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A New Theorem on the Absolute Riesz Summability Factors

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Abstract. In [5], we proved a main theorem dealing with absolute Riesz summability factors of infinite series using a quasi-δ-power increasing sequence. In this paper, we generalize that theorem by using a general class of power increasing sequences instead of a quasi-δ-power increasing sequence. This theorem also includes some new and known results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We write $\mathcal{BV}_O = \mathcal{BV} \cap \mathcal{C}_O$, where $\mathcal{C}_O = \{ x = (x_k) \in \Omega \ : \ \lim_k |x_k| = 0 \}$, $\mathcal{BV}$=$\{ x = (x_k) \in \Omega \ : \ \sum_k |x_k - x_{k+1}| < \infty \}$ and Ω being the space of all real-valued sequences. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^\delta (\log n)^\sigma, \sigma \geq 0, 0 < \delta < 1\}$ (see [10]). If we take σ=0, then we get a quasi-δ-power increasing sequence (see [9]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n and t_n the nth $(C, 1)$ means of the sequence (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $| C, 1 |_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n - u_{n-1} \mid^k = \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n \mid^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$V_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

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defines the sequence (V_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid V_n - V_{n-1} \mid^k < \infty. \tag{4}$$

In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability reduces to $\mid C, 1 \mid_k$ summability.

2.Known result

In [5], we have proved the following theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series.

Theorem 2.1 Let $(\lambda_n) \in \mathcal{BV}_O$ and let (X_n) be a quasi-δ-power increasing sequence for some δ $(0 < \delta < 1)$ and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{7}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{8}$$

If

$$\sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty \tag{9}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{10}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{11}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k, k \geq 1$.

It should be noted that if we take (X_n) as an almost increasing sequence, then we get a result which was proved in [4]. Also it should be remarked that we can take $(\lambda_n) \in \mathcal{BV}$ instead of $(\lambda_n) \in \mathcal{BV}_O$ and it is sufficient to prove Theorem 2.1.

3. The main result

The aim of this paper is to generalize Theorem 2.1 by using a quasi-f-power increasing sequence instead of a quasi-δ-power increasing sequence. Now we shall prove the following theorem.

Theorem 3.1 Let $(\lambda_n) \in \mathcal{BV}$ and let (X_n) be a quasi-f-power increasing sequence for some δ $(0 < \delta < 1)$ and $\sigma \geq 0$. If the conditions (5)-(11) are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

It should be noted that if we take σ=0, then we get Theorem 2.1.

We require the following lemmas for the proof of our theorem.

Lemma 3. 2 ([3]) If the conditions (10) and (11) are satisfied, then $\Delta(P_n/p_n n^2) = O(1/n^2)$.

Lemma 3. 3 ([6]) Except for the condition $(\lambda_n) \in \mathcal{BV}$ under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem, we have the following;

$$nX_n\beta_n = O(1), \tag{12}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{13}$$

4. Proof of Theorem 3.1 Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{vp_v}. \tag{14}$$

Then, for $n \geq 1$ we have that

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v} \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}. \end{aligned}$$

Using Abel's transformation, we get that

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\ &+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v (v+1) \frac{t_v}{v^2 p_v} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta(P_v / v^2 p_v) \\ &+ \lambda_n t_n (n+1)/n^2 \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{15}$$

When $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, and so we get that

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid t_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \\ &\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid p_v \mid t_v \mid^k \frac{1}{P_v} \frac{1}{v^k} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \mid \lambda_v \mid\mid t_v \mid^k \frac{1}{v^k} \\ &= O(1) \sum_{v=1}^{m} v^{k-1} \frac{1}{v^k} \mid \lambda_v \mid\mid t_v \mid^k \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{r=1}^{v} \frac{\mid t_r \mid^k}{r} \\
&+ O(1) \mid \lambda_m \mid \sum_{v=1}^{m} \frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1)
\end{aligned}
$$

as $m \to \infty$, by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3.
Now, by using (10), we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,2} \mid^k &= O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} \mid \Delta\lambda_v \mid p_v \mid t_v \mid \right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k (\beta_v)^k \mid t_v \mid^k p_v \\
&\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k (\beta_v)^k \mid t_v \mid^k p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} (\beta_v)^k \mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} (v\beta_v)^{k-1} v\beta_v \frac{1}{v^k} \mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m} v^{k-1} v\beta_v \frac{1}{v^k} \mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m} v\beta_v \frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{\mid t_r \mid^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} \frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3. 3. The other parts of the proof can be done similar as in [5] by using Lemma 3. 2 and Lemma 3. 3 and therefore we omitted it. If we take $p_n = 1$ for all values of n, then we get a new result dealing with $\mid C, 1 \mid_k$ summability factors of infinite series.

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