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Complete Moment Convergence for Weighted Sums of Negatively Orthant Dependent Random Variables

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Abstract. In this paper, the complete moment convergence and the integrability of the supremum for weighted sums of negatively orthant dependent (NOD, in short) random variables are presented. As applications, the complete convergence and the Marcinkiewicz-Zygmund type strong law of large numbers for NOD random variables are obtained. The results established in the paper generalize some corresponding ones for independent random variables and negatively associated random variables.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins [7] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins [7] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [5] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One can refer to Baum and Katz [2] for instance. On the other hand, Chow [4] introduced the concept of complete moment convergence, which is more general than complete convergence.

The concept of complete moment convergence was introduced by Chow [4] as follows: let $\{Z_n, n \geq 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If $\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty$ for all $\varepsilon > 0$, then $\{Z_n, n \geq 1\}$ is called to converge in the sense of complete moment convergence. It is well known that the complete moment convergence can imply complete convergence.

Since the concept of complete moment convergence was introduced by Chow [4], many applications have been found. See for example, Sung [15] established general methods for obtaining the complete

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moment convergence for sums of random variables satisfying the Marcinkiewicz-Zygmund type moment inequality. Liang et al. [10] provided necessary and sufficient conditions for complete moment convergence of negatively associated (NA, in short) random variables. Wu et al. [26] studied the complete moment convergence for ρ^*-mixing random variables. Yang et al. [27] investigated complete convergence for moving average process based on asymptotically almost negatively associated (AANA, in short) sequence. Yang et al. [28] studied complete convergence for moving average process based on martingale differences. Wang and Hu [19] established the equivalence of the complete convergence and complete moment convergence for a class of random variables. Wang and Hu [20] studied the complete convergence and complete moment convergence for martingale difference sequence. The main purpose of the present investigation is to study the complete moment convergence for weighted sums of negatively orthant dependent random variables.

Now, let us recall the definitions of negatively associated random variables and negatively orthant dependent random variables.

Definition 1.1. *A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively associated (NA, in short) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \cdots, n\}$,*

$$Cov\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is NA if every finite subcollection is NA.

Definition 1.2. *A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively orthant dependent (NOD, in short) if*

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

Since the concepts of NA and NOD sequences were introduced by Joag-Dev and Proschan [8], many applications have been found. Obviously, independent random variables are NOD. Joag-Dev and Proschan [8] pointed out that NA random variables are NOD. So we can see that NOD is weaker than NA. A number of limit theorems for NOD random variables have been established by many authors. See for example, Taylor et al. [17] studied strong law of large numbers, Volodin [18] established the Kolmogorov exponential inequality, Amini and Bozorgnia [1], and Wu [23] obtained complete convergence for NOD random varialbes, Ko and Kim [9] established almost convergence for weighted sums of NOD random variables, Shen [11] studied the strong limit theorems for arrays of rowwise NOD random variables, Sung [16] established exponential inequalities for NOD random variables, Wu [25] and Wang et al. [21] obtained the complete convergence theorem for weighted sums of arrays of rowwise NOD random variables, Shen [12] studied the strong convergence rate for weighted sums of arrays of rowwise NOD random variables, and so forth.

The main purpose of the present investigation is to provide the complete moment convergence for weighted sums of NOD random variables. We will present some simple conditions to prove the complete moment convergence. The techniques used in the paper are the truncation method and the Rosenthal type inequality for NOD random variables.

Definition 1.3. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

Throughout the paper, $I(A)$ stands for the indicator function of set A and C denotes a positive constant which may be different in various places. Denote $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

2. Main Results

Our main results are as follows.

