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Multi-Generalized 2-Normed Space

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Abstract. In this paper, we introduce the concepts of multi-generalized 2-normed space and dual multi-generalized 2-normed space and we then investigate some results related to them. We also prove that, if $(E, \|.,.\|)$ is a generalized 2-normed space, $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ is a sequence of generalized 2-norms on E^k $(k \in \mathbb{N})$ such that for each $x, y \in E$, $\|x, y\|_1 = \|x, y\|$ and for each $k \in \mathbb{N}$ axioms $(MG1)$, $(MG2)$ and $(MG4)$($(DG4)$) of (dual) multi-generalized 2-normed space are true, then $\{(E^k, \|.,.\|_k), k \in \mathbb{N}\}$ is a (dual) multi-generalized 2-normed space. Finally we deal with an application of a dual multi-generalized 2-normed space defined on a proper commutative H^*-algebra.

1. Introduction and Preliminaries

The notion of (dual) multi-normed spaces which are somewhat similar to the operator sequence spaces, was initiated by H. G. Dales and M. E. Polyakov in [5]. That provides a suitable supply for the study of multi-normed spaces together with many examples. Some results of (dual) multi-normed spaces are stable under generalized 2-normed spaces [12]. In this paper we use these properties to discover new ones for (dual) multi-generalized 2-normed spaces. In [12], Z. Lewandowska introduced a generalization of Gähler 2-normed space [7, 18], under the name of generalized 2-normed space. After that she published some papers on this issue (e.g. [9–11]). In the following lines, we present some definitions and examples which will be utilized in the sequel.

Definition 1.1. *(see [12]) Let X and Y be linear spaces over the field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). A function $\|.,.\| : X \times Y \to [0, \infty)$ is called a generalized 2-norm on $X \times Y$ if it satisfies the following conditions,*
(i) $\|\alpha x, y\| = \|x, \alpha y\| = |\alpha|\|x, y\|$ for all $\alpha \in \mathbb{K}$ and $x \in X, y \in Y$;
(ii) $\|x, y_1 + y_2\| \le \|x, y_1\| + \|x, y_2\|$ for all $x \in X, y_1, y_2 \in Y$;
(iii) $\|x_1 + x_2, y\| \le \|x_1, y\| + \|x_2, y\|$ for all $x_1, x_2 \in X, y \in Y$.
The pair $(X \times Y, \|.,.\|)$ is called a generalized 2-normed space. If $X = Y$, then the generalized 2-normed space will be denoted by $(X, \|.,.\|)$.

Example 1.2. *(see [11]) Let X be a real linear space having two seminorms $\|.\|_1$ and $\|.\|_2$. Then $(X, \|.,.\|)$ is a generalized 2-normed space with the generalized 2-norm defined by $\|x, y\| = \|x\|_1\|y\|_2$ where $x, y \in X$.*

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A sequence $\{x_n\}_n$ in a generalized 2-normed space $(X, \|.,.\|)$ is said to be a 2-Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, u\| = 0$ for all $u \in X$. In addition, $\{x_n\}_n$ is called 2-convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, u\| = 0$ for all $u \in X$. A generalized 2-normed space is called generalized 2-Banach space if every 2-Cauchy sequence is 2-convergent. Since Lewandowska up to now there are many mathematicians worked on generalized 2-normed spaces and developed it in several directions, see [1, 3, 16, 17]) and references cited therein.

The notion of (dual) multi-normed space first was introduced in [5]. This concept has some connections with operator spaces and Banach lattices.

Let $(E, \|.\|)$ be a complex normed space. We denote by E^k ($k \in \mathbb{N}$), the linear space $E \oplus \ldots \oplus E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. Following notations and terminologies of [5], we denote by $\mathbb{N}_k$ the set $\{1, \ldots, k\}$ and by ς_k the group of permutations on k symbols. For $\sigma \in \varsigma_k$, $x = (x_1, \ldots, x_k) \in E^k$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$ define $A_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ and $M_\alpha(x) = (\alpha_1 x_1, \ldots, \alpha_k x_k)$. Let $n \in \mathbb{N}$, we set $x^{[n]} = (x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k) \in E^{nk}$, where $x^{[n]}$ consists of n copies of each block $(x_1, \ldots, x_k)$.

Take $k \in \mathbb{N}$ and let S be a subset of $\mathbb{N}_k$. For $(x_1, \ldots, x_k) \in E^k$, we set $Q_S(x_1, \ldots, x_k) = (y_1, \ldots, y_k)$, where $y_i = x_i$ ($i \notin S$) and $y_i = 0 \quad (i \in S)$. Thus Q_S is the projection onto the complement of S.

Definition 1.3. *(see [5]) Let $(E, \|.\|)$ be a complex (respectively, real) normed space, and take $n \in \mathbb{N}$. A multi-norm of level n on $\{E^k, k \in \mathbb{N}_n\}$ is a sequence $\{\|.\|_k\} = \{\|.\|_k,\ k \in \mathbb{N}_n\}$ such that $\|.\|_k$ is a norm on E^k for each $k \in \mathbb{N}_n$, such that $\|x\|_1 = \|x\|$ for each $x \in E$ (so that $\|.\|_1$ is the initial norm), and such that the following axioms $(MN1)$-$(MN4)$ are satisfied for each $k \in \mathbb{N}_n$ with $k \geq 2$:*

$(MN1)$ *for each $\sigma \in \varsigma_k$ and $x \in E^k$, $\|A_\sigma(x)\|_k = \|x\|_k$;*

$(MN2)$ *for each $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ (respectively, each $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$) and $x \in E^k$,*

$$\|M_\alpha(x)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|)\|x\|_k;$$

$(MN3)$ *for each $x_1, \ldots, x_{k-1} \in E$, $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$;*

$(MN4)$ *for each $x_1, \ldots, x_{k-1} \in E$, $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$.*

In this case, $\{(E^k, \|.\|_k), k \in \mathbb{N}_n\}$ is a multi-normed space of level n. A multi-norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence

$$\{\|.\|_k\} = \{\|.\|_k,\ k \in \mathbb{N}\}$$

such that $\{\|.\|_k,\ k \in \mathbb{N}_n\}$ is a multi-norm of level n for each $n \in \mathbb{N}$. In this case, $\{(E^n, \|.\|_n), n \in \mathbb{N}\}$ is a multi-normed space. Moreover, if axiom $(MN4)$ replaced by the following axiom, then it is called a dual multi-norm and $\{(E^n, \|.\|_n), n \in \mathbb{N}\}$ is called a dual multi-normed space.

