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Constraint Qualifications in Nonsmooth Multiobjective Optimization Problem

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Abstract. A multiobjective optimization problem (MOP) with inequality and equality constraints is considered where the objective and inequality constraint functions are locally Lipschitz and equality constraint functions are differentiable. Burachik and Rizvi [J. Optim. Theory Appl. 155, 477-491 (2012)] gave Guignard and generalized Abadie regularity conditions for a differentiable programming problem and derived Karush-Kuhn-Tucker (KKT) type necessary optimality conditions. In this paper, we have defined the nonsmooth versions of Guignard and generalized Abadie regularity conditions given by Burachik and Rizvi and obtained KKT necessary optimality conditions for efficient and weak efficient solutions of (MOP). Further several constraint qualifications sufficient for the above newly defined constraint qualifications are introduced for (MOP) with no equality constraints. Relationships between them are presented and examples are constructed to support the results.

1. Introduction

In optimization theory KKT conditions are very significant in order to find an optimal solution. These optimality conditions may fail if the constraint functions do not satisfy certain conditions. These conditions are known as constraint qualifications. In vector optimization, very frequently conditions are also imposed on all or some of the components of objective function along with the constraints in order that KKT conditions hold at an efficient or weak efficient point. These conditions are generally referred to as regularity conditions in vector optimization but some authors like Maeda [10] and Li [8] have also used the word constraint qualifications. If the Lagrange multiplier corresponding to a component of an objective function is positive then it indicates an active role of that component in identifying the solutions of the problem. When at least one (all) component(s) of the objective function is (are) active in determining the optimal solution then weak (strong) KKT conditions are said to hold. In literature, researchers introduced regularity conditions/constraint qualifications in order to get weak (strong) KKT conditions.

In this direction, Maeda [10] was the first to introduce generalized Guignard constraint qualification (GGCQ) and obtained strong KKT conditions using (GGCQ) for a differentiable programming problem with only inequality constraints. In addition, several other constraint qualifications are proposed in [10] and all of them are shown to be sufficient for (GGCQ). Further, Preda and Chitescu [11], Li [8], Chandra

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et al.[3], Giorgi et al.[5, 6] and many others have extended the work of Maeda [10] in different formats of objective and constraint functions. All the above authors have used the Q^i sets given by Maeda [10] to define different constraint qualifications.

In 2009, Rizvi et al.[12] introduced generalized Guignard constraint qualification (GGCQ) by using M^i sets which are easily determinable than the sets Q^i as they depend only on a single component (i) of the objective function. Rizvi et al.[12] then derived strong KKT conditions for an efficient solution of a differentiable programming problem using (GGCQ). Recently Burachik and Rizvi [2] used the M^i sets introduced by Rizvi et al.[12] to define regularity conditions named as Guignard regularity condition (GRC) and generalized Abadie regularity condition (GARC). Weak KKT conditions for an efficient solution using (GRC) and strong KKT conditions for Geoffrion proper efficient solution using (GARC) are obtained in [2] for a multiobjective programming problem with only inequality constraints where the functions are assumed to be continuously differentiable.

Motivated by the above research work, in this paper we define nonsmooth versions of (GRC) and (GARC) introduced by Burachik and Rizvi [2]. These newly defined versions are called generalized Guignard constraint qualification (GGCQ) and generalized Abadie constraint qualification (GACQ) respectively. (GACQ) introduced here is easily determinable than (GARC). The results of this paper are more general than the results existing in literature in the sense that here we use M^i sets which are easily determinable than the sets Q^i and the functions involved are locally Lipschitz.

This paper is organized in 4 sections as follows:
Section 2 includes some notations and definitions which will be used throughout the paper. Section 3 considers a multiobjective optimization problem (MOP) with both inequality and equality constraints. Two constraint qualifications (GACQ) and (GGCQ) are introduced and weak KKT conditions for weak efficient and efficient solutions of (MOP) are obtained using these newly defined constraint qualifications. Section 4 considers (MOP) with only inequality constraints and the new problem is called (MP). Several constraint qualifications are proposed for (MP) and relationship amongst them is investigated.

Examples are constructed to illustrate the results obtained.

2. Notations and Preliminaries

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.
For any $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$, we define
(1) $x = y$ iff $x_i = y_i \ \forall i = 1, \dots, n$,
(2) $x < y$ iff $x_i < y_i \ \forall i = 1, \dots, n$,
(3) $x \leqq y$ iff $x_i \leq y_i \ \forall i = 1, \dots, n$,
(4) $x \leq y$ iff $x \leqq y$ and $x \neq y$.
Note: If $x, y \in \mathbb{R}$, then we use $x \leq y$ to denote x is less than or equal to y.

Let S be a subset of $\mathbb{R}^n$. Then clS, coS, $coneS$ and $linS$ denote the closure, convex hull, cone generated and subspace generated by S respectively. Let $B_\delta(x_0)$ be the open ball centered at x_0 and radius $\delta > 0$, $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

Definition 2.1. Let S be a non empty subset of $\mathbb{R}^n$. The tangent cone to S at $\bar{x} \in clS$ is the set defined by

$$T(S;\bar{x}) = \{d \in \mathbb{R}^n \mid \exists \{d_n\} \to d, \{t_n\} \downarrow 0 \ s.t. \ \bar{x} + t_n d_n \in S \ \forall n\}.$$

Two equivalent expressions of tangent cone are given as

$$T(S;\bar{x}) = \{d \in \mathbb{R}^n \mid \exists \{x_n\} \subseteq S \text{ with } x_n \to \bar{x}, \{t_n\} \subseteq \mathbb{R}_{++} \ s.t. \ t_n(x_n - \bar{x}) \to d\},$$

$$T(S;\bar{x}) = \{d \in \mathbb{R}^n \mid \exists \{x_n\} \subseteq S \text{ with } x_n \to \bar{x}, \{t_n\} \downarrow 0 \ s.t. \ {t_n}^{-1}(x_n - \bar{x}) \to d\}.$$

The set $T(S;\bar{x})$ is a non empty closed cone but not necessarily convex. If S is convex, then $T(S;\bar{x})$ is convex.

Definition 2.2. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz* at a point $x_0 \in \mathbb{R}^n$ if there exist real numbers $k > 0$ and $\delta > 0$ such that for all $x, y \in B_\delta(x_0)$, we have

$$|\phi(x) - \phi(y)| \leq k\|x - y\|.$$

ϕ is said to be locally Lipschitz on $\mathbb{R}^n$ if it is locally Lipschitz at each point of $\mathbb{R}^n$.

Definition 2.3. [4] Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then the *Clarke's generalized directional derivative* of ϕ at $x_0 \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $\phi^o(x_0; d)$ and is given as

$$\phi^o(x_0; d) = \limsup_{\substack{x \to x_0 \\ t \downarrow 0}} \frac{\phi(x + td) - \phi(x)}{t}.$$

It is well known that $v \to \phi^o(x_0; v)$ is a continuous sublinear function on $\mathbb{R}^n$.
The *Clarke's generalized subdifferential* of a locally Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ is given by

$$\partial^c \phi(x_0) = \{\xi \in \mathbb{R}^n : \phi^o(x_0; d) \geq \langle \xi, d \rangle \ \forall\, d \in \mathbb{R}^n\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a vector valued function. Then f is said to be locally Lipschitz on $\mathbb{R}^n$ if each $f_i, i = 1, 2, \ldots, p$ is locally Lipschitz on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a locally Lipschitz function. Then the Clarke's generalized directional derivative of f at $x_0 \in \mathbb{R}^n$ in the direction d is given by

$$f^o(x_0; d) = ({f_1}^o(x_0; d), {f_2}^o(x_0; d), \ldots, {f_p}^o(x_0; d)).$$

The Clarke's generalized subdifferential of f at $x_0 \in \mathbb{R}^n$ is the set

$$\partial^c f(x_0) = \partial^c f_1(x_0) \times \partial^c f_2(x_0) \times \ldots \times \partial^c f_p(x_0).$$

Now we recall the definitions of ∂^c-pseudoconvex and ∂^c-quasiconvex functions which are the non-smooth generalizations of usual notions of pseudoconvex and quasiconvex functions for differentiable functions.