Theorem 2.1. *Let $r > 1$, $0 < p < 2$ and $pr > 1$. Assume that $\{X_n, n \geq 1\}$ is a sequence of mean zero NOD random variables which is stochastically dominated by a random variable X such that $EX^2 < \infty$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ ba an array of real numbers. For some $q > \max\{\frac{2p(r-1)}{(2-p)}, pr\}$, we assume that $E|X|^{pr}\log^q(1+|X|) < \infty$ and*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n). \tag{2.1}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p}\right)^+ < \infty. \tag{2.2}$$

Remark 2.1. It can be found that the complete moment convergence can imply complete convergence. Let the conditions of Theorem 2.1 hold. Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/p}\right) < \infty. \tag{2.3}$$

In fact, it can be checked that for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p}\right)^+ \\
= \ & \sum_{n=1}^{\infty} n^{r-2-1/p} \int_0^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p} > t\right) dt \\
\geq \ & \sum_{n=1}^{\infty} n^{r-2-1/p} \int_0^{\varepsilon n^{1/p}} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p} > t\right) dt \\
\geq \ & \varepsilon \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > 2\varepsilon n^{1/p}\right).
\end{aligned}$$

So (2.2) implies (2.3).

Corollary 2.1. *Let $r > 1$, $0 < p < 2$ and $pr > 1$. Assume that $\{X_n, n \geq 1\}$ is a sequence of mean zero NOD random variables which is stochastically dominated by a random variable X such that $EX^2 < \infty$. Let $\{a_n, n \geq 1\}$ be a sequence of real numbers. For some $q > \max\{\frac{2p(r-1)}{(2-p)}, pr\}$, we assume that $E|X|^{pr}\log^q(1+|X|) < \infty$ and*

$$\sum_{i=1}^{n} |a_i|^q = O(n). \tag{2.4}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_iX_i\right| - \varepsilon n^{1/p}\right)^+ < \infty, \tag{2.5}$$

and for $1 < r < 2$,

$$\sum_{n=1}^{\infty} n^{r-2} E\left(\sup_{k\geq n}\left|\frac{1}{k^{1/p}}\sum_{i=1}^{k} a_iX_i\right| - \varepsilon\right)^+ < \infty. \tag{2.6}$$

On the other hand, for any $0 < p < 1$ and $r = 1/p$, we obtain the following result.

Theorem 2.2. *Let $0 < p < 1$ and $\{X_n, n \geq 1\}$ be a sequence of mean zero NOD random variables which is stochastically dominated by a random variable X such that $E|X|\log^3(1+|X|) < \infty$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ ba an array of real numbers such that*

$$\sum_{i=1}^{n} |a_{ni}|^2 = O(n). \tag{2.7}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p}\right)^+ < \infty. \tag{2.8}$$

Corollary 2.2. *Let $0 < p < 1$ and $\{X_n, n \geq 1\}$ be a sequence of mean zero NOD random variables which is stochastically dominated by a random variable X such that $E|X|\log^3(1+|X|) < \infty$. Let $\{a_n, n \geq 1\}$ be a sequence of real numbers such that*

$$\sum_{i=1}^{n} |a_i|^2 = O(n). \tag{2.9}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_iX_i\right| - \varepsilon n^{1/p}\right)^+ < \infty, \tag{2.10}$$

and for $\frac{1}{2} < p < 1$,

$$\sum_{n=1}^{\infty} n^{1/p-2} E\left(\sup_{k\geq n}\left|\frac{1}{k^{1/p}}\sum_{i=1}^{k} a_iX_i\right| - \varepsilon\right)^+ < \infty. \tag{2.11}$$

In particular, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{1/p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_iX_i\right| > \varepsilon n^{1/p}\right) < \infty. \tag{2.12}$$

Remark 2.2. Let the conditions of Corollary 2.1 hold. Denote $S_n = \sum_{i=1}^{n} a_iX_i$ for $n \geq 1$. Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right) < \infty, \tag{2.13}$$

and for $1 < r < 2$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon\right) < \infty. \tag{2.14}$$

In fact, it can be checked that for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p}\right)^+ \\
= & \sum_{n=1}^{\infty} n^{r-2-1/p} \int_0^{\infty} P\left(\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p} > t\right)dt \\
\ge & \sum_{n=1}^{\infty} n^{r-2-1/p} \int_0^{\varepsilon n^{1/p}} P\left(\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p} > t\right)dt \\
\ge & \ \varepsilon \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}|S_k| > 2\varepsilon n^{1/p}\right).
\end{aligned}$$

So (2.5) implies (2.13).