$(DM4)$ *for each $x_1, \ldots, x_{k-1} \in E$, $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, 2x_{k-1})\|_{k-1}$.*

Example 1.4. *(see [5]) Let $(E, \|.\|)$ be a normed space. For each $k \in \mathbb{N}$, put*

(i) $\|(x_1, \ldots, x_k)\|_k^1 = \max_{i\in\mathbb{N}_k} \|x_i\|$,

(ii) $\|(x_1, \ldots, x_k)\|_k^2 = \sum_{i=1}^{k} \|x_i\|$,

where $x_1, \ldots, x_k$ are in E. Then $\{(E^k, \|.\|_k^1), k \in \mathbb{N}\}$ is a multi-normed space and $\{(E^k, \|.\|_k^2), k \in \mathbb{N}\}$ is a dual multi-normed space.

Suppose that $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is a (dual) multi-normed space. The following property is almost immediate consequence of the axioms.

$$\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\| \ (x_1, \ldots, x_k \in E).$$

It follows from the above assertion that, if $(E, \|.\|_1)$ is a Banach space, then $(E^k, \|.\|_k)$ is a Banach space for each $k = 2, 3, ...$, in this case, $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is called a (dual) multi-Banach space.

By now, many authors have already contributed to the theoretical development of the theory of multi-normed spaces (e.g. see [6, 13–15]). In the present work we demonstrate the concept of (dual) multi-normed space in the framework of generalized 2-normed spaces. We also provide many examples together with an application of a dual multi-generalized 2-normed space defined on a proper commutative H^*-algebra [2, 4, 8, 19]. We will describe H^*-algebras in more details in the section 4. This paper is organized as follows: In section 2, we introduce the concept of (dual) multi-geneneralized 2-normed spaces and describe some results concerned with these new ones. In section 3, we show that if $(E, \|.,.\|)$ is a generalized 2-normed space, $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ is a sequence of generalized 2-norms on E^k ($k \in \mathbb{N}$) such that for each $x, y \in E$, $\|x, y\|_1 = \|x, y\|$ and for each $k \in \mathbb{N}$ axioms $(MG1)$, $(MG2)$ and $(MG4)$($(DG4)$) of (dual) multi-generalized 2-normed space are true, then $\{(E^k, \|.,.\|_k), k \in \mathbb{N}\}$ is a (dual) multi-generalized 2-normed space. In section 4, we give an application of a dual multi-generalized 2-normed space. Throughout this paper, we mean by $\mathbb{T}$ and by $\mathbb{S}$ the unit ball and the closed unit ball of $\mathbb{C}$ respectively, more precisely $\mathbb{T} = \{\alpha \in \mathbb{C}, |\alpha| = 1\}$ and $\mathbb{S} = \{\alpha \in \mathbb{C}, |\alpha| \leq 1\}$.

2. (Dual) Multi-Generalized 2-Normed Space

In this section we introduce a (dual) multi-generalized 2-normed space and investigate some properties of it. For this, we need the following definition.

Definition 2.1. *Let $(E, \|.,.\|)$ be a generalized 2-normed space (over the field $\mathbb{K}$). A special generalized 2-norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $\|.,.\|_k$ is a generalized 2-norm on E^k, $\|x, y\|_1 = \|x, y\|$ for each $x, y \in E$ and the following axioms $(MG1)$-$(MG3)$ are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:*
$(MG1)$ for each $\sigma \in \varsigma_k$ and $x, y \in E^k$, $\|A_\sigma(x), A_\sigma(y)\|_k = \|x, y\|_k$;
$(MG2)$ for each $\alpha_1, \dots, \alpha_k \in \mathbb{K}$ and $x, y \in E^k$, $\|M_\alpha(x), y\|_k = \|x, M_\alpha(y)\|_k \leq (\max_{1\leq i\leq k}|\alpha_i|)\|x, y\|_k$;
$(MG3)$ for each $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1} \in E$,

$$\|(x_1, \dots, x_{k-1}, 0), (y_1, \dots, y_{k-1}, 0)\|_k = \|(x_1, \dots, x_{k-1}), (y_1, \dots, y_{k-1})\|_{k-1}.$$

Now consider two following more axioms.
$(MG4)$ for each $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1} \in E$,

$$\|(x_1, \dots, x_{k-1}, x_{k-1}), (y_1, \dots, y_{k-1}, y_{k-1})\|_k = \|(x_1, \dots, x_{k-1}), (y_1, \dots, y_{k-1})\|_{k-1}.$$

$(DG4)$ for each $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1} \in E$,

$$\|(x_1, \dots, x_{k-1}, x_{k-1}), (y_1, \dots, y_{k-1}, y_{k-1})\|_k = \|(x_1, \dots, 2x_{k-1}), (y_1, \dots, y_{k-1})\|_{k-1}.$$

A special generalized 2-norm is said to be a (dual) multi-generalized 2-norm if it is equipped with the axiom $(MG4)$ ($(DG4)$). In this case, $\{(E^k, \|.,.\|_k), k \in \mathbb{N}\}$ is called a (dual) multi-generalized 2-normed space.

We give the definition in the case where the index set is $\mathbb{N}$. If the index set is $\mathbb{N}_k$ ($k \in \mathbb{N}$), then special, multi- and dual multi-generalized 2-normed spaces are of level k.

Remark 2.2. *It is readily verified from the axioms $(MG2)$ and $(MG3)$, that*

$$\|(x_1, \dots, x_k, 0), (y_1, \dots, y_k, y_{k+1})\|_{k+1} = \|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k,$$

where $x_1, \dots, x_k, y_1, \dots, y_{k+1} \in E$. Indeed, we have

$$\begin{aligned}
\|(x_1, \dots, x_k, 0), (y_1, \dots, y_k, y_{k+1})\|_{k+1} &= \|M_{(1,\dots,1,0)}(x_1, \dots, x_k, 0), (y_1, \dots, y_k, y_{k+1})\|_{k+1} \\
&= \|(x_1, \dots, x_k, 0), M_{(1,\dots,1,0)}(y_1, \dots, y_k, y_{k+1})\|_{k+1} \\
&= \|(x_1, \dots, x_k, 0), (y_1, \dots, y_k, 0)\|_{k+1} \\
&= \|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k.
\end{aligned}$$

Example 2.3. *Let $(E, \|.,.\|)$ be a non-zero generalized 2-normed space. For each $k \in \mathbb{N}$, set*
(i) $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k^1 = \max\{\|x_1, y_1\|, \dots, \|x_k, y_k\|\}$,
(ii) $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k^2 = \sum_{i=1}^{k} \|x_i, y_i\|$,
where $(x_1, \dots, x_k), (y_1, \dots, y_k) \in E^k$. Then, $\{(E^k, \|.,.\|_k^1), k \in \mathbb{N}\}$ is a multi-generalized 2-normed space and $\{(E^k, \|.,.\|_k^2)$, $k \in \mathbb{N}\}$ is a dual multi-generalized 2-normed space.