Definition 2.4. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then ϕ is said to be ∂^c*-quasiconvex* at $x_0 \in \mathbb{R}^n$ if for all $x \in \mathbb{R}^n$, we have

$$\phi(x) \leq \phi(x_0) \quad \Rightarrow \quad \phi^o(x_0; x - x_0) \leq 0$$

or

$$\phi^o(x_0; x - x_0) > 0 \quad \Rightarrow \quad \phi(x) > \phi(x_0).$$

ϕ is said to be ∂^c-quasiconvex on $\mathbb{R}^n$ if it is ∂^c-quasiconvex at each element of $\mathbb{R}^n$. The function ϕ is said to be ∂^c-quasiconcave at x_0 if $-\phi$ is ∂^c-quasiconvex at x_0.

Definition 2.5. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then ϕ is said to be ∂^c*-pseudoconvex* at $x_0 \in \mathbb{R}^n$ if for all $x \in \mathbb{R}^n$, we have

$$\phi(x) < \phi(x_0) \quad \Rightarrow \quad \phi^o(x_0; x - x_0) < 0.$$

The function ϕ is said to be ∂^c-pseudoconcave at x_0 if $-\phi$ is ∂^c-pseudoconvex at x_0.

Definition 2.6. A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be quasilinear if it is both quasiconvex and quasiconcave.

Now from [1], we have the following lemma:

Lemma 2.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be ∂^c-quasiconvex. Then ϕ is quasiconvex.*

It is well known that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex iff α-lower level set $L_\alpha(f) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is a convex set for each $\alpha \in \mathbb{R}$. Therefore lower level sets of a ∂^c-quasiconvex function are also convex.

3. Constraint Qualification and Kuhn Tucker Conditions

We consider the following multiobjective optimization problem:

$$\text{(MOP)} \qquad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad g(x) \leqq 0,$$
$$h(x) = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are locally Lipschitz functions on $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^r$ is differentiable function on $\mathbb{R}^n$. Let $I = \{1, 2, \ldots, p\}$, $J = \{1, 2, \ldots, m\}$, $K = \{1, 2, \ldots, r\}$, $S = \{x \in \mathbb{R}^n : g(x) \leqq 0, h(x) = 0\}$ be the feasible set of (MOP) and $J(x) = \{j \in J : g_j(x) = 0\}$ be the set of active indices of g at x.

Definition 3.1. A point $\bar{x} \in S$ is said to be an efficient solution of (MOP) if there exists no $y \in S$ such that

$$f(y) \leq f(\bar{x}).$$

Definition 3.2. A point $\bar{x} \in S$ is said to be a weak efficient solution of (MOP) if there exists no $y \in S$ such that

$$f(y) < f(\bar{x}).$$

Now we discuss the Karush-Kuhn-Tucker necessary optimality conditions for the problem (MOP). To obtain these KKT conditions for a multiobjective programming problem, one may need to impose some conditions not only on the constraints but also on some or all of the objective function components. In literature these conditions are referred to as constraint qualifications or regularity conditions. There are several types of constraint qualifications which are imposed on the constraints like Abadie, Guignard, Slater, Mangasarian-Fromovitz, etc.

T.Maeda [10] first generalized the Guignard constraint qualification and called it generalized Guignard constraint qualification (GGCQ). For a fixed $\bar{x} \in S$, Maeda [10] considered the sets of following type:

$$Q^i(\bar{x}) = \{x \in \mathbb{R}^n : f_s(x) \leq f_s(\bar{x}), s = 1, 2, \ldots, p, s \neq i, g(x) \leqq 0, h(x) = 0\}, \quad i = 1, \ldots, p.$$

$$Q(\bar{x}) = \{x \in \mathbb{R}^n : f(x) \leqq f(\bar{x}), g(x) \leqq 0, h(x) = 0\}.$$

In Maeda [10], (GGCQ) was dependent on the sets Q^i. Later on sets M^i were introduced which depend only on a single component of objective function. For a fixed $\bar{x} \in S$, Rizvi et al.[12] defined M^i as the set

$$M^i(\bar{x}) = \{x \in \mathbb{R}^n : f_i(x) \leq f_i(\bar{x}), g(x) \leqq 0, h(x) = 0\}, \quad i = 1, \ldots, p$$

and

$$M(\bar{x}) = \{x \in \mathbb{R}^n : f(x) \leqq f(\bar{x}), g(x) \leqq 0, h(x) = 0\}.$$

Now we give the definitions of following linearizing cones given in relation to (MOP).

$$L(M^i(\bar{x}); \bar{x}) = \{d \in \mathbb{R}^n : f_i^o(\bar{x}; d) \leq 0, g_j^o(\bar{x}; d) \leq 0 \ \forall j \in J(\bar{x}), \nabla h_k(\bar{x}) d = 0 \ \forall k \in K\}, \ i = 1, \ldots, p.$$

$$L(M(\bar{x}); \bar{x}) = \{d \in \mathbb{R}^n : f_i^o(\bar{x}; d) \leq 0 \ \forall i \in I, g_j^o(\bar{x}; d) \leq 0 \ \forall j \in J(\bar{x}), \nabla h_k(\bar{x}) d = 0 \ \forall k \in K\}.$$

Lemma 3.1. *Let $\bar{x} \in S$. Then $L(M(\bar{x}); \bar{x})$ is a closed convex cone.*

Proof. The lemma can easily be proved by using the fact that $v \to f_i^o(\bar{x}; v)$ $\forall i \in I$, $v \to g_j^o(\bar{x}; v)$ $\forall j \in J(\bar{x})$ are continuous sublinear functions and $\nabla h_k(\bar{x})(\cdot)$ $\forall k \in K$ are linear functions. □

Lemma 3.2. *Let $\bar{x}$ be a feasible solution of (MOP) and that f_i, g_j, $i \in I$ and $j \in J(\bar{x})$ be ∂^c-quasiconvex functions respectively at $\bar{x}$. Then*

$$\bigcap_{i=1}^{p} cl\ co\ T(M^i(\bar{x}); \bar{x}) \subseteq L(M(\bar{x}); \bar{x}).$$

Proof. We know that

$$L(M(\bar{x});\bar{x}) = \bigcap_{i=1}^{p} L(M^i(\bar{x});\bar{x}).$$

Now we first show that for each $i = 1,\dots,p$,

$$T(M^i(\bar{x});\bar{x}) \subseteq L(M^i(\bar{x});\bar{x}).$$

Let $d \in T(M^i(\bar{x});\bar{x})$. Then there exist sequences $\{x_n\} \subseteq M^i(\bar{x})$ and $\{t_n\} \subseteq \mathbb{R}_{++}$ such that

$$x_n \to \bar{x} \text{ and } t_n(x_n - \bar{x}) \to d.$$

Assume that $t_n(x_n - \bar{x}) = d_n$ for each n. Then for all $j \in J(\bar{x})$ and $n \in \mathbb{N}$, we get

$$g_j(x_n) = g_j\left(\bar{x} + \frac{d_n}{t_n}\right) \leq 0 = g_j(\bar{x})$$

and $$f_i(x_n) = f_i\left(\bar{x} + \frac{d_n}{t_n}\right) \leq f_i(\bar{x}).$$

Using above two inequalities and ∂^c-quasiconvexity of $f_i, g_j, j \in J(\bar{x})$, we get

$$g_j^o(\bar{x};d) \leq 0 \ \ \forall\, j \in J(\bar{x}), \tag{3.1}$$

$$f_i^o(\bar{x};d) \leq 0 \tag{3.2}$$

as $f_i^o(\bar{x};\cdot)$ and $g_j^o(\bar{x};\cdot)$, $j \in J(\bar{x})$ are positively homogenous and continuous functions. Now for $k \in K$, we have

$$h_k(x_n) = h_k\left(\bar{x} + \frac{d_n}{t_n}\right) = 0 = h_k(\bar{x}).$$

Because h_k is differentiable for each $k \in K$, therefore we have

$$h_k\left(\bar{x} + \frac{d_n}{t_n}\right) = h_k(\bar{x}) + \nabla h_k(\bar{x})\left(\frac{d_n}{t_n}\right) + o\left(\frac{\|d_n\|}{t_n}\right), \text{ where } \lim_{n\to\infty} \frac{o\left(\frac{\|d_n\|}{t_n}\right)}{\left(\frac{\|d_n\|}{t_n}\right)} = 0$$

$$\Rightarrow \qquad \nabla h_k(\bar{x})\left(\frac{d_n}{t_n}\right) + \frac{o\left(\frac{\|d_n\|}{t_n}\right)}{\frac{\|d_n\|}{t_n}}\frac{\|d_n\|}{t_n} = 0.$$