Meanwhile, inspired by the proof of Theorem 12.1 of Gut [6], it can be checked that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\ge n}\left|\frac{S_k}{k^{1/p}}\right| > 2^{2/p}\varepsilon\right) = \sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1} n^{r-2} P\left(\sup_{k\ge n}\left|\frac{S_k}{k^{1/p}}\right| > 2^{2/p}\varepsilon\right) \\
\le & \sum_{m=1}^{\infty} P\left(\sup_{k\ge 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right| > 2^{2/p}\varepsilon\right)\sum_{n=2^{m-1}}^{2^m-1} 2^{m(r-2)} \le \sum_{m=1}^{\infty} 2^{m(r-1)} P\left(\sup_{k\ge 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right| > 2^{2/p}\varepsilon\right) \\
= & \sum_{m=1}^{\infty} 2^{m(r-1)} P\left(\sup_{l\ge m}\max_{2^{l-1}\le k<2^l}\left|\frac{S_k}{k^{1/p}}\right| > 2^{2/p}\varepsilon\right) \le \sum_{m=1}^{\infty} 2^{m(r-1)} \sum_{l=m}^{\infty} P\left(\max_{1\le k\le 2^l}|S_k| > \varepsilon 2^{(l+1)/p}\right) \\
= & \sum_{l=1}^{\infty} P\left(\max_{1\le k\le 2^l}|S_k| > \varepsilon 2^{(l+1)/p}\right)\sum_{m=1}^{l} 2^{m(r-1)} \le C\sum_{l=1}^{\infty} 2^{l(r-1)} P\left(\max_{1\le k\le 2^l}|S_k| > \varepsilon 2^{(l+1)/p}\right) \\
= & \ 2^{2-r}C\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} 2^{(l+1)(r-2)} P\left(\max_{1\le k\le 2^l}|S_k| > \varepsilon 2^{(l+1)/p}\right) \\
\le & \ 2^{2-r}C\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} n^{r-2} P\left(\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right) \quad (\text{since } r < 2) \\
\le & \ 2^{2-r}C\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right).
\end{aligned}$$

Combining (2.13) with the inequality above, we obtain (2.14) immediately. □

Remark 2.3. Since $r > 1$, it can be seen by (2.13) that

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right) \le \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right) < \infty.$$

By the inequality above and the standard method, we can obtain the Marcinkiewicz-Zygmund type strong law of large numbers for NOD sequence as follows:

$$n^{-1/p}\sum_{i=1}^{n} a_i X_i \to 0 \ \ a.s., \ \text{ as } \ n \to \infty.$$

3. Preliminary Lemmas

The following lemmas are our basic techniques to prove the main results. The first one is a basic property for NOD random variables.

Lemma 3.1. *(cf. Bozorgnia et al. [3]). Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables, and let $\{f_n, n \geq 1\}$ be a sequence of nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of NOD random variables.*

The next one is the Rosenthal type inequality for NOD random variables which can be found in Wu [24].

Lemma 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^q < \infty$ for some $q \geq 2$. Then there exists a positive constant C_q depending only on q such that*

$$E\left(\max_{1\leq i\leq n}\left|\sum_{i=1}^{n} X_i\right|^q\right) \leq C_q \log^q n \left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i{}^2\right)^{q/2}\right\}, \quad n \geq 1.$$

The following one is a basic property for stochastic domination. For the proof, one can refer to Wu [22], Shen and Wu [13], or Shen et al. [14].

Lemma 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$\begin{aligned} E|X_n|^\alpha I(|X_n| \leq b) &\leq C_1[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)], \\ E|X_n|^\alpha I(|X_n| > b) &\leq C_2 E|X|^\alpha I(|X| > b), \end{aligned}$$

where C_1 and C_2 are positive constants.

The last one is the moment inequality for the maximum partial sum of random variables, which plays an important role to prove the main results of the paper.