Example 2.4. *Let $(E, \|.\|)$ be an H^*-algebra (for the definition see section 4). Define a generalized 2-norm on E^k ($k \in \mathbb{N}$) by setting $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \sum_{i=1}^{k} \|x_i y_i\|$, then $\{(E^k, \|.,.\|_k), k \in \mathbb{N}\}$ is a dual multi-generalized 2-normed space.*

Example 2.5. *(see [5]) Let $\{(E^k, \|.,.\|_k^\alpha), k \in \mathbb{N}\}_\alpha$ be a family of (dual) multi-generalized 2-normed spaces. For each $k \in \mathbb{N}$ and $x_1, \dots, x_k, y_1, \dots, y_k \in E$, define*

$$\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \sup_\alpha \|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k^\alpha.$$

Then $\{(E^k, \|.,.\|_k), k \in \mathbb{N}\}$ is a (dual) multi-generalized 2-normed space, too.

Inspired by the examples of [5] we give some examples show that axioms $(MG1)$-$(MG4)$ (-$(DG4)$) are independent of each other.

Example 2.6. *Let $(E, \|.,.\|)$ be a non-zero generalized 2-normed space. Set $\|x, y\|_1 = \|x, y\|$ ($x, y \in E$).*
(I) For each $k \in \mathbb{N} - \{1\}$, set $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \max\{\|x_1, y_1\|, \frac{\|x_2, y_2\|}{2}, \dots, \frac{\|x_k, y_k\|}{k}\}$, where $(x_1, \dots, x_k)$, $(y_1, \dots, y_k) \in E^k$. Then it is immediately checked that $\|.,.\|_k$ is a generalized 2-norm on E^k and that $\{\|.,.\|_k\}_{k \in \mathbb{N}}$ satisfies $(MG2), (MG3)$ and $(MG4)$. However, take $x, y \in E$ with $\|x, y\| = 1$. Then $\|(2x, 3x), (2y, 4y)\|_2 = 6$, but $\|(3x, 2x), (4y, 2y)\|_2 = 12$. Thus $\|.,.\|_2$ does not satisfy axiom $(MG1)$.
(II) Set $\|(x_1, x_2), (y_1, y_2)\|_2 = \max\{\|x_1, y_1\|, 2\|x_2, y_2\|\}$, where $(x_1, x_2), (y_1, y_2) \in E^2$. Then it is immediately checked that $\|.,.\|_2$ is a generalized 2-norm on E^2 and that $\|.,.\|_2$ satisfies $(MG2), (MG3)$ and $(DG4)$. However, we claim that $\|.,.\|_2$ does not satisfy axiom $(MG1)$. For this, similar previous part take $x, y \in E$ with $\|x, y\| = 1$. Then $\|(2x, 3x), (2y, 4y)\|_2 = 24$, but $\|(3x, 2x), (4y, 2y)\|_2 = 12$, and so $\|.,.\|_2$ does not satisfy axiom $(MG1)$.

Example 2.7. *(III) Let $E = \mathbb{R}$ and $k \in \mathbb{N}$. Define $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \max\{|(x_i - x_j)(y_i - y_j)|,\ i, j \in \mathbb{N}_k \cup \{0\}, x_0, y_0 = 0\}$, where $x_1, \dots, x_k, y_1, \dots, y_k \in E$. We observe that $\{(E^k, \|.,.\|_k),\ k \in \mathbb{N}\}$ is a sequence of generalized 2-normed spaces, and $(MG1), (MG3)$ and $(MG4)$ are true. However we claim that $(MG2)$ does not hold, because obviously $\|M_\alpha(x), y\|_k \neq \|x, M_\alpha(y)\|_k$ ($x, y \in E^k, \alpha \in \mathbb{R}^k$) and moreover, $4 = \|(1, -1), (-1, 1)\|_2 \nleq \|(1, 1), (-1, 1)\|_2 = 1$ giving the claim.*
(IV) Let $(E, \|.,.\|)$ be a non-zero complex generalized 2-normed space. For each $k \in \mathbb{N}$, $x_1, \dots, x_k, y_1, \dots, y_k \in E$, define $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \max\{\|\eta_i x_i, \varepsilon_j y_j\|,\ i, j \in \mathbb{N}_k, \eta_i, \varepsilon_j \in \mathbb{T}\}$. Clearly, $\|.,.\|_k$ is a generalized 2-norm on E^k and axioms $(MG1), (MG3)$ and $(MG4)$ hold. Also $\|M_\alpha(x), y\|_k$, $\|x, M_\alpha(y)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|x, y\|_k$ for each $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$ and $x, y \in E^k$, but evidently $\|M_\alpha(x), y\|_k \neq \|x, M_\alpha(y)\|_k$.
(V) Let $(E, \|.,.\|)$ be a non-zero generalized 2-normed space. For each $k \in \mathbb{N}$, $x_1, \dots, x_k, y_1, \dots, y_k \in E$, define

$$\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \sup\{\max_{i \in \mathbb{N}_k} \|\sum_{j=1}^{k} \eta_j x_j, y_i\|,\ \eta_1, \dots, \eta_k \in \mathbb{S}\}.$$

Clearly, $\|.,.\|_k$ is a generalized 2-norm on E^k and $(MG1), (MG3)$ and $(DG4)$ hold. For $(DG4)$, we have

$$
\begin{aligned}
&\|(x_1,\dots,x_{k-1},x_{k-1}),(y_1,\dots,y_{k-1},y_{k-1})\|_k\\
&\quad= \sup\{\max_{i\in\mathbb{N}_{k-1}}\|\eta_1x_1+\cdots+\eta_{k-1}x_{k-1}+\eta_kx_{k-1},y_i\|,\ \eta_1,\dots,\eta_{k-1},\eta_k\in\mathbb{S}\}\\
&\quad= \sup\{\max_{i\in\mathbb{N}_{k-1}}\|\eta_1x_1+\cdots+\frac{\eta_{k-1}+\eta_k}{2}(2x_{k-1}),y_i\|,\ \eta_1,\dots,\frac{\eta_{k-1}+\eta_k}{2}\in\mathbb{S}\}\\
&\quad= \|(x_1,\dots,2x_{k-1}),(y_1,\dots,y_{k-1})\|_{k-1}.
\end{aligned}
$$