Multiplying by t_n and taking limit as $n \to \infty$, we get

$$\nabla h_k(\bar{x})d = 0 \quad \forall\, k \in K, \tag{3.3}$$

because $\{d_n\}$ is bounded.
Using (3.1), (3.2) and (3.3), we get

$$d \in L(M^i(\bar{x});\bar{x}).$$

Therefore, $$T(M^i(\bar{x});\bar{x}) \subseteq L(M^i(\bar{x});\bar{x}) \tag{3.4}$$

$$\Rightarrow \qquad cl\ co\ T(M^i(\bar{x});\bar{x}) \subseteq cl\ co\ L(M^i(\bar{x});\bar{x}).$$

Now as $L(M^i(\bar{x});\bar{x})$ is closed and convex, we get

$$cl\ co\ T(M^i(\bar{x});\bar{x}) \subseteq L(M^i(\bar{x});\bar{x}). \tag{3.5}$$

Since we have proved (3.5) for arbitrary $i \in I$, we get

$$\textstyle\bigcap_{i=1}^{p} cl\ co\ T(M^i(\bar{x});\bar{x}) \subseteq \bigcap_{i=1}^{p} L(M^i(\bar{x});\bar{x}) = L(M(\bar{x});\bar{x}).$$

Hence the result is proved. □

In general the reverse inclusions in (3.4) and (3.5) of above lemma do not hold as can be seen from the following example.

Example 3.1. Consider the problem

$$\begin{array}{lll} \text{(MOP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ & \text{s.t.} & g(x) \leq 0, \\ & & h(x) = 0, \end{array}$$

where $f_1, f_2, g, h : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = \begin{cases} x^2, & x < 1 \\ 1, & x \geq 1 \end{cases}, \; f_2(x) = \begin{cases} -1, & x < -1 \\ x^3, & x \geq -1 \end{cases}, \; g(x) = \begin{cases} -x, & x < 0 \\ -x^3, & x \geq 0 \end{cases}, \; h(x) = \begin{cases} -x^2, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

Here feasible set $S = [0, \infty)$. So, let us take $\bar{x} = 0$. Now $M^1(0) = M^2(0) = \{0\}$. Therefore,

$$T(M^i(0); 0) = \{0\} = cl\ co\ T(M^i(0); 0), \quad i = 1, 2.$$

Since $f_1^o(0; d) = f_2^o(0; d) = 0\ \forall d \in \mathbb{R}, g^o(0; d) \leq 0\ \forall d \geq 0$ and $\nabla h(0) = 0$, therefore

$$L(M^i(0); 0) = \mathbb{R}_+, \ i = 1, 2.$$

Hence it is easy to see that the reverse inclusions in (3.4) and (3.5) are not satisfied at $\bar{x} = 0$ for any of the indices 1 or 2.

Therefore on the lines of Burachik and Rizvi [2], in order to obtain necessary conditions for a feasible solution of (MOP) to be an efficient/weak efficient solution, we assume the following:

$$L(M^i(\bar{x}); \bar{x}) \subseteq cl\ co\ T(M^i(\bar{x}); \bar{x}) \quad \text{for at least one } i \in I \tag{3.6}$$

and

$$L(M^i(\bar{x}); \bar{x}) \subseteq T(M^i(\bar{x}); \bar{x}) \quad \text{for at least one } i \in I. \tag{3.7}$$

(3.6) and (3.7) are called (GGCQ) and (GACQ) respectively. These constraint qualifications are nonsmooth analogues of Guignard and generalized Abadie regularity conditions given in Burachik and Rizvi [2]. If $K = \emptyset$, then (GGCQ) reduces to Guignard regularity condition given by Burachik and Rizvi [2] in continuously differentiable case. It is easy to see that

$$\text{(GACQ)} \;\Rightarrow\; \text{(GGCQ)}.$$

First we prove the following theorem in order to prove the necessary conditions for efficient solution of (MOP).

Theorem 3.1. *Let $\bar{x}$ be an efficient solution of (MOP). Suppose that (GACQ) holds at $\bar{x}$ and for each $i \in I$, f_i is ∂^c-quasiconcave at $\bar{x}$, then the system*

$$f^o(\bar{x}; d) < 0, \tag{3.8}$$

$$g_j^o(\bar{x}; d) \leq 0 \ \ \forall\ j \in J(\bar{x}), \tag{3.9}$$

$$\nabla h(\bar{x}) d = 0, \tag{3.10}$$

has no solution $d \in \mathbb{R}^n$.

Proof. Let us suppose that the system given by (3.8)-(3.10) has a solution $d \in \mathbb{R}^n$. Then we have

$$f_i^o(\bar{x}; d) < 0\ \forall\ i \in I, \tag{3.11}$$

$$g_j^o(\bar{x}; d) \leq 0\ \forall\ j \in J(\bar{x}),$$
$$\nabla h_k(\bar{x}) d = 0\ \forall\ k \in K.$$

By the definition of linearizing cone to $M^i(\bar{x})$ at the point $\bar{x}$, we get that

$$d \in L(M^i(\bar{x}); \bar{x}) \quad \forall\, i \in I.$$

As (GACQ) holds at $\bar{x}$, we have

$$d \in T(M^{\hat{i}}(\bar{x}); \bar{x}) \quad \text{say for some } \hat{i} \in I.$$

By the definition of tangent cone, there exist sequences $\{x_n\} \subseteq M^{\hat{i}}(\bar{x})$ and $\{t_n\} \subseteq \mathbb{R}_{++}$ such that

$$x_n \to \bar{x} \ \text{ and } \ t_n(x_n - \bar{x}) \to d.$$

Let $t_n(x_n - \bar{x}) = d_n \ \forall\, n$. Then $d_n \to d$. Since $\{x_n\} \subseteq M^{\hat{i}}(\bar{x})$ therefore for each $n \in \mathbb{N}$ we have

$$f_{\hat{i}}(x_n) \leq f_{\hat{i}}(\bar{x}).$$

We now claim that for each $n \in \mathbb{N}$, $\exists r \neq \hat{i}$ such that

$$f_r(x_n) \geq f_r(\bar{x}). \tag{3.12}$$

If the claim is not true, then there exists at least one $n_0 \in \mathbb{N}$ for which we have

$$f_r(x_{n_0}) < f_r(\bar{x}) \ \ \forall\, r \neq \hat{i},$$

which contradicts the efficiency of $\bar{x}$. Hence for each $n \in \mathbb{N}$, we have at least one index $r \neq \hat{i}$ such that (3.12) is true. Let for each $n \in \mathbb{N}$, I_n denotes the set of all such indices. Then $I_n \neq \emptyset$ and $I_n \subseteq I \setminus \{\hat{i}\}$ for each $n \in \mathbb{N}$. Consider the mapping $\psi : \mathbb{N} \to P$, where P denotes the power set of the finite set $I \setminus \{\hat{i}\}$ and is defined as $\psi(n) = I_n$. Now the set $\mathbb{N}$ is infinite whereas P is finite, so there exists a subset A in the range of ψ at which infinite number of elements of $\mathbb{N}$ are mapped. Let this infinite set be $\mathbb{N}_0$, then for each $j \in A$, we have

$$f_j(x_n) \geq f_j(\bar{x}) \ \ \forall\, n \in \mathbb{N}_0.$$

For a particular $j_0 \in A$, we have

$$f_{j_0}(x_n) \geq f_{j_0}(\bar{x}) \ \ \forall\, n \in \mathbb{N}_0 \subseteq \mathbb{N}.$$