Lemma 3.4. *(cf. Sung [15]). Let $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ be sequences of random variables. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_i + Z_i)\right| - \varepsilon a\right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}} E\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right|^q + E\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Z_i\right|.$$

4. Proofs of the Main Results

Proof of Theorem 2.1. For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$Y_{ni} = -n^{1/p} I(X_i < -n^{1/p}) + X_i I(|X_i| \leq n^{1/p}) + n^{1/p} I(X_i > n^{1/p}),$$

$$Y^*_{ni} = n^{1/p} I(X_i < -n^{1/p}) - n^{1/p} I(X_i > n^{1/p}) + X_i I(|X_i| > n^{1/p})$$

and

$$\tilde{Y}_{ni} = Y_{ni} - EY_{ni}.$$

Obviously, it has $X_i = Y^*_{ni} + EY_{ni} + \tilde{Y}_{ni}$. Applying Lemma 3.4 with $a = n^{1/p}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| - \varepsilon n^{1/p}\right)^+ \\ \leq\ & \sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} Y^*_{ni}\right|\right) + \sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} EY_{ni}\right|\right) \\ & + C\sum_{n=1}^{\infty} n^{r-2-q/p} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} \tilde{Y}_{ni}\right|^q\right) \\ \doteq\ & H + I + J. \end{aligned} \tag{4.1}$$

By (2.1) and Hölder's inequality, we have for $1 \le k \le q$ that

$$\sum_{i=1}^{n} |a_{ni}|^k \le (\sum_{i=1}^{n} |a_{ni}|^q)^{k/q} (\sum_{i=1}^{n} 1)^{1-k/q} \le Cn. \tag{4.2}$$

For H, noting that $|Y^*_{ni}| \le |X_i| I(|X_i| > n^{1/p})$, we have by (4.2)(taking $k = 1$), Lemma 3.3 and $E[|X|^{rp} \log^q (1+|X|)] < \infty$ that

$$\begin{aligned} H &\le C \sum_{n=1}^{\infty} n^{r-2-1/p} \sum_{i=1}^{n} |a_{ni}| E\left(|X_i| I(|X_i| > n^{1/p})\right) \\ &\le C \sum_{n=1}^{\infty} n^{r-2-1/p} \sum_{i=1}^{n} |a_{ni}| E|X| I(|X| > n^{1/p}) \\ &= C \sum_{n=1}^{\infty} n^{r-1-1/p} \sum_{m=n}^{\infty} E|X| I(m < |X|^p \le m+1) \\ &= C \sum_{m=1}^{\infty} E|X| I(m < |X|^p \le m+1) \sum_{n=1}^{m} n^{r-1-1/p} \\ &\le C \sum_{m=1}^{\infty} m^{r-1/p} E|X| I(m < |X|^p \le m+1) \\ &\le CE|X|^{rp} < \infty. \end{aligned} \tag{4.3}$$

Meanwhile, noting that $EX_n = 0, n \ge 1$, we get that

$$\begin{aligned} EY_{ni} &= E[-n^{1/p} I(X_i < -n^{1/p}) + X_i I(|X_i| \le n^{1/p}) + n^{1/p} I(X_i > n^{1/p})] \\ &= E[-n^{1/p} I(X_i < -n^{1/p}) - X_i I(|X_i| > n^{1/p}) + n^{1/p} I(X_i > n^{1/p})]. \end{aligned} \tag{4.4}$$

Consequently, combining (4.4) with the proof of (4.3), one has that

$$I \le 3 \sum_{n=1}^{\infty} n^{r-2-1/p} \sum_{i=1}^{n} |a_{ni}| E\left(|X_i| I(|X_i| > n^{1/p})\right) \le CE|X|^{rp} < \infty. \tag{4.5}$$

Noting that $a_{ni} = a^+_{ni} - a^-_{ni}$, we have by C_r inequality that

$$\begin{aligned} J &= \sum_{n=1}^{\infty} n^{r-2-q/p} E\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} a_{ni} \tilde{Y}_{ni}\right|^q\right) \\ &\le C \sum_{n=1}^{\infty} n^{r-2-q/p} E\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} a^+_{ni} \tilde{Y}_{ni}\right|^q\right) + C \sum_{n=1}^{\infty} n^{r-2-q/p} E\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} a^-_{ni} \tilde{Y}_{ni}\right|^q\right). \end{aligned}$$