Further for nonzero $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{C}^k$,

$$
\begin{aligned}
\frac{1}{\max\limits_{i\in\mathbb{N}_k}|\alpha_i|}\|M_\alpha(x_1,\dots,x_k),(y_1,\dots,y_k)\|_k &= \frac{1}{\max\limits_{i\in\mathbb{N}_k}|\alpha_i|}\sup\{\max_{i\in\mathbb{N}_k}\|\sum_{j=1}^k\alpha_j\eta_jx_j,y_i\|,\eta_1,\dots,\eta_k\in\mathbb{S}\}\\
&= \sup\{\max_{i\in\mathbb{N}_k}\|\sum_{j=1}^k\eta'_jx_j,y_i\|,\ \eta'_j\in\mathbb{S},j\in\mathbb{N}_k\}\\
&= \|(x_1,\dots,x_k),(y_1,\dots,y_k)\|_k,
\end{aligned}
$$

where $\eta'_j=\frac{1}{\max\limits_{i\in\mathbb{N}_k}|\alpha_i|}\eta_j\alpha_j$ ($j\in\mathbb{N}_k$). This equality gives us the second part of (MG2). *Similarly one can quickly checked that*

$$\|(x_1,\dots,x_k),M_\alpha(y_1,\dots,y_k)\|_k\le\max_{1\le i\le k}|\alpha_i|\|(x_1,\dots,x_k),(y_1,\dots,y_k)\|_k,$$

but trivially $\|M_\alpha(x_1,\dots,x_k),(y_1,\dots,y_k)\|_k\neq\|(x_1,\dots,x_k),M_\alpha(y_1,\dots,y_k)\|_k$ and so (MG2) *does not hold in general.*
(VI) *Suppose that $E=\mathbb{C}$ and $\|z_1,z_2\|=2|z_1z_2|$ ($z_1,z_2\in E$). Then $(E,\|.,.\|)$ is a generalized* 2*-normed space. Assume that $\|(z_1,z_2),(w_1,w_2)\|_2=2|z_1w_1+z_2w_2|$, where $(z_1,z_2),(w_1,w_2)\in E^2$. It is a generalized* 2*-norm on E^2 such that satisfies in the axioms* (MG1), (MG3), (DG4) *and for each $(\alpha_1,\alpha_2)\in\mathbb{C}^2$, $\|M_\alpha(z_1,z_2),(w_1,w_2)\|_2=\|(z_1,z_2),M_\alpha(w_1,w_2)\|_2$. However the second part of axiom* (MG2) *does not hold. For instance, we have $4=\|(1,i),(1,-i)\|_2\nleq\|(1,1),(1,-i)\|_2=2\sqrt{2}$.*

Example 2.8. (VII) *Let $E=\mathbb{C}$, $\|x,y\|=|xy|$ and $\|(x_1,x_2),(y_1,y_2)\|_2=\frac{1}{2}(|x_1y_1|+|x_2y_2|)$, where $x,y,x_1,x_2,y_1,y_2\in E$. It is not hard to see that $(E,\|.,.\|)$ and $(E^2,\|.,.\|_2)$ are generalized* 2*-normed spaces and* (MG1), (MG2), (MG4) *are true but* (MG3) *is not.*
(VIII) *Suppose that $E=\mathbb{R}^2$ and $\|(x_1,y_1),(x_2,y_2)\|=|x_1y_2-y_1x_2|$, then $(E,\|.,.\|)$ is a generalized* 2*-normed space (see [18]). Define*
$\|((x_1,y_1),(x_2,y_2)),((z_1,w_1),(z_2,w_2))\|_2=2\,max\{\|(x_1,y_1),(z_1,w_1)\|,\|(x_2,y_2),(z_2,w_2)\|\}$.
We observe that $(E^2,\|.,.\|_2)$ is a generalized 2*-normed space and axioms* (MG1), (MG2) *are true. The calculation $\|((x_1,y_1),(x_1,y_1)),((z_1,w_1),(z_1,w_1))\|_2=2\|(x_1,y_1),(z_1,w_1)\|=\|2(x_1,y_1),(z_1,w_1)\|$ shows that* (DG4) *is also valid. On the other hand* (MG3) *does not hold, since $\|((1,1),(0,0)),((-1,1),(0,0))\|_2=4$ but $\|(1,1),(-1,1)\|=2$.*

Example 2.9. (IX) *Let $E=\mathbb{C}$, $\|x,y\|=|xy|$ and $\|(x_1,x_2),(y_1,y_2)\|_2=|x_1y_1|+|x_2y_2|$, where $x,y,x_1,x_2,y_1,y_2\in E$. It is immediately verified that $(E,\|.,.\|)$ and $(E^2,\|.,.\|_2)$ are generalized* 2*-normed spaces and* (MG1), (MG2), (MG3) *are true but* (MG4) *is not.*
(X) *Let $E=\mathbb{R}$. For $k\in\mathbb{N}$, and $x_1,\dots,x_k,y_1,\dots,y_k\in E$, define $\|(x_1,\dots,x_k),(y_1,\dots,y_k)\|_k=(\sum_{i=1}^k|x_iy_i|^2)^{\frac{1}{2}}$. Then $\{\|.,.\|_k,k\in\mathbb{N}\}$ is a special generalized* 2*-norm on $\{E^k,k\in\mathbb{N}\}$, but both of axioms* (MG4) *and* (DG4) *are not true.*

The four presented examples in the above are just in level 2. In the following lemma we assume $(E,\|.,.\|)$ is a generalized 2-normed space and $\{(E^k,\|.,.\|_k),k\in\mathbb{N}\}$ is a special generalized 2-normed space with $\|x,y\|_1=\|x,y\|$ for all $x,y\in E$. The proof is trivial and so is omitted (see [5, pp. 44-47]).

Lemma 2.10. *Let $j, k \in \mathbb{N}, x_1, \ldots, x_{j+k}, y_1, \ldots, y_{j+k} \in E$ and $\eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k \in \mathbb{T}$. Then*
(i) $\|(\eta_1 x_1, \ldots, \eta_k x_k), (\xi_1 y_1, \ldots, \xi_k y_k)\|_k = \|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_k$.
(ii) $\|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_k \leq \|(x_1, \ldots, x_k, x_{k+1}), (y_1, \ldots, y_k, y_{k+1})\|_{k+1}$.
(iii) $\|(x_1, \ldots, x_j, x_{j+1}, \ldots, x_{j+k}), (y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+k})\|_{j+k} \leq \|(x_1, \ldots, x_j), (y_1, \ldots, y_j)\|_j +$ $\|(x_{j+1}, \ldots, x_{j+k}), (y_{j+1}, \ldots, y_{j+k})\|_k$.
(iv) $\max_{i \in \mathbb{N}_k} \|x_i, y_i\| \leq \|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_k \leq \sum_{i=1}^{k} \|x_i, y_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i, y_i\|$.

The last part of the above lemma guides us to the the following result.