Since f_{j_0} is ∂^c-quasiconcave at $\bar{x}$, therefore we obtain

$$f^o_{j_0}(\bar{x}; d) \geq 0,$$

as $f^o_{j_0}(\bar{x}; \cdot)$ is positively homogenous and continuous. This contradicts (3.11). Hence given system has no solution. □

Theorem 3.2. *Let $\bar{x}$ be an efficient solution of (MOP) at which (GACQ) holds and for each $i \in I$, f_i be ∂^c- quasiconcave at $\bar{x}$. Assume that*

$$D = cone\ co \left\{ \bigcup_{j \in J(\bar{x})} \partial^c g_j(\bar{x}) \right\} + lin\{\nabla h_k(\bar{x}) : k \in K\} \tag{3.13}$$

is closed. Then there exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$, $\lambda \geq 0$, $\mu \geqq 0$ such that

$$\sum_{i=1}^{p} \lambda_i f^o_i(\bar{x}; v) + \sum_{j=1}^{m} \mu_j g^o_j(\bar{x}; v) + \sum_{k=1}^{r} \nu_k \nabla h_k(\bar{x}) v \geq 0 \ \ \forall\, v \in \mathbb{R}^n, \tag{3.14}$$

$$\mu_j g_j(\bar{x}) = 0 \ \ \forall j \in J. \tag{3.15}$$

Proof. Since $\bar{x}$ is an efficient solution of (MOP) at which all the conditions of Theorem 3.1 hold, therefore by Theorem 3.1, the system given by (3.8) - (3.10) has no solution $d \in \mathbb{R}^n$.
As $f^o_i(\bar{x}; \cdot), i \in I$ and $g^o_j(\bar{x}; \cdot), j \in J(\bar{x})$ are sublinear functions, $\nabla h_k(\bar{x})(\cdot), k \in K$ are linear functions and D is closed, therefore by Theorem 3.13 in Jimenez and Novo [7], there exist $\lambda \geq 0$, $\mu \geqq 0$ such that

$$\sum_{i=1}^{p} \lambda_i f^o_i(\bar{x}; v) + \sum_{j \in J(\bar{x})} \mu_j g^o_j(\bar{x}; v) + \sum_{k=1}^{r} \nu_k \nabla h_k(\bar{x}) v \geq 0 \ \ \forall\, v \in \mathbb{R}^n.$$

Taking $\mu_j = 0$ for $j \notin J(\bar{x})$, we get the desired result. □

Now the next two theorems give necessary optimality conditions for weak efficient solution of problem (MOP) by using (GGCQ).

Theorem 3.3. *Let $\bar{x}$ be a weak efficient solution of (MOP) at which (GGCQ) holds. Suppose that for each $i \in I$, f_i is ∂^c- quasiconcave at $\bar{x}$ and feasible set S is convex, then the system given by (3.8) - (3.10) has no solution $d \in \mathbb{R}^n$.*

Proof. Let us suppose that the system given by (3.8)-(3.10) has a solution $d \in \mathbb{R}^n$. Then

$$d \in L(M^i(\bar{x}); \bar{x}) \quad \forall\, i \in I.$$

Since (GGCQ) holds at $\bar{x}$, therefore

$$d \in cl\ co\ T(M^{\hat{i}}(\bar{x}); \bar{x}) \quad \text{say for some } \hat{i} \in I. \tag{3.16}$$

Now we know that for each $i \in I$, $M^i(\bar{x}) \subseteq S$. Therefore in particular for $\hat{i} \in I$,

$$M^{\hat{i}}(\bar{x}) \subseteq S$$

$$\Rightarrow \qquad cl\ co\ T(M^{\hat{i}}(\bar{x}); \bar{x}) \subseteq cl\ co\ T(S; \bar{x}). \tag{3.17}$$

Since S is convex, therefore $T(S; \bar{x})$ is convex. Also as the tangent cone is a closed cone, we have

$$cl\ co\ T(S; \bar{x}) = T(S; \bar{x}).$$

Using (3.16), (3.17) and above equality, we obtain $d \in T(S; \bar{x})$. By definition of tangent cone, there exist sequences $\{x_k\} \subseteq S$ and $\{t_k\} \subseteq \mathbb{R}_{++}$ such that

$$x_k \to \bar{x} \quad \text{and} \quad t_k(x_k - \bar{x}) \to d.$$

Let $d_k = t_k(x_k - \bar{x})\ \forall k \in \mathbb{N}$, then $d_k \to d$.
From (3.8), we have that

$$f_i^o(\bar{x}; d) < 0\ \forall\ i \in I.$$

As $f_i^o(\bar{x}; \cdot)$ is continuous and sublinear for each $i \in I$, therefore for sufficiently large values of k, we get

$$f_i^o(\bar{x}; \frac{d_k}{t_k}) < 0\ \forall\ i \in I.$$

Since $f_i's$, for all $i \in I$ are ∂^c-quasiconcave, therefore we obtain that for sufficiently large values of k,

$$f_i(x_k) < f_i(\bar{x})\ \forall\ i \in I,$$

which contradicts the fact that $\bar{x}$ is a weak efficient solution of (MOP). Hence system given by (3.8)-(3.10) has no solution. □

Theorem 3.4. *Let $\bar{x}$ be a weak efficient solution of (MOP) at which all the conditions of Theorem 3.3 are satisfied. Also assume that the cone given by (3.13) is closed. Then there exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$, $\lambda \geq 0$, $\mu \geqq 0$ such that (3.14) and (3.15) hold.*

Proof. Using Theorem 3.3, proof follows on the lines of Theorem 3.2. □

Now we give the following example to illustrate Theorems 3.3 and 3.4.

Example 3.2. Consider the problem

$$\begin{array}{lrl} \text{(MOP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ & \text{s.t.} & g(x) \leq 0, \\ & & h(x) = 0, \end{array}$$

where $f_1, f_2, g, h : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = \begin{cases} -x, & x \leq 0 \\ -3x, & x > 0 \end{cases},\ f_2(x) = \min\{-|x|, -x^2\},\ g(x) = \begin{cases} x^3, & x \leq 0 \\ -2x, & x > 0 \end{cases},\ h(x) = \begin{cases} 0, & x \leq 0 \\ -x^2, & x > 0. \end{cases}$$

Here feasible set $S = (-\infty, 0]$ which is a convex set and each feasible point is a weak efficient solution. So, let us take $\bar{x} = 0$. It is easy to check by using definitions that f_1, f_2 are ∂^c-quasiconcave functions at $\bar{x}$. Also as $f_2^o(0; d) \leq 0$ for $d = 0$, $g^o(0; d) \leq 0\ \forall d \geq 0$, $\nabla h(0) = 0$ and $M^2(0) = (-\infty, 0]$, therefore

$$L(M^2(0); 0) = \{0\} \quad \text{and} \quad cl\ co\ T(M^2(0); 0) = \mathbb{R}_-$$

showing that (GGCQ) is satisfied at $\bar{x}$. Thus, all the conditions of Theorem 3.3 are satisfied at $\bar{x} = 0$ and it can be seen that the system given by (3.8)-(3.10) has no solution $d \in \mathbb{R}$ as $f_2^o(0; d) \nless 0$ for any $d \in \mathbb{R}$. Therefore the conclusion of Theorem 3.3 holds.
Now the cone D given by (3.13) is $\mathbb{R}_-$ here which is obviously closed and there exist multipliers $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu = 1$, $\nu \in \mathbb{R}$ such that (3.14) and (3.15) hold. Therefore necessary optimality conditions are satisfied.

Remark 3.1. Since $\bar{x} = 0$ is an efficient solution of (MOP) at which (GACQ) is satisfied, therefore above example supports Theorems 3.1 and 3.2 also.

Remark 3.2. In Theorems 3.3 and 3.4, necessary optimality conditions are proved for a weak efficient solution of problem (MOP) under (GGCQ). Since every efficient solution is a weak efficient solution, therefore conditions (3.14) and (3.15) hold true for efficient solution as well under (GGCQ).