Hence, without loss of generality, we may assume that $a_{ni} \ge 0$ for all $1 \le i \le n$ and $n \ge 1$. Obviously, by Lemma 3.1, we can find that for fixed $n \ge 1$, $\{a_{ni}\tilde{Y}_{ni}, 1 \le i \le n\}$ are still NOD random variables with mean zero. Therefore, applying Lemma 3.2, we can check that

$$\begin{aligned} J &= \sum_{n=1}^{\infty} n^{r-2-q/p} E\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} a_{ni} \tilde{Y}_{ni}\right|^q\right) \\ &\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \log^q n \left(\sum_{i=1}^{n} |a_{ni}|^2 E\tilde{Y}^2_{ni}\right)^{q/2} + C \sum_{n=1}^{\infty} n^{r-2-q/p} \log^q n \sum_{i=1}^{n} |a_{ni}|^q E|\tilde{Y}_{ni}|^q \\ &\doteq CJ_1 + CJ_2. \end{aligned} \tag{4.6}$$

Since $q > (r-1)/(1/p-1/2)$, it can be seen by (4.2)(taking $k=2$), Lemma 3.3 and $EX^2 < \infty$ that

$$\begin{aligned} J_1 &= \sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2E\tilde{Y}_{ni}^2\right)^{q/2} \le C\sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2EY_{ni}^2\right)^{q/2} \\ &\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2EX_i^2\right)^{q/2} \le C\sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2EX^2\right)^{q/2} \\ &\le C\sum_{n=1}^{\infty} n^{r-2-q/p+q/2}\log^q n < \infty. \end{aligned} \tag{4.7}$$

For J_2, it follows from Lemma 3.3 that

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\sum_{i=1}^{n}|a_{ni}|^qE|\tilde{Y}_{ni}|^q \le C\sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\sum_{i=1}^{n}|a_{ni}|^qE|Y_{ni}|^q \\ &\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\log^q n\sum_{i=1}^{n}|a_{ni}|^qE\left[|X_i|^qI(|X_i|\le n^{1/p}) + n^{q/p}I(|X_i|>n^{1/p})\right] \\ &\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\log^q nE\left[|X|^qI(|X|\le n^{1/p})\right] + C\sum_{n=1}^{\infty} n^{r-1}\log^q nP(|X|>n^{1/p}) \\ &\doteq CJ_{21} + CJ_{22}. \end{aligned} \tag{4.8}$$

Since $q > pr$ and $E[|X|^{rp}\log^q(1+|X|)] < \infty$, one has

$$\begin{aligned} J_{21} &= \sum_{n=1}^{\infty} n^{r-1-q/p}\log^q n\sum_{m=1}^{n}E\left[|X|^qI((m-1)^{1/p}<|X|\le m^{1/p})\right] \\ &= \sum_{m=1}^{\infty}E\left[|X|^qI((m-1)^{1/p}<|X|\le m^{1/p})\right]\sum_{n=m}^{\infty} n^{r-1-q/p}\log^q n \\ &\le C\sum_{m=1}^{\infty}E\left[|X|^{rp}|X|^{q-pr}I((m-1)^{1/p}<|X|\le m^{1/p})\right]m^{r-q/p}\log^q(1+m) \\ &\le CE|X|^{rp}\log^q(1+|X|) < \infty. \end{aligned} \tag{4.9}$$

For J_{22}, it follows from $pr > 1$ that

$$\begin{aligned} J_{22} &= \sum_{n=1}^{\infty} n^{r-1-1/p}\log^q nE\left[|X|I(|X|>n^{1/p})\right] \\ &= C\sum_{n=1}^{\infty} n^{r-1-1/p}\log^q n\sum_{m=n}^{\infty}E\left[|X|I(m^{1/p}<|X|\le(m+1)^{1/p})\right] \\ &= C\sum_{m=1}^{\infty}E\left[|X|I(m<|X|^p\le(m+1))\right]\sum_{n=1}^{m} n^{r-1-1/p}\log^q n \\ &\le C\sum_{m=1}^{\infty}E\left[|X|I(m<|X|^p\le(m+1))\right]m^{r-1/p}\log^q(1+m) \\ &\le CE|X|^{rp}\log^q(1+|X|) < \infty. \end{aligned} \tag{4.10}$$