Corollary 2.11. *Suppose that $\{\|., .\|_k\}_{k \in \mathbb{N}}$ is a family of (dual) multi-generalized 2-norms on $\{E^k, k \in \mathbb{N}\}$, and $(E, \|., .\|_1)$ is a generalized 2-Banach space. Then for each $k \in \mathbb{N}$, $(E^k, \|., .\|_k)$ is a generalized 2-Banach space, too. In this case, $\{(E^k, \|., .\|_k), k \in \mathbb{N}\}$ is called a (dual) multi-generalized 2-Banach space.*

Lemma 2.12. *Let $\{(E^k, \|., .\|_k), k \in \mathbb{N}\}$ be a multi-generalized 2-normed space and $x_1, \ldots, x_{k-2}$, $x', x'', y_1, \ldots, y_{k-2}, y', y''$ be in E. Then*

$$\|(X, x', x''), (Y, y', y'')\|_k \leq \|(X, x', x''), (Y, y', y')\|_k + \|(X, x', x''), (Y, y'', y'')\|_k,$$

where $X = x_1, ..., x_{k-2}$, $Y = y_1, ..., y_{k-2}$.

Proof. Applying Lemma 2.10 and axiom $(MG1)$, we deduce that

$$\begin{aligned} \|(X, x', x''), (Y, y', y'')\|_k &\leq \|(X, x'), (Y, y')\|_{k-1} + \|x'', y''\| \\ &\leq \|(X, x', x''), (Y, y', y')\|_k + \|(x'', X, x'), (y'', Y, y'')\|_k \\ &= \|(X, x', x''), (Y, y', y')\|_k + \|(X, x', x''), (Y, y'', y'')\|_k, \end{aligned}$$

Therefore we get the desired result. □

The following lemma is a version of [5, Lemma 2.16] in the framework of multi-generalized 2-normed spaces.

Lemma 2.13. *Let $\{(E^k, \|., .\|_k), k \in \mathbb{N}\}$ be a multi-generalized 2-normed space, $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be in E^k, $x_{k+1}, x_{k+2}, y_{k+1}, y_{k+2}$ be in E and $a, b, p, q \in [0, 1]$ with $a + b = 1$, $p + q = 1$. Then*

$$\begin{aligned} &\|(x, ax_{k+1} + bx_{k+2}, ax_{k+1} + bx_{k+2}), (y, py_{k+1} + qy_{k+2}, py_{k+1} + qy_{k+2})\|_{k+2} \\ &\leq \|X, (y, y_{k+1}, y_{k+1})\|_{k+2} + \|X, (y, y_{k+2}, y_{k+2})\|_{k+2}, \end{aligned}$$

where $X = (x_1, ..., x_{k+2})$.

Proof. We have $(x, ax_{k+1}+bx_{k+2}, ax_{k+1}+bx_{k+2}) = a^2(x, x_{k+1}, x_{k+1})+ab(x, x_{k+1}, x_{k+2})+ab(x, x_{k+2}, x_{k+1})+b^2(x, x_{k+2}, x_{k+2})$. Similar relation holds when $x, x_{k+1}, x_{k+2}, a, b$ substitute with $y, y_{k+1}, y_{k+2}, p, q$, respectively. Applying Lemmata 2.10 and 2.12 and also axiom $(MG1)$, it follows that

$$\begin{aligned} &\|(x, ax_{k+1} + bx_{k+2}, ax_{k+1} + bx_{k+2}), (y, py_{k+1} + qy_{k+2}, py_{k+1} + qy_{k+2})\|_{k+2} \\ &\leq (a + b)^2 \|(x, x_{k+1}, x_{k+2}), (y, py_{k+1} + qy_{k+2}, py_{k+1} + qy_{k+2})\|_{k+2} \\ &= \|X, p^2(y, y_{k+1}, y_{k+1}) + pq(y, y_{k+1}, y_{k+2}) + pq(y, y_{k+2}, y_{k+1}) + q^2(y, y_{k+2}, y_{k+2})\|_{k+2} \\ &\leq p^2 \|X, (y, y_{k+1}, y_{k+1})\|_{k+2} + 2pq \|X, (y, y_{k+1}, y_{k+1})\|_{k+2} \\ &+ 2pq \|X, (y, y_{k+2}, y_{k+2})\|_{k+2} + q^2 \|X, (y, y_{k+2}, y_{k+2})\|_{k+2} \\ &= (p^2 + 2pq) \|X, (y, y_{k+1}, y_{k+1})\|_{k+2} + (q^2 + 2pq) \|X, (y, y_{k+2}, y_{k+2})\|_{k+2} \\ &\leq (p + q)^2 (\|X, (y, y_{k+1}, y_{k+1})\|_{k+2} + \|X, (y, y_{k+2}, y_{k+2})\|_{k+2}) \\ &= \|X, (y, y_{k+1}, y_{k+1})\|_{k+2} + \|X, (y, y_{k+2}, y_{k+2})\|_{k+2}. \end{aligned}$$

Note that the the second inequality in the above relation holds by Lemma 2.12. So the proof is complete. □

By slightly modification in the proof of [5, Lemmata 2.19, 2.22], and using Lemma 2.10, one gets the following proposition.

Proposition 2.14. *Let $\{(E^k, \|.,.\|_k), k \in \mathbb{N}\}$ be a dual multi-generalized 2-normed space and k' and n be arbitrary fixed elements in $\mathbb{N}$. Then for each $x_1, \ldots, x_{k'+n}, y_1, \ldots, y_{k'+1} \in E$, we have*
(i) $\|(x_1, \ldots, x_{k'}, x_{k'+1} + x_{k'+2} + \ldots + x_{k'+n}), (y_1, \ldots, y_{k'}, y_{k'+1})\|_{k'+1}$

$$\leq \|(x_1, \ldots, x_{k'}, x_{k'+1}, \ldots, x_{k'+n}), (y_1, \ldots, y_{k'}, y_{k'+1}, \ldots, y_{k'+1})\|_{k'+n}.$$

(ii) $\|(x_1, \ldots, x_{k'-2}, x_{k'-1} + x_{k'}), (y_1, \ldots, y_{k'-2}, y_{k'-1} + y_{k'})\|_{k'-1}$

$$\leq \|(x_1, \ldots, x_{k'-2}, x_{k'-1}, x_{k'}), (\alpha_1 y_1, \ldots, \alpha_{k'-2} y_{k'-2}, y_{k'-1}, y_{k'-1})\|_{k'}$$
$$+\|(x_1, \ldots, , x_{k'-2}, x_{k'-1}, x_{k'}), (\beta_1 y_1, \ldots, \beta_{k'-2} y_{k'-2}, y_{k'}, y_{k'})\|_{k'},$$

where $\alpha_i, \beta_i \geq 0$ and $\alpha_i + \beta_i = 1$, for each $i \in \mathbb{N}_{k'-2}$.
(iii) $\sup\{\|(\xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_{k'} x_{k'}), (\eta_1 y_1 + \ldots + \eta_{k'} y_{k'})\|, \xi_1, \ldots, \xi_{k'}, \eta_1, \ldots, \eta_{k'} \in \mathbb{T}\}$

$$\leq \|(x_1, \ldots, x_{k'}), (y_1, \ldots, y_1)\|_{k'} + \|(x_1, \ldots, x_{k'}), (y_2, \ldots, y_2)\|_{k'} + \ldots + \|(x_1, \ldots, x_{k'}), (y_{k'}, \ldots, y_{k'})\|_{k'}.$$

(iv) $\|(\alpha_1 x, \ldots, \alpha_k x), (y, \ldots, y)\|_k = \sum_{i=1}^{k} |\alpha_i| \|x, y\|$, *where $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and $x, y \in E$.*

3. Main Result

We are now in a position to state the main result of this note which is a version of [5, Proposition 2.7] in the framework of (dual) multi-generalized 2-normed spaces. We bring this result in two cases multi- and dual multi-generalized 2-normed spaces separately, because of avoiding long proof.