Remark 3.3. If $\bar{x}$ is a weak efficient solution of (MOP) at which (GACQ) is satisfied and also $f_i, i \in I$ are ∂^c-quasiconcave at $\bar{x}$, then moving on the same lines as in Theorem 3.3, it can be proved that the system given by (3.8)-(3.10) has no solution $d \in \mathbb{R}^n$. Further if we assume that cone D given by (3.13) is closed, then moving on the lines of Theorem 3.2, we get that (3.14) and (3.15) hold. Hence KKT conditions are satisfied at a weak efficient solution of (MOP) under (GACQ).

Remark 3.4. If in (MOP), h_k is quasilinear for each $k \in K$ and g_j is ∂^c-quasiconvex for each $j \in J$, then by using Lemma 2.1 and the fact that lower level sets of a quasiconvex function are convex sets, we get that feasible set S is convex.

We now give the following example to illustrate the importance of Theorem 3.3 as the more relaxed constraint qualification (GGCQ) is used here. We will show in this example that there are cases where only (GGCQ) is satisfied but not (GACQ).

Example 3.3. Consider the problem

$$\begin{array}{lll} \text{(MOP)} & \text{Minimize} & f(x, y) = (f_1(x, y), f_2(x, y)) \\ & \text{s.t.} & g(x, y) \leq 0, \end{array}$$

where $f_1, f_2, g : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$f_1(x, y) = (|y| - x)(|y| - 2x),\ f_2(x, y) = \left(|y| - \frac{x}{2}\right)\left(|y| - \frac{x}{4}\right) \text{ and } g(x, y) = |y| - 2x.$$

Here feasible set $S = \{(x, y) \in \mathbb{R}^2 : |y| - 2x \leq 0\}$ which is a convex set and $\bar{x} = (0, 0)$ is an efficient as well as weak efficient solution of (MOP). Now from the definition of g, we can see that $\partial^c g(\bar{x}) = co\{(-2, -1), (-2, 1)\}$ and hence

$$g^o(\bar{x}; (d_1, d_2)) = \begin{cases} -2d_1 + d_2, & d_2 \geq 0, d_1 \in \mathbb{R} \\ -2d_1 - d_2, & d_2 < 0, d_1 \in \mathbb{R}. \end{cases}$$

Similarly, as $\partial^c f_i(\bar{x}) = \{(0, 0)\}$ for $i = 1, 2$, therefore

$$f_i^o(\bar{x}; (d_1, d_2)) = 0 \ \ \forall\ (d_1, d_2) \in \mathbb{R}^2.$$

Hence, for $i = 1, 2$

$$\begin{aligned} L(M^i(\bar{x}); \bar{x}) &= \{(d_1, d_2) \in \mathbb{R}^2 : f_i^o(\bar{x}; (d_1, d_2)) \leq 0, g^o(\bar{x}; (d_1, d_2)) \leq 0\} \\ &= \{(d_1, d_2) \in \mathbb{R}^2 : d_2 \leq 2d_1 \text{ and } d_2 \geq -2d_1\} \end{aligned}$$

and it is shown in Fig.1.

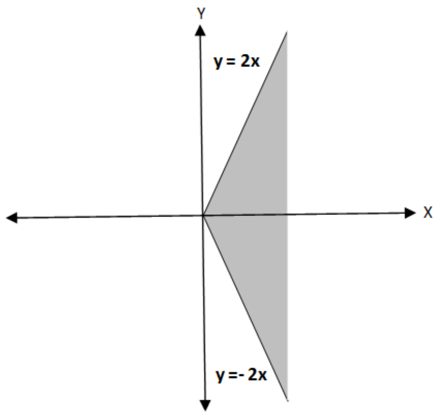

Fig.1 Shaded region represents $L(M^i(\bar{x});\bar{x}),\ i = 1,2$

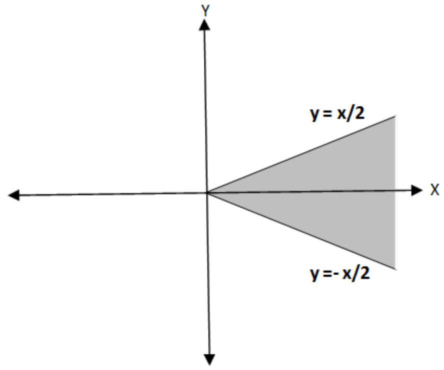

Fig.2 Shaded region represents $cl\ co\ T(M^2(\bar{x});\bar{x})$

Now,

$$M^1(\bar{x}) = \{(x,y) \in S : (|y| - x)(|y| - 2x) \leq 0\},$$

$$M^2(\bar{x}) = \{(x,y) \in S : \left(|y| - \tfrac{x}{2}\right)\left(|y| - \tfrac{x}{4}\right) \leq 0\}$$

and the corresponding regions have been shown in Fig.3 and Fig.4 respectively.

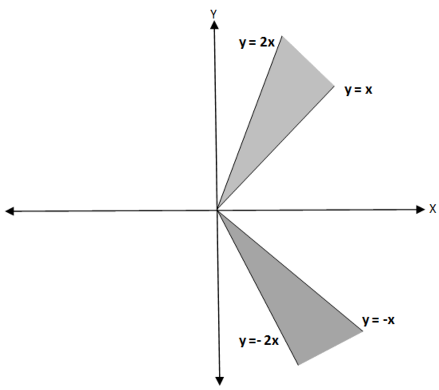

Fig.3 Shaded region represents $M^1(\bar{x})$

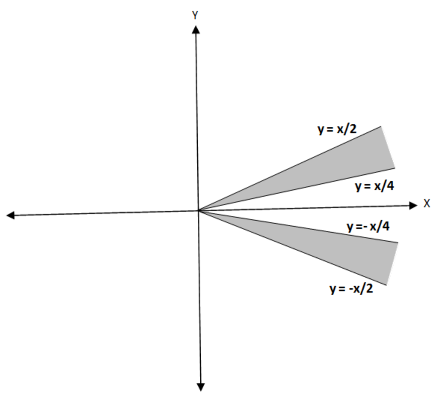

Fig.4 Shaded region represents $M^2(\bar{x})$

It is easy to see from Fig.3 and Fig.4 that $T(M^1(\bar{x});\bar{x}) = M^1(\bar{x})$ and $T(M^2(\bar{x});\bar{x}) = M^2(\bar{x})$ and both are non convex sets. So, $cl\ co\ T(M^2(\bar{x});\bar{x})$ is shown in Fig.2 and obviously from figures Fig.1, Fig.2 and Fig.4, we can see that

$$L(M^2(\bar{x});\bar{x}) \not\subseteq T(M^2(\bar{x});\bar{x}) \ \text{ and } \ L(M^2(\bar{x});\bar{x}) \not\subseteq cl\ co\ T(M^2(\bar{x});\bar{x}).$$

From Fig.3, it can be seen that $cl\ co\ T(M^1(\bar{x});\bar{x})$ is as shown in Fig.1. Therefore

$$L(M^1(\bar{x});\bar{x}) = cl\ co\ T(M^1(\bar{x});\bar{x}) \ \text{ but } \ L(M^1(\bar{x});\bar{x}) \not\subseteq T(M^1(\bar{x});\bar{x}).$$

Hence (GGCQ) is satisfied at $\bar{x}$ but (GACQ) does not hold at $\bar{x}$. Also using Definition 2.4, it is easy to check that f_1, f_2 are ∂^c-quasiconcave at $\bar{x}$. Therefore, all the conditions of Theorem 3.3 are satisfied and the system given by (3.8)-(3.10) has no solution $d \in \mathbb{R}^2$. Now as the cone D given by (3.13) is

$$D = \left\{(d_1, d_2) \in \mathbb{R}^2 : d_2 \geq \frac{1}{2}d_1 \text{ and } d_2 \leq -\frac{1}{2}d_1\right\},$$

which is obviously a closed cone, therefore by Theorem 3.4, there exist multipliers $\lambda_1 = 1$, $\lambda_2 = 2$ and $\mu = 0$ for which (3.14) and (3.15) hold. Hence KKT conditions are satisfied at $\bar{x}$.

