^{\alpha}-t_2^{\alpha}| + 2(t_2-t_1)^{\alpha}] \leq \frac{3N(1+(\eta^*)^{\lambda})(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero, therefore $F$ is equicontinuous. As a consequence of Step1-3 together with Arzela-Ascoli theorem, we can conclude that $F : C \to C$ is continuous and completly continuous.

*Step 4.*A priori bounds.
Now it remains to show that set

$$E(F) = \{y \in C : y = \lambda Fy, \lambda \in (0,1)\}$$

is bounded.
Let $y \in E(F)$, then $y = \lambda Fy$ for some $\lambda \in (0,1)$, Thus for each $t \in J$, we have

$$y(t) = \lambda\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds + \frac{\eta\Gamma(\beta+1)}{\Gamma(\beta+1)-\eta\tau^{\alpha}}\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,y(s))ds\right)$$

For each $t \in J$, we have

$$\begin{aligned}
|y(t)| &\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\alpha}|}\frac{\lambda}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|f(s,y(s))|ds \\
&\leq \frac{N\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+|y(s)|^{\lambda})ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\alpha}|}\frac{N\lambda}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}(1+|y(s)|^{\lambda})ds \\
&\leq \frac{N\lambda}{\Gamma(\alpha+1)} + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\alpha}|}\frac{N\lambda}{\Gamma(\alpha+\beta+1)} \\
&\quad + \frac{N\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)|^{\lambda}ds + \frac{|\eta|\Gamma(\beta+1)}{|\Gamma(\beta+1)-\eta\tau^{\alpha}|}\frac{N\lambda}{\Gamma(\alpha+\beta)}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}|y(s)|^{\lambda}ds.
\end{aligned}$$

By Lemma 3.1, there exists a $M^* > 0$ such that

$$|y(t)| \leq M^*, \quad t \in J.$$

Thus for every $t \in J$, we have

$$||y||_{\infty} \leq M^*$$

This shows that the set $E(F)$ is bounded.
As a consequence of Shaefer's fixed point theorem we deduce that $F$ has a fixed point which is a solution of the fractional BVP (1). □

## 5. Example

In this section we give an example to illustrate our main results. Consider the following fractional boundary value problem

$$\begin{aligned}
{}^{c}\mathcal{D}^{\frac{1}{2}}y(t) &= \frac{1}{2\sqrt{t-\frac{1}{2}}}\frac{|y|}{1+|y|} - 1 + \sin^2 t, \quad t \in [0,1] \qquad (12)\\
y(0) &= \sqrt{3}I^{\frac{1}{2}}y(\frac{1}{3})
\end{aligned}$$

Hence, $\alpha = \beta = \frac{1}{2}, \eta = \sqrt{3}$, $\tau = \frac{1}{3}$ and $f(t,y) = \frac{1}{2\sqrt{t-\frac{1}{2}}}\frac{|y|}{1+|y|} - 1 + \sin^2 t$. As $\eta = \sqrt{3} \neq \Gamma(\beta+1)/\tau^{\beta} = \Gamma(\frac{3}{2})/(\frac{1}{3})^{\frac{1}{2}}$ and for all $y_1, y_2 \in [0,\infty), t \in [0,1]$, We have

$$\begin{aligned}
|f(t,y_1) - f(t,y_2)| &= \frac{1}{2\sqrt{t-\frac{1}{2}}}\left|\frac{y_1}{1+y_2} - \frac{y_2}{1+y_2}\right| \\
&= \frac{1}{2\sqrt{t-\frac{1}{2}}}\frac{|y_1-y_2|}{(1+y_1)(1+y_2)} \leq \frac{1}{2\sqrt{t-\frac{1}{2}}}|y_1-y_2|
\end{aligned}$$

Obviously, for all $y \in [0, \infty)$ and each $t \in [0, 1]$,

$$
\begin{aligned}
|f(t, y)| &= \left| \frac{1}{2\sqrt{t-\frac{1}{2}}} \frac{|y|}{1+|y|} - 1 + \sin^2 t \right| \\
&\leq \left| \frac{1}{2\sqrt{t-\frac{1}{2}}} \frac{|y|}{1+|y|} \right| \leq \frac{1}{2\sqrt{t-\frac{1}{2}}}.
\end{aligned}
$$

For $t \in [0, 1], q \in (0, \alpha)$, let $m(t) = h(t) = \frac{1}{2\sqrt{t-\frac{1}{2}}} \in L^{\frac{1}{q}}(J, \mathbb{R}_+), M = \|\frac{1}{2\sqrt{t-\frac{1}{2}}}\|_{L^{\frac{1}{q}}(J,\mathbb{R}_+)}$. Choosing suitable $q \in (0, \alpha)$, one can arrive at the following inequality

$$
\Omega_{\alpha,q} = \left( \frac{2^{2-q}}{(\frac{1-2q}{1-q})^{1-q}} + \frac{\sqrt{3\pi}}{2(\sqrt{\pi}-1)} \right) M < 1.
$$

Thus all the assumption in Theorem 4.1 are satisfied, our results can be applied to the problem 12.

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