Theorem 3.1. *Let $(E, \|.,.\|)$ be a generalized 2-normed space. Let $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ be a sequence such that $\|.,.\|_k$ is a generalized 2-norm on E^k for each $k \in \mathbb{N}$ and $\|x, y\|_1 = \|x, y\|$ for all $x, y \in E$. Also axioms $(MG1)$, $(MG2)$ and $(MG4)$ are satisfied for each $k \in \mathbb{N}$. Then $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ is a multi-generalized 2-norm on $\{E^k, k \in \mathbb{N}\}$.*

Proof. By Definition 2.1, it is enough to show that axiom $(MG3)$ holds. For, let $k \in \mathbb{N}$, $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be in E^k such that $\|x, y\|_k = 1$. Set $\alpha = \|(x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, 0)\|_{k+1}$, so that $\alpha \leq 1$. Indeed, by axioms $(MG2)$ and $(MG4)$, we have

$$\begin{aligned}
\alpha &= \|M_{(1,\ldots,1,0)}(x_1, \ldots, x_k, x_k), M_{(1,\ldots,1,0)}(y_1, \ldots, y_k, y_k)\|_{k+1} \\
&\leq \|(x_1, \ldots, x_k, x_k), (y_1, \ldots, y_k, y_k)\|_{k+1} \\
&= \|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_k \\
&= 1.
\end{aligned}$$

Let n be any arbitrary fixed element in $\mathbb{N}$, take $x^{[n+2]}, y^{[n+2]} \in E^{(n+2)k}$, by $(MG1)$, $(MG4)$, $\|x^{[n+2]}, y^{[n+2]}\|_{(n+2)k} = \|x, y\|_k = 1$ (1). For $1 \leq i \leq n+2$, let B_i be the subset $\{(i-1)k+1, \ldots, ik\}$ of $\mathbb{N}_{(n+2)k}$, and let Q_{B_i} be a projection onto the complement of B_i. We thus find that $\|Q_{B_i}(x^{[n+2]}), Q_{B_j}(y^{[n+2]})\|_{(n+2)k} = \|Q_{B_i\cup B_j}(x^{[n+2]}), Q_{B_i\cup B_j}(y^{[n+2]})\|_{(n+2)k}$ (2), by $(MG2)$. Applying again axioms $(MG1)$ and $(MG4)$ we deduce that (2) is equal to α. Further, $\sum_{i=1}^{n+2} Q_{B_i}(x^{[n+2]}) = (n+1)x^{[n+2]}$ and $\sum_{j=1}^{n+2} Q_{B_j}(y^{[n+2]}) = (n+1)y^{[n+2]}$ and it follows from (1) that

$$\begin{aligned}
(n+1)^2 &= (n+1)^2 \|x^{[n+2]}, y^{[n+2]}\|_{(n+2)k} \\
&= \|(n+1)x^{[n+2]}, (n+1)y^{[n+2]}\|_{(n+2)k} \\
&\leq \sum_{i,j=1}^{n+2} \|Q_{B_i}(x^{[n+2]}), Q_{B_j}(y^{[n+2]})\|_{(n+2)k} \\
&= (n+2)^2 \alpha.
\end{aligned}$$

Therefore $\alpha \geq \frac{(n+1)^2}{(n+2)^2}$. Letting n tends to infinity, we obtain that $\alpha = 1$ and our goal is achieved. $\square$

Theorem 3.2. *Let $(E, \|.,.\|)$ be a generalized 2-normed space, $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ be a sequence such that $\|.,.\|_k$ be a generalized 2-norm on E^k for each $k \in \mathbb{N}$ and $\|x, y\|_1 = \|x, y\|$ for each $x, y \in E$. Also $(MG1)$, $(MG2)$ and $(DG4)$ are satisfied for each $k \in \mathbb{N}$. Then $\{\|.,.\|_k\}_{k\in\mathbb{N}}$ is a dual multi-generalized 2-norm on $\{E^k, k \in \mathbb{N}\}$.*

Proof. Let $k \in \mathbb{N}$, and $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k)$ be in E^k. For convenience, by β we denote the real number $\|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_k$ and by α the real number $\|(x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, 0)\|_{k+1}$. If $\beta = 0$, then

$$\begin{aligned}
0 \leq \alpha &= \|(x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, 0)\|_{k+1} \\
&= \|M_{(1,\ldots,1,0)}(x_1, \ldots, x_k, x_k), M_{(1,\ldots,1,0)}(y_1, \ldots, y_k, y_k)\|_{k+1} \\
&\leq \|(x_1, \ldots, x_k, x_k), (y_1, \ldots, y_k, y_k)\|_{k+1} \quad (MG2) \\
&= \|(x_1, \ldots, 2x_k), (y_1, \ldots, y_k)\|_k \quad (DG4) \\
&\leq 2\|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_k \quad (MG2) \\
&= 2\beta = 0.
\end{aligned}$$

It forces that $\alpha = 0$ too. Now assume that β is nonzero and n is an arbitrary fixed element of $\mathbb{N}$, then $x^{[2^n]}, y^{[2^n]}$ are in $E^{(2^n)k}$ and so by axioms $(MG1)$ and $(DG4)$, $\|x^{[2^n]}, y^{[2^n]}\|_{(2^n)k} = 2^n\beta$ (3). For $i = 1, \ldots, 2^n$, let B_i be the subset $\{(i-1)k+1, \ldots, ik\}$ of $\mathbb{N}_{(2^n)k}$, and let Q_{B_i} be a projection onto the complement of B_i. From $(MG2)$, it yields that $\|Q_{B_i}(x^{[2^n]}), Q_{B_j}(y^{[2^n]})\|_{(2^n)k} = \|Q_{B_i\cup B_j}(x^{[2^n]}), Q_{B_i\cup B_j}(y^{[2^n]})\|_{(2^n)k}$ (4).
Using $(MG1)$, $(MG2)$ and $(DG4)$ we deduce that the equality (4) is less than or equal to $2^n\alpha$. Further, $\sum_{i=1}^{2^n} Q_{B_i}(x^{[2^n]}) = (2^n-1)x^{[2^n]}$ and $\sum_{j=1}^{2^n} Q_{B_j}(y^{[2^n]}) = (2^n-1)y^{[2^n]}$ and it follows from (3) that