There may exist a multiobjective programming problem where all the conditions of Theorem 3.1 are satisfied but feasible set is not convex. In such cases Theorem 3.1 can only be applied to get KKT conditions as shown in the next example.

Example 3.4. Consider the problem

$$\begin{array}{lll} \text{(MOP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ & \text{s.t.} & g(x) = (g_1(x), g_2(x)) \leqq 0, \end{array}$$

where $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}, \quad f_2(x) = \begin{cases} -x, & x < 0 \\ -2x, & x \ge 0 \end{cases}, \quad g_1(x) = \begin{cases} x, & x < 0 \\ 2x^2, & 0 \le x \le 1 \\ 3 - x, & x > 1 \end{cases}, \quad g_2(x) = -|x| - 1.$$

Here feasible set $S = (-\infty, 0] \bigcup [3, \infty)$ which is obviously a non convex set. It is easy to see that $\bar{x} = 0$ is an efficient solution of (MOP) and f_1, f_2 are ∂^c-quasiconcave at $\bar{x}$. Also

$$L(M^2(0); 0) = \{0\} \text{ and } T(M^2(0); 0) = cl\ co\ T(M^2(0); 0) = \{0\},$$

showing that (GACQ) and (GGCQ) are satisfied at $\bar{x}$. Hence Theorem 3.3 can't be applied here as feasible set is not convex but Theorem 3.1 can be applied and we get that system given by (3.8)-(3.10) has no solution. Now the cone D given by (3.13) is $\mathbb{R}_+$ which is a closed set, therefore by Theorem 3.2, there exist multipliers $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = 1$ and $\mu_2 = 0$ such that (3.14) and (3.15) hold. Hence KKT conditions are satisfied at $\bar{x}$.

Remark 3.5. Since tangent cone always contain the origin 0, therefore if for any $\bar{x} \in S$, $L(M^i(\bar{x}); \bar{x}) = \{0\}$ for at least one $i \in I$, then (GACQ) and (GGCQ) are automatically satisfied at $\bar{x}$.

Now we give conditions that allow us to make sure that the cone given by (3.13) is closed.

Proposition 3.1. *Consider the following:*

(*a*) *If*

$$0 \notin co\left(\bigcup_{j \in J(\bar{x})} \partial^c g_j(\bar{x})\right) + lin\{\nabla h_k(\bar{x}) : k \in K\} \tag{3.18}$$

then the cone D is closed.

(*b*) *If* $C(S; \bar{x}) = \{d \in \mathbb{R}^n : g_j^o(\bar{x}; d) < 0\ \forall j \in J(\bar{x}), \nabla h_k(\bar{x})d = 0\ \forall k \in K\} \neq \emptyset$, *then the cone D is closed.*

Proof. Part (a) follows from proposition 3.6 in [7] by using the fact that $g_j^o(\bar{x}; \cdot)$, $j \in J(\bar{x})$ are sublinear functions, $\nabla h_k(\bar{x})(\cdot)$, $k \in K$ are linear functions and for each $j \in J(\bar{x})$, $\partial g_j^o(\bar{x}; \cdot)(0) = \partial^c g_j(\bar{x})$ where $\partial g_j^o(\bar{x}; \cdot)(0)$ denotes the convex subdifferential of a convex function $g_j^o(\bar{x}; \cdot)$ at 0.

If (b) holds, then (3.18) is also satisfied. To prove it let us suppose that (b) holds but (3.18) does not hold. So, let $d \in C(S; \bar{x})$. Then

$$g_j^o(\bar{x}; d) < 0\ \forall\ j \in J(\bar{x}), \tag{3.19}$$

$$\nabla h_k(\bar{x})d = 0\ \forall\ k \in K. \tag{3.20}$$

Since (3.18) does not hold, therefore

$$0 = \sum_{i=1}^{n+1} \mu_i \zeta_i + \sum_{k \in K} \nu_k \nabla h_k(\bar{x}), \tag{3.21}$$

where $\mu_i \geq 0, \sum_{i=1}^{n+1} \mu_i = 1$, $\zeta_i \in \cup_{j\in J(\bar{x})}\partial^c g_j(\bar{x})$, $i = 1, \ldots, n+1$ and $\nu_k \in \mathbb{R}\ \forall k \in K$. Now (3.21) implies that

$$0 = \sum_{i=1}^{n+1} \mu_i \langle \zeta_i, d\rangle + \sum_{k\in K} \nu_k \langle \nabla h_k(\bar{x}), d\rangle . \tag{3.22}$$

Since $\zeta_i \in \cup_{j\in J(\bar{x})}\partial^c g_j(\bar{x})$ for each $i = 1, \ldots, n+1$, therefore for each i, $\zeta_i \in \partial^c g_j(\bar{x})$ for at least one $j \in J(\bar{x})$ say $\partial^c g_{j_i}(\bar{x})$. Then using (3.19) and (3.20), (3.22) gives

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n+1} \mu_i g^o_{j_i}(\bar{x}; d) + \sum_{k\in K} \nu_k \langle \nabla h_k(\bar{x}), d\rangle \\ &< 0, \end{aligned}$$

which is absurd. Therefore, if (b) holds then (3.18) is true and hence conclusion follows from part(a). □

We now give the following example to illustrate Remark 3.5 and Proposition 3.1.

Example 3.5. Consider the problem

$$\begin{array}{lll} \text{(MOP)} \quad \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ \qquad\qquad \text{s.t.} & g(x) = (g_1(x), g_2(x)) \leqq 0, \\ & h(x) = 0, \end{array}$$

where $f_1, f_2, g_1, g_2, h : \mathbb{R} \to \mathbb{R}$ are given by $f_1(x) = -|x|$,

$$f_2(x) = \begin{cases} -x^3, & x < 1 \\ -1, & x \geq 1 \end{cases}, \ g_1(x) = \begin{cases} x - x^2, & x < 0 \\ 2x, & x \geq 0, \end{cases}, \ g_2(x) = \begin{cases} -x-1, & x < 0 \\ -2x-1, & x \geq 0 \end{cases}, \ h(x) = \begin{cases} 0, & x < 0 \\ -x^3, & x \geq 0. \end{cases}$$

Here feasible set $S = [-1, 0]$. Let us consider $\bar{x} = 0$, then it can be seen that

$$L(M^1(0); 0) = \{0\}, \ \ T(M^1(0); 0) = cl\ co\ T(M^1(0); 0) = \mathbb{R}_-.$$

Hence both (GACQ) and (GGCQ) are satisfied at $\bar{x} = 0$ as $L(M^1(0); 0) = \{0\}$ showing that Remark 3.5 holds. It can be verified here that $L(M^2(0); 0) \neq \{0\}$.
Since $\partial^c g_1(0) = [1, 2]$ and $\nabla h(0) = 0$, therefore cone $D = \mathbb{R}_+$ which is obviously closed and we can see that both the conditions (a) and (b) given in Proposition 3.1 are satisfied at $\bar{x}$ as right hand side of (3.18) is $[1, 2]$ and every $d < 0$ is an element of $C(S; \bar{x})$.

4. Constraint Qualifications Sufficient for (GACQ) and hence for (GGCQ)

In this section, we consider the problem (MOP) with $K = \emptyset$ and call it as a problem (MP) which is defined as

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) \\ & \text{subject to} & g(x) \leqq 0. \end{array}$$

Let $X = \{x \in \mathbb{R}^n : g(x) \leqq 0\}$ be the feasible set of problem (MP) and $x \in X$.

On the lines of Li and Zhang [9], we propose some constraint qualifications for the problem (MP) which need less computation than the constraint qualifications given in [9] and show that these qualifications are sufficient for (GACQ) and hence (GGCQ) considered in this paper with $K = \emptyset$. Consequently, they validate also the Kuhn-Tucker necessary optimality conditions for efficient and weak efficient solutions of the problem (MP).