Therefore, (2.2) follows from (4.1)–(4.10) immediately. The proof is completed. □

Proof of Corollary 2.1. Similarly to the proof of Theorem 2.1, we obtain (2.5) immediately. It is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} E\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| - \varepsilon 2^{2/p}\right)^{+} \\
= & \sum_{n=1}^{\infty} n^{r-2} \int_0^{\infty} P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right) dt \\
= & \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} n^{r-2} \int_0^{\infty} P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right) dt \\
\leq & 2^{2-r} \sum_{m=1}^{\infty} \int_0^{\infty} P\left(\sup_{k\geq 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right) dt \sum_{n=2^{m-1}}^{2^m-1} 2^{m(r-2)} \\
\leq & 2^{2-r} \sum_{m=1}^{\infty} 2^{m(r-1)} \int_0^{\infty} P\left(\sup_{k\geq 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right) dt \\
= & 2^{2-r} \sum_{m=1}^{\infty} 2^{m(r-1)} \int_0^{\infty} P\left(\sup_{l\geq m}\max_{2^{l-1}\leq k\leq 2^l}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right) dt \\
\leq & 2^{2-r} \sum_{m=1}^{\infty} 2^{m(r-1)} \sum_{l=m}^{\infty} \int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > (\varepsilon 2^{2/p} + t) 2^{(l-1)/p}\right) dt \\
= & 2^{2-r} \sum_{l=1}^{\infty} \int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > (\varepsilon 2^{2/p} + t) 2^{(l-1)/p}\right) dt \sum_{m=1}^{l} 2^{m(r-1)} \\
\leq & 2^{2-r} \sum_{l=1}^{\infty} 2^{l(r-1)} \int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > (\varepsilon 2^{2/p} + t) 2^{(l-1)/p}\right) dt \quad \left(\text{let } s = 2^{(l-1)/p} t\right) \\
\leq & C \sum_{l=1}^{\infty} 2^{l(r-1-1/p)} \int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \varepsilon 2^{(l+1)/p} + s\right) ds \\
= & 2^{2+1/p-r} C \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} 2^{(l+1)(r-2-1/p)} \int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \varepsilon 2^{(l+1)/p} + s\right) ds \\
\leq & 2^{2+1/p-r} C \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} n^{r-2-1/p} \int_0^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p} + s\right) ds \quad (\text{since } r < 2) \\
\leq & 2^{2+1/p-r} C \sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}|S_k| - \varepsilon n^{1/p}\right)^{+}, \qquad (4.11)
\end{aligned}$$

where $S_n = \sum_{i=1}^{n} a_i X_i$ for $n \geq 1$.

Therefore, (2.6) follows from (2.5) and (4.11) immediately. □

Proof of Theorem 2.2. Similarly to the proof of Theorem 2.1, and applying Lemma 3.4 with $a = n^{1/p}$ and $q = 2$, we can obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/p}\right)^+ \\
\le\ & \sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}Y^*_{ni}\right|\right) + \sum_{n=1}^{\infty} n^{-2}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}EY_{ni}\right|\right) + C\sum_{n=1}^{\infty} n^{-2-1/p} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\tilde{Y}_{ni}\right|^2\right) \\
\doteq\ & H^* + I^* + CJ^*.
\end{aligned} \tag{4.12}$$