$$\begin{aligned}
(2^n-1)^2 &= \frac{(2^n-1)^2\|x^{[2^n]}, y^{[2^n]}\|_{(2^n)k}}{2^n\beta} \\
&= \frac{\|(2^n-1)x^{[2^n]}, (2^n-1)y^{[2^n]}\|_{(2^n)k}}{2^n\beta} \\
&= \frac{\|\sum_{i=1}^{2^n} Q_{B_i}(x^{[2^n]}), \sum_{j=1}^{2^n} Q_{B_j}(y^{[2^n]})\|_{(2^n)k}}{2^n\beta} \\
&\leq \frac{\sum_{i,j=1}^{2^n}\|Q_{B_i}(x^{[2^n]}), Q_{B_j}(y^{[2^n]})\|_{(2^n)k}}{2^n\beta} \\
&\leq \frac{(2^n)^2 2^n\alpha}{2^n\beta} \\
&= \frac{(2^n)^2\alpha}{\beta}.
\end{aligned}$$

Therefore $\alpha \geq \frac{(2^n-1)^2\beta}{(2^n)^2}$. Since this is true for any n, so letting $n \to \infty$, then $\alpha \geq \beta$.
For the reverse direction assume that $x = (x_1, \ldots, x_k, 0)$ and $y = (y_1, \ldots, y_k, 0)$. Then $\|x^{[2^n]}, y^{[2^n]}\|_{2^n(k+1)} = 2^n\alpha$. For $i = 1, \ldots, 2^n$, let $C_i = \{i(k+1)-k, \ldots, i(k+1)\}$ and let Q_{C_i} be a projection onto the complement of C_i. Next, put
$X_1 = (x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k, 0, \ldots, 0)$,
$Y_1 = (y_1, \ldots, y_k, \ldots, y_1, \ldots, y_k, 0, \ldots, 0)$,
where the number of repetitions of each item x_i and y_i, $i = 1, \ldots, k$ is $2^n - 2$ and also zero has repeated $(2^n - 2) + 2(k + 1)$ times in each of X_1 and Y_1.

$X_2 = (x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k, 0, \ldots, 0)$,
$Y_2 = (y_1, \ldots, y_k, \ldots, y_1, \ldots, y_k, 0, \ldots, 0)$,
where the number of repetitions of each item x_i and y_i, $i = 1, \ldots, k$ is $2^n - 2$ and also zero has repeated $2k$ times in each of X_2 and Y_2.
Finally, set $\gamma = (1, \ldots, 1, 0, \ldots, 0)$, where 1 has repeated $(2^n - 2)k$ times and zero has repeated $2k$ times. Then

$$\begin{aligned} \|Q_{C_i}(x^{[2^n]}), Q_{C_j}(y^{[2^n]})\|_{2^n(k+1)} &= \|Q_{C_i \cup C_j}(x^{[2^n]}), Q_{C_i \cup C_j}(y^{[2^n]})\|_{2^n(k+1)} \\ &= \|X_1, Y_1\|_{2^n(k+1)} \\ &= \|X_2, Y_2\|_{2^n k} \\ &= \|M_\gamma x^{[2^n]}, M_\gamma y^{[2^n]}\|_{2^n k} \\ &\leq 2^n \beta. \quad \text{(by(MG2))} \end{aligned}$$

It is easily verified that $\sum_{i=1}^{2^n} Q_{C_i}(x^{[2^n]}) = (2^n - 1)x^{[2^n]}$ and $\sum_{j=1}^{2^n} Q_{C_j}(y^{[2^n]}) = (2^n - 1)y^{[2^n]}$. It follows that

$$\begin{aligned} (2^n - 1)^2 &= \frac{\|(2^n - 1)x^{[2^n]}, (2^n - 1)y^{[2^n]}\|_{2^n(k+1)}}{2^n \alpha} \\ &= \frac{\|\sum_{i=1}^{2^n} Q_{C_i}(x^{[2^n]}), \sum_{j=1}^{2^n} Q_{C_j}(y^{[2^n]})\|_{2^n(k+1)}}{2^n \alpha} \\ &\leq \frac{\sum_{i,j=1}^{2^n} \|Q_{C_i}(x^{[2^n]}), Q_{C_j}(y^{[2^n]})\|_{2^n(k+1)}}{2^n \alpha} \\ &\leq \frac{(2^n)^2 2^n \beta}{2^n \alpha}. \end{aligned}$$

Hence, $\alpha \leq \frac{2^{2n}}{(2^n-1)^2}\beta$. Letting $n \to \infty$, we conclude that $\alpha \leq \beta$. Therefore $\alpha = \beta$ and so we get our desired result. □

4. Application

In this section we give an application of multi-generalized 2-normed spaces. For this purpose, it is convenient to make a few observation about H^*-algebras (see [2]).

Definition 4.1. *An H^*-algebra, introduced by W. Ambrose [2] in the associative case, is a Banach algebra A, satisfying the following conditions:*
(i) A is itself a Hilbert space under an inner product $\langle ., . \rangle$;
(ii) For each a in A there is an element a^* in A, the so-called adjoint of a, such that we have both $\langle ab, c \rangle = \langle b, a^*c \rangle$ and $\langle ab, c \rangle = \langle a, cb^* \rangle$ for all $b, c \in A$. Recall that $A_0 = \{a \in A,\ aA = \{0\}\} = \{a \in A : Aa = \{0\}\}$ is called the annihilator ideal of A. A proper H^*-algebra is an H^*-algebra with zero annihilator ideal. Ambrose proved that an H^*-algebra is proper if and only if every element has a unique adjoint. The trace-class $\tau(A)$ of A is defined by the set $\tau(A) = \{ab,\ a, b \in A\}$. The trace functional tr on $\tau(A)$ is defined by $tr(ab) = \langle a, b^* \rangle = \langle b, a^* \rangle = tr(ba)$ for each $a, b \in A$, in particular $tr(aa^*) = \langle a, a \rangle = \|a\|^2$, for all $a \in A$. A nonzero element $e \in A$ is called a projection, if it is self-adjoint and idempotent. In addition, if $eAe = \mathbb{C}e$, then it is called a minimal projection. For example each simple H^*-algebra (an H^*-algebra without nontrivial closed two-sided ideals) contains minimal projections. Two idempotents e and e' are doubly orthogonal if $\langle e, e' \rangle = 0$ and $ee' = e'e = 0$. Suppose that e is a minimal projection in a commutative, proper H^*-algebra A, then $Ae = eAe = \mathbb{C}e$. Recall that if $\{e_i\}_{i \in I}$ is a maximal family of doubly orthogonal minimal projections in a proper H^*-algebra A, then A is the direct sum of the minimal left ideals Ae_i or the minimal right ideals e_iA [2, Theorem 4.1]. If M is a subset of an H^*-algebra A, then we mean by $M^\perp$ the orthogonal complement of M. For more details on H^*-algebras, see [4, 19] and references cited therein.