(GACQ) Generalized Abadie Constraint Qualification

$$L(M^i(\bar{x}); \bar{x}) \subseteq T(M^i(\bar{x}); \bar{x}) \quad \text{for at least one } i \in I.$$

(GGCQ) Generalized Guignard Constraint Qualification

$$L(M^i(\bar{x}); \bar{x}) \subseteq cl\ co\ T(M^i(\bar{x}); \bar{x}) \quad \text{for at least one } i \in I.$$

(GCCQ) Generalized Cottle Constraint Qualification
For at least one $i \in I$, the system

$$f_i^o(x; d^i) < 0, \tag{4.1}$$

$$g_j^o(x; d^i) < 0, \quad j \in J(x) \tag{4.2}$$

has a solution $d^i \in \mathbb{R}^n$.

(GSCQ) Generalized Slater Constraint Qualification
For at least one $i \in I$, the system

$$f_i(y^i) < f_i(x),$$

$$g_j(y^i) < g_j(x), \quad j \in J(x)$$

has a solution $y^i \in \mathbb{R}^n$. Also for each $i \in I$ and $j \in J(x)$, f_i, g_j are ∂^c-pseudoconvex at x respectively.

(GLCQ) Generalized Linear Constraint Qualification
The functions f_i, for at least one $i \in I$ and g_j, for all $j \in J(x)$ are ∂^c-pseudoconcave at x.

(GLOCQ) Generalized Linear Objective Constraint Qualification
For each $i \in I$, f_i is ∂^c-pseudoconcave at x and for at least one $k \in I$, the system

$$f_k^o(x; d^k) \leq 0,$$

$$g_j^o(x; d^k) < 0, \quad j \in J(x)$$

has a solution $d^k \in \mathbb{R}^n$.

(GMFCQ) Generalized Mangasarian Fromovitz Constraint Qualification
For at least one $i \in I$, the system

$$f_i^o(x; v^i) \leq 0, \tag{4.3}$$

$$g_j^o(x; v^i) < 0, \quad j \in J(x) \tag{4.4}$$

has a solution $v^i \in \mathbb{R}^n$ and $f_i^o(x; u^i) < 0$ for some $u^i \in \mathbb{R}^n$.

Remark 4.1. By definition, it follows that (GACQ) $\Rightarrow$ (GGCQ).

Proposition 4.1. *(GCCQ) implies (GACQ).*

Proof. Let (GCCQ) holds at $x \in X$. Then, for at least one $j \in I$, (4.1) and (4.2) have a solution d^j. Now let $d \in L(M^j(x); x)$. Then we show that

$$d \in T(M^j(x); x).$$

Since d^j is a solution to the jth system given in (GCCQ), therefore we have

$$f_j^o(x; d^j) < 0,$$

$$g_i^o(x; d^j) < 0, \quad i \in J(x).$$

Since $f_j^o(x; \cdot)$ and $g_i^o(x; \cdot)$, $i \in J(x)$ are sublinear and $d \in L(M^j(x); x)$, therefore for any fixed real number $\bar{t} > 0$, we have

$$f_j^o(x; d + \bar{t}d^j) \leq f_j^o(x; d) + \bar{t} f_j^o(x; d^j) < 0, \tag{4.5}$$

$$g_i^o(x; d + \bar{t}d^j) \leq g_i^o(x; d) + \bar{t}g_i^o(x; d^j) < 0, \quad i \in J(x). \tag{4.6}$$

Using the definition of Clarke's generalized directional derivative, we have from inequalities (4.5) and (4.6) that there exists a sequence $\{t_n\}$ with $t_n \downarrow 0$ such that for all sufficiently large n

$$f_j(x + t_n(d + \bar{t}d^j)) < f_j(x), \tag{4.7}$$

$$g_i(x + t_n(d + \bar{t}d^j)) < g_i(x) = 0, \quad i \in J(x). \tag{4.8}$$

Now as $g_i(x) < 0$ for $i \notin J(x)$ and $g_i,\ i \in J$ are locally Lipschitz continuous, therefore for sufficiently large values of n, we get that

$$g_i(x + t_n(d + \bar{t}d^j)) < 0, \quad i \notin J(x). \tag{4.9}$$

On using the definition of $M^j(x)$, inequalities (4.7)-(4.9) give that for all n sufficiently large

$$x + t_n(d + \bar{t}d^j) \in M^j(x),$$

which by definition of tangent cone implies that

$$d + \bar{t}d^j \in T(M^j(x); x).$$

Since $\bar{t} > 0$ is arbitrary, therefore

$$\{d + td^j : t > 0, t \in \mathbb{R}\} \subseteq T(M^j(x); x).$$

As the set $T(M^j(x); x)$ is closed and $d \in cl\ \{d + td^j : t > 0, t \in \mathbb{R}\}$, we get that

$$d \in T(M^j(x); x).$$

Hence $L(M^j(x); x) \subseteq T(M^j(x); x)$ for a particular $j \in I$. Thus (GACQ) holds at x. □

Proposition 4.2. (*GSCQ*) *implies* (*GCCQ*).

Proof. Let (GSCQ) holds at $x \in X$, so that the ith system given in (GSCQ) has a solution y^i. Then

$$f_i(y^i) < f_i(x),$$
$$g_j(y^i) < g_j(x), \quad j \in J(x).$$

Since for each $i \in I$ and $j \in J(x)$, f_i and g_j are ∂^c-pseudoconvex at x respectively, therefore we have

$$f_i^o(x; y^i - x) < 0,$$
$$g_j^o(x; y^i - x) < 0, \quad j \in J(x)$$

showing that $d^i = y^i - x$ is a solution to the ith system given by (4.1) and (4.2) in (GCCQ). Hence the result is proved. □

Proposition 4.3. (*GLCQ*) *implies* (*GACQ*).

Proof. Let (GLCQ) holds at $x \in X$. Suppose that $i_0 \in I$ for which f_{i_0} is ∂^c-pseudoconcave at x and $d \in L(M^{i_0}(x); x)$. Then it is sufficient to prove that $d \in T(M^{i_0}(x); x)$. Since $d \in L(M^{i_0}(x); x)$, therefore we have

$$f_{i_0}^o(x; d) \leq 0,$$

$$g_j^o(x; d) \leq 0, \quad j \in J(x).$$

Now for any sequence $\{t_n\}$ with $t_n \downarrow 0$, we get

$$f_{i_0}^o(x; x + t_n d - x) = f_{i_0}^o(x; t_n d) \leq 0,$$

$$g_j^o(x; x + t_n d - x) = g_j^o(x; t_n d) \le 0, \quad j \in J(x)$$

as $f_{i_0}^o(x; \cdot)$ and $g_j^o(x; \cdot)$, $j \in J(x)$ are sublinear on $\mathbb{R}^n$. Since (GLCQ) holds at x, therefore f_{i_0} and g_j, $j \in J(x)$ are ∂^c-pseudoconcave at x which implies that

$$f_{i_0}(x + t_n d) \le f_{i_0}(x), \tag{4.10}$$

$$g_j(x + t_n d) \le g_j(x) = 0, \quad j \in J(x). \tag{4.11}$$

Also as $g_j(x) < 0$ for all $j \in J \setminus J(x)$ and g_j's are locally Lipschitz continuous for all $j \in J$, therefore for all n sufficiently large, we get that

$$g_j(x + t_n d) < 0, \quad j \in J \setminus J(x). \tag{4.12}$$

By using the definition of $M^{i_0}(x)$, we obtain from (4.10)-(4.12) that for sufficiently large values of n

$$x + t_n d \in M^{i_0}(x)$$

which gives that $d \in T(M^{i_0}(x); x)$. □

Proposition 4.4. *(GLOCQ) implies (GACQ).*

Proof. Let (GLOCQ) holds at $x \in X$. Suppose that $k \in I$ for which the system given in (GLOCQ) has a solution d^k. First we prove that $d^k \in T(M^k(x); x)$.
Since d^k is a solution to the system given in (GLOCQ), therefore by definition of Clarke's generalized directional derivative, there is a sequence $\{t_n\}$ with $t_n \downarrow 0$ such that for all n sufficiently large

$$g_j(x + t_n d^k) < g_j(x) = 0, \quad j \in J(x). \tag{4.13}$$

Again since d^k is a solution to the system in (GLOCQ) and $f_k^o(x; \cdot)$ is sublinear, therefore