Similarly to the proof of (4.3), we have by $E|X|\log^3(1+|X|) < \infty$ that

$$\begin{aligned}
H^* &\le C\sum_{n=1}^{\infty} n^{-1} E[|X|I(|X| > n^{1/p})] = C\sum_{n=1}^{\infty} n^{-1}\sum_{m=n}^{\infty} E[|X|I(m < |X|^p \le m+1)] \\
&= C\sum_{m=1}^{\infty} E[|X|I(m < |X|^p \le m+1)]\sum_{n=1}^{m} n^{-1} \\
&\le C\sum_{m=1}^{\infty} \log(1+m)E[|X|I(m < |X|^p \le m+1)] \\
&\le CE[|X|\log(1+|X|)] < \infty.
\end{aligned} \tag{4.13}$$

Meanwhile, similarly to the proofs of (4.5) and (4.13), we have

$$I^* \le C\sum_{n=1}^{\infty} n^{-1} E[|X|I(|X| > n^{1/p})] \le CE[|X|\log(1+|X|)] < \infty. \tag{4.14}$$

On the other hand, without loss of generality, we assume that $a_{ni} \ge 0$ for all $1 \le i \le n$ and $n \ge 1$. Similarly to the proof of (4.6), we have by (2.7) that

$$\begin{aligned}
J^* &= \sum_{n=1}^{\infty} n^{-2-1/p} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\tilde{Y}_{ni}\right|^2\right) \le C\sum_{n=1}^{\infty} n^{-2-1/p}\log^2 n \sum_{i=1}^{n} |a_{ni}|^2 E|\tilde{Y}_{ni}|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-2-1/p}\log^2 n \sum_{i=1}^{n} |a_{ni}|^2 |EY_{ni}|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-2-1/p}\log^2 n \sum_{i=1}^{n} |a_{ni}|^2 E\left[|X_i|^2 I(|X_i| \le n^{1/p}) + n^{2/p} I(|X_i| > n^{1/p})\right] \\
&\le C\sum_{n=1}^{\infty} n^{-1-1/p}\log^2 n E\left[|X|^2 I(|X| \le n^{1/p})\right] + C\sum_{n=1}^{\infty} n^{-1+1/p}\log^2 n P(|X| > n^{1/p}) \\
&\doteq CJ_1^* + CJ_2^*.
\end{aligned} \tag{4.15}$$

Since $E|X|\log^3(1+|X|) < \infty$, one has

$$\begin{aligned}
J_1^* &= \sum_{n=1}^{\infty} n^{-1-1/p}\log^2 n \sum_{m=1}^{n} E\left[|X|^2 I((m-1)^{1/p} < |X| \le m^{1/p})\right] \\
&= \sum_{m=1}^{\infty} E\left[|X|^2 I((m-1)^{1/p} < |X| \le m^{1/p})\right]\sum_{n=m}^{\infty} n^{-1-1/p}\log^2 n \\
&\le C\sum_{m=1}^{\infty} E\left[|X|^2 I((m-1)^{1/p} < |X| \le m^{1/p})\right] m^{-1/p}\log^2(1+m) \\
&\le CE|X|\log^2(1+|X|) < \infty.
\end{aligned} \tag{4.16}$$

For J_2^*, it has

$$\begin{aligned} J_2^* &\le \sum_{n=1}^{\infty} n^{-1}\log^2 nE\left[|X|I(|X|>n^{1/p})\right] \\ &= C\sum_{n=1}^{\infty} n^{-1}\log^2 n\sum_{m=n}^{\infty} E\left[|X|I(m^{1/p}<|X|\le(m+1)^{1/p})\right] \\ &= C\sum_{m=1}^{\infty} E\left[|X|I(m<|X|^p\le(m+1))\right]\sum_{n=1}^{m} n^{-1}\log^2 n \\ &\le C\sum_{m=1}^{\infty} E\left[|X|I(m<|X|^p\le(m+1))\right]\log^3(1+m) \\ &\le CE|X|\log^3(1+|X|)<\infty. \end{aligned} \tag{4.17}$$

Hence, (2.8) follows from (4.12)-(4.17) immediately. □

Proof of Corollary 2.2. Similarly to the proof of Theorem 2.2, we obtain (2.10) immediately. Meanwhile, for $0 < p < 1$, combining (2.10) and (4.11), we obtain (2.11) immediately. Finally, by the proof of (2.13) in Remark 2.2, (2.12) also holds. □

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