Example 4.2. *Let $(E, \|.\|)$ be an H^*-algebra. We know that E^k $(k \in \mathbb{N})$ is an H^*-algebra where the linear operations are considered componentwise and moreover $\langle (x_1, \dots, x_k), (y_1, \dots, y_k) \rangle = \sum_{i=1}^{k} \langle x_i, y_i \rangle$, $(x_1, \dots, x_k)^* = (x_1^*, \dots, x_k^*)$. Define a generalized 2-norm on E^k by setting $\|(x_1, \dots, x_k), (y_1, \dots, y_k)\|_k = \sum_{i=1}^{k} |\langle x_i, y_i \rangle|$. Then $\{(E^k, \|., .\|_k), k \in \mathbb{N}\}$ is a dual multi-generalized 2-normed space. Furthermore we can improve the axiom $(MG3)$ as follow:*
$(MG'3)$ Let $(E, \|.\|)$ be a proper commutative H^*-algebra, $\{e_i\}_{i \in I}$ be a maximal family of doubly orthogonal minimal projections in E, and $\{(E^k, \|., .\|_k), k \in \mathbb{N}\}$ be the dual multi-generalized 2-normed space as the above example. For each $x = (x_1, \dots, x_{k-1}, x_k)$ and $y = (y_1, \dots, y_{k-1}, y_k)$ in E^k, if $x_k y_k = 0$, then

$$\|(x_1, \dots, x_{k-1}, x_k), (y_1, \dots, y_{k-1}, y_k)\|_k = \|(x_1, \dots, x_{k-1}), (y_1, \dots, y_{k-1})\|_{k-1}.$$

The last equality is true by the definition of $\|., .\|_k$ and the equality $|\langle x_k, y_k \rangle| = tr(x_k y_k^*) = 0$. Note that if $y_k = \sum_i \lambda_i e_i$ $(\lambda_i \in \mathbb{C})$, then $y_k^* = \sum_i \overline{\lambda_i} e_i$. By virtue of this fact one can see that $x_k y_k^* = 0$ too.

Definition 4.3. *Let $(E, \|.\|)$ be a proper commutative H^*-algebra, $\{e_i\}_{i \in I}$ be a maximal family of doubly orthogonal minimal projections in E, and x be an arbitrary element in E. The least ideal of E containing x, is called x-ideal of E and it is denoted by I_x. Now if $x = \sum_i \lambda_i e_i$ for some $\lambda_i \in \mathbb{C}$, then clearly I_x generated by e_i s' with nonzero coefficient which appear in the expansion of x in terms of $\{e_i\}_{i \in I}$.*

Theorem 4.4. *Suppose that $(E, \|.\|)$ is a commutative proper H^*-algebra, $\{(E^k, \|., .\|_k), k \in \mathbb{N}\}$ is the dual multi-generalized 2-normed space as Example 4.2, and $k \in \mathbb{N}$. Let $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ be in E^k.*
(i) If there is at least $i \in \mathbb{N}_k$ in which $x_i y_i \neq 0$ and I_{x_i} or I_{y_i} is not the whole of E, then there exists $k_0 \in \mathbb{N}_k$ and a nonzero element $z = (z_1, \dots, z_{k_0}) \in E^{k_0}$ with $z_i \neq x_i, y_i$, $(i = 1, \dots, k_0)$ and $\|(x_1 z_1, \dots, x_k z_k), (y_1, \dots, y_k)\|_k = \|(x_1 z_1, \dots, x_{k_0} z_{k_0}), (y_1, \dots, y_{k_0})\|_{k_0} = 0$ (5).
(ii) If $I_{\sum_{i=1}^k x_i}$ or $I_{\sum_{i=1}^k y_i}$ are not equal whole of E, then we can select equal components for z in the preceding part.

Proof. (i) By $(MG1)$ and $(MG'3)$, there exists $k_0 \in \mathbb{N}_k$ such that $\|(x_1, \dots, x_{k_0}, \dots, x_k), (y_1, \dots, y_{k_0}, \dots, y_k)\|_k = \|(x_1, \dots, x_{k_0}), (y_1, \dots, y_{k_0})\|_{k_0}$ and $x_i y_i \neq 0$ $(i = 1, \dots k_0)$. Now if by assumption $I_{x_i}{}^{\perp} \cup I_{y_i}{}^{\perp} \neq \{0\}$ for some $i = 1, \dots k_0$, then it suffices to take z_i any nonzero element of this set, otherwise get $z_i = 0$. Clearly in the first case $|\langle x_i z_i, y_i \rangle| = 0$, since if $z_i \in I_{x_i}{}^{\perp}$, then $z_i x_i \in I_{x_i} \cap I_{x_i}{}^{\perp} = \{0\}$ and if $z_i \in I_{y_i}{}^{\perp}$ then $\langle x_i z_i, y_i \rangle = \langle x_i, y_i z_i^* \rangle = 0$, the last equality holds by virtue of the fact that $I_{y_i}{}^{\perp}$ is a self adjoint ideal and $y_i z_i^* \in I_{y_i} \cap I_{y_i}{}^{\perp} = \{0\}$. Take $z = (z_1, \dots, z_{k_0}) \in E^{k_0}$, by the above results z is nonzero and also fulfills condition (5). Next we are going to show the $z_i \neq x_i, y_i$ for $i = 1, \dots, k_0$. This is obvious if $z_i = 0$ (note that x_i and y_i are nonzero for each $i = 1, \dots k_0$). In the case that z_i is nonzero, first let $z_i \in I_{x_i}{}^{\perp}$. Then $z_i \neq x_i$ and $x_i y_i \neq 0$ implies that y_i does not belong to $I_{x_i}{}^{\perp}$, so $z_i \neq y_i$. A similar argument shows that $z_i \neq x_i, y_i$, if $z_i \in I_{y_i}{}^{\perp}$.
(ii) It is enough to get $z_i's$ $(i = 1, \dots, k_0)$ equal to an arbitrary element of $(I_{\sum_{i=1}^k x_i})^{\perp} \cup (I_{\sum_{i=1}^k y_i})^{\perp}$. Evidently $I_{\sum_{i=1}^k x_i}$ is the ideal generated by all minimal projections e_i's that appear in the expansion $x_i s'$ $(i = 1, \dots, k)$ with nonzero coefficients. This fact causes that $I_{x_i} \subseteq I_{\sum_{i=1}^k x_i}$. Thus the result follows by the preceding part. □

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