$$f_k^o(x; x + t_n d^k - x) = f_k^o(x; t_n d^k) \le 0.$$

As f_i for each $i \in I$ is ∂^c-pseudoconcave at x, we get

$$f_k(x + t_n d^k) \le f_k(x). \tag{4.14}$$

Since $g_j(x) < 0$ for all $j \in J \setminus J(x)$ and as g_j's are locally Lipschitz continuous for all $j \in J$, therefore for all n sufficiently large, we have,

$$g_j(x + t_n d^k) < 0, \quad j \in J \setminus J(x). \tag{4.15}$$

From (4.13)-(4.15), we get that for sufficiently large values of n

$$x + t_n d^k \in M^k(x)$$

which yields that $d^k \in T(M^k(x); x)$. As d^k is an arbitrary solution to the kth system, therefore every solution of that system is an element of $T(M^k(x); x)$.
Now let us suppose that some $\bar{d} \in L(M^k(x); x)$. Since $\bar{d} \in L(M^k(x); x)$ and d^k is a solution to the system given in (GLOCQ), $\bar{d} + t_n d^k$ also solves that system as

$$f_k^o(x; \bar{d} + t_n d^k) \le f_k^o(x; \bar{d}) + t_n f_k^o(x; d^k) \le 0,$$

$$g_j^o(x; \bar{d} + t_n d^k) \le g_j^o(x; \bar{d}) + t_n g_j^o(x; d^k) < 0, \quad j \in J(x)$$

and hence $\bar{d} + t_n d^k \in T(M^k(x); x)$. Since $T(M^k(x); x)$ is closed and $\bar{d}$ is the limit point of the sequence $\{\bar{d} + t_n d^k\}$, we get that

$$\bar{d} \in T(M^k(x); x).$$

Thus $L(M^k(x); x) \subseteq T(M^k(x); x)$ for a particular $k \in I$ and hence (GACQ) holds at x. □

Proposition 4.5. (*GMFCQ*) *holds iff* (*GCCQ*) *holds.*

Proof. It is easy to see by definition that if (GCCQ) holds then (GMFCQ) holds. Now let (GMFCQ) holds at $x \in X$. Then, we may assume that the solution exists for the ith system. So, suppose that, $f_i^o(x; u^i) < 0$ and the system given by (4.3) and (4.4) has a solution v^i.
Since $f_i^o(x; \cdot)$ is sublinear and $g_j^o(x; \cdot)$, $j \in J(x)$ is continuous, we get that for sufficiently small real number $t > 0$

$$f_i^o(x; v^i + tu^i) < 0,$$
$$g_j^o(x; v^i + tu^i) < 0, \quad j \in J(x).$$

Hence, $v^i + tu^i$ solves the ith system in (GCCQ) which shows that (GCCQ) holds at x. Thus

$$\text{(GMFCQ)} \Leftrightarrow \text{(GCCQ)}.$$

□

Now the following examples illustrate the implications shown between the different constraint qualifications given above.

Example 4.1. Consider the problem

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ & \text{s.t.} & g(x) = (g_1(x), g_2(x)) \leqq 0, \end{array}$$

where $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = \begin{cases} x, & x < 1 \\ 1, & x \geq 1 \end{cases}, \; f_2(x) = \begin{cases} 0, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}, \; g_1(x) = \begin{cases} -x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}, \; g_2(x) = \begin{cases} -x^3, & x < 1 \\ -x, & x \geq 1. \end{cases}$$

Now for the above problem (MP), feasible set $X = [0, \infty)$. Let $\bar{x} = 1$. Then

$$M^1(1) = \{x \in X : f_1(x) \leq f_1(1)\} = [0, \infty), \quad T(M^1(1); 1) = \mathbb{R}.$$
$$L(M^1(1); 1) = \{d \in \mathbb{R} : f_1^o(1; d) \leq 0\} = \mathbb{R}_-$$

as none of the constraint functions is active at $\bar{x} = 1$. It can be easily seen that

1. (GACQ) and hence (GGCQ) is satisfied here for $i = 1$ and it can be verified that both of them are satisfied for $i = 2$ also.
2. (GCCQ) is not satisfied at $\bar{x} = 1$ as neither $f_1^o(1; d) < 0$ nor $f_2^o(1; d) < 0$ for any $d \in \mathbb{R}$ showing that
$$\text{(GACQ)} \not\Rightarrow \text{(GCCQ)}$$
3. (GSCQ) is not satisfied at $\bar{x} = 1$ as by Proposition (4.2), (GSCQ) $\Rightarrow$ (GCCQ). It is also due to the fact that f_1, f_2 are not ∂^c-pseudoconvex at $\bar{x} = 1$.
4. (GLCQ) holds as f_1 is ∂^c-pseudoconcave at $\bar{x} = 1$.
5. (GLOCQ) is satisfied at $\bar{x} = 1$ as $f_i, i = 1, 2$ are ∂^c-pseudoconcave at $\bar{x} = 1$ and both the systems given in (GLOCQ) have a solution, although we need it only for one system.
6. (GMFCQ) does not hold at $\bar{x} = 1$ because of the reason given in (2) and also as (GCCQ) does not hold at $\bar{x} = 1$, therefore by Proposition 4.5, (GMFCQ) does not hold at $\bar{x} = 1$.

Example 4.2. Consider the problem

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ & \text{s.t.} & g(x) = (g_1(x), g_2(x)) \leqq 0, \end{array}$$

where $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = \begin{cases} 2, & x < -1 \\ -x + 1, & -1 \leq x \leq 1 \\ (x-1)^2, & x > 1 \end{cases}, \; f_2(x) = -|x|, \; g_1(x) = \begin{cases} -x^3, & x < 1 \\ -x, & x \geq 1 \end{cases}, \; g_2(x) = \begin{cases} \frac{1}{3}(1 - x), & x < 1 \\ 1 - x, & x \geq 1. \end{cases}$$

Now for the above problem (MP), feasible set $X = [1, \infty)$. Let us take $\bar{x} = 1$. Then moving on the same lines as in Example 4.1, we can see that (GACQ),(GGCQ),(GCCQ),(GLCQ) and (GMFCQ) are satisfied at $\bar{x} = 1$ but (GSCQ) and (GLOCQ) do not hold at $\bar{x} = 1$ as f_2 is not ∂^c-pseudoconvex and f_1 is not ∂^c-pseudoconcave respectively at $\bar{x} = 1$ which shows that

$$\text{(GCCQ)} \not\Rightarrow \text{(GSCQ)} \quad \text{and} \quad \text{(GACQ)} \not\Rightarrow \text{(GLOCQ)}$$

Remark 4.2. In Example 4.2, (GACQ),(GGCQ),(GCCQ),(GLCQ) and (GMFCQ) are satisfied at $\bar{x} = 1$ for the index i = 2.

We summarize the relationship between above constraint qualifications in Fig.5.

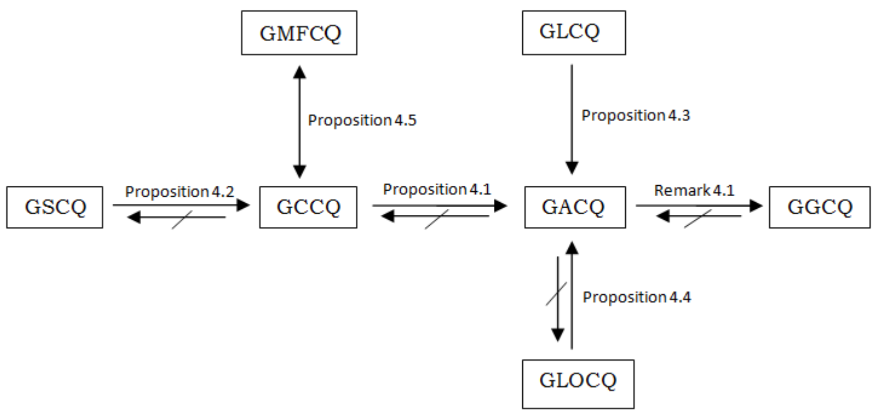

Fig.5 Relations between the constraint qualifications

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