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Some Applications of Mittag-Leffler Function in the Unit Disk

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Abstract. In this paper we introduce an operator associated with generalized Mittag-Leffler function in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. By using this operator and the virtue of differential subordination, we obtain interesting results. Some applications of our results are also obtained.

1. Introduction

The Mittag-Leffler function $E_\alpha(z)$ $(z \in \mathbb{C})$ ([10], [11]) is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\alpha \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0).$$

Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found *e.g.* in [1], [2], [3], [4], [5], [7], [12], [13],[14], [15], [16], [19], [20] and [21].

Moreover, Srivastava and Tomovski [18] introduced the function $E^{\gamma,k}_{\alpha,\beta}(z)$ $(z \in \mathbb{C})$ in the form

$$E^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}\ z^n}{\Gamma(\alpha n + \beta)\ n!}, \tag{1.1}$$
$$(\alpha, \beta, \gamma \ \in\ \mathbb{C};\ \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k) - 1\}; \mathrm{Re}(k) > 0)\ .$$

where $(\gamma)_n$ is the Pochhammer symbol:

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1, & n = 0 \\ \gamma(\gamma + 1)...(\gamma + n - 1) & \end{cases}.$$

Srivastava and Tomovski [18] proved that the function $E^{\gamma,k}_{\alpha,\beta}(z)$ defind by (1.1) is an entire function in the complex z-plane .

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Let A denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Noting that, by using a similar proof which is used by Srivastava and Tomovski [18, Theorem 1, P 201] we find that, if

$$\mathrm{Re}(\alpha) \geq 0 \text{ when } \mathrm{Re}(k) = 1 \text{ with } \beta \neq 0,$$

then, the power series in the defining equation (1.1) is still analytic and converges absolutely in $\mathbb{U}$ for all $\gamma \in \mathbb{C}$.

Now, we define the function $Q_{\alpha,\beta}^{\gamma,k}(z)$ by

$$Q_{\alpha,\beta}^{\gamma,k}(z) = \frac{\Gamma(\alpha+\beta)}{(\gamma)_k}\left(E_{\alpha,\beta}^{\gamma,k}(z) - \frac{1}{\Gamma(\beta)}\right) \quad (z \in \mathbb{U}), \tag{1.3}$$

$$(\beta, \gamma \in \mathbb{C};\ \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k)-1\};\mathrm{Re}(k) > 0;$$
$$\mathrm{Re}(\alpha) = 0 \text{ when } \mathrm{Re}(k) = 1 \text{ with } \beta \neq 0).$$

Throughout this paper, unless otherwise indicated, the conditions on the four complex parameters α, β, γ and k will be as follows:

$$\beta, \gamma \in \mathbb{C},\ \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k)-1\},\ \mathrm{Re}(k) > 0 \text{ and}$$
$$\mathrm{Re}(\alpha) = 0 \textit{ when } \mathrm{Re}(k) = 1 \text{ with } \beta \neq 0.$$

Moreover, let $f(z) \in A$. Denote by $\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f) : A \to A$ the operator is defined by

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z) = Q_{\alpha,\beta}^{\gamma,k}(z) * f(z) \quad (z \in \mathbb{U}), \tag{1.4}$$

where the symbol $(*)$ denotes the *Hadamard product* (*or convolution*).

We note that

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+nk)\,\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\,\Gamma(\beta+\alpha n)\,n!} a_n z^n.$$

Also, noting that:

1. $\mathcal{H}_{0,\beta}^{1,1}(f)(z) = f(z)$.
2. $\mathcal{H}_{0,\beta}^{2,1}(f)(z) = \frac{1}{2}\left(f(z) + zf'(z)\right)$.
3. $\mathcal{H}_{0,\beta}^{0,1}(f)(z) = \int_0^z \frac{1}{t} f(t)dt$.
4. $\mathcal{H}_{1,0}^{1,1}(\frac{z}{1-z}) = ze^z$.
5. $\mathcal{H}_{1,1}^{1,1}(\frac{z}{1-z}) = e^z - 1$.
6. $\mathcal{H}_{2,1}^{1,1}(\frac{z}{1-z}) = -2 + \cosh(\sqrt{z})$.

Remark 1.1. *It is noteworthy to mention that, the Srivastava–Wright operator [17] (see also [6]) which is defined by the Fox-Wright generalization ${}_q\Psi_s$ of the hypergeometric ${}_qF_s$ function also generalized the Mittag-Leffler function.*

2. Some Definitions and Lemmas

In our paper we use the following definitions.

Definition 2.1. *Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be subordinate to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.*

Definition 2.2. *Let $\Psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ be analytic in domain $\mathbb{D}$, and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in \mathbb{D}$ when $z \in \mathbb{U}$, then we say that $p(z)$ satisfies a first order differential subordination if:*

$$\Psi(p(z), zp'(z); z) \prec h(z) \quad (z \in \mathbb{U}). \tag{2.1}$$

The univalent function $q(z)$ is called dominant of the differential subordination (2.1), *if $p(z) \prec q(z)$ for all $p(z)$ satisfying* (2.1), *if $\tilde{q}(z) \prec q(z)$ for all dominant of* (2.1), *then we say that $\tilde{q}(z)$ is the best dominant of* (2.1).

By using the definition of $\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)$ which is defined by (1.4), we can prove the following lemma:

Lemma 2.1. *If $f(z) \in A$ $(z \in \mathbb{U})$, then*

$$z\left(\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)\right)' = \left(\frac{\gamma+k}{k}\right)\left(\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)\right) - \frac{\gamma}{k}\left(\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)\right) \tag{2.2}$$

and

$$\alpha z\left(\mathcal{H}_{\alpha,\beta+1}^{\gamma,k}(f)(z)\right)' = (\alpha+\beta)\left(\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)\right) - \beta\left(\mathcal{H}_{\alpha,\beta+1}^{\gamma,k}(f)(z)\right). \tag{2.3}$$

Remark 2.1. *Putting $f(z) = \frac{z}{1-z}$ and $k = q \in (0,1) \cup \mathbb{N}$, in* (2.3) *we have the result due to Shukla [16, Theorem 2.1, P. 800].*

Using (2.2), (2.3) and mathematical induction, we get the following lemmas:

Lemma 2.2. *If $f(z) \in A$ $(z \in \mathbb{U})$ and $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, then*

$$\mathcal{H}_{\alpha,\beta}^{\gamma+m,k}(f)(z) = \frac{k^m}{(\gamma+k)_m}(zD + \frac{\gamma}{k})^m\, \mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z) \quad (D := \frac{d}{dz}), \tag{2.4}$$

where $(zD+\frac{\gamma}{k})^m = (zD+\frac{\gamma}{k})\circ(zD+\frac{\gamma}{k})\circ...\circ(zD+\frac{\gamma}{k})$ to $m-$times and $\circ$ denotes the composition $(I\circ J)(f)(z) = I(J(f(z)))$.

Lemma 2.3. *If $z \in \mathbb{U}$, $f \in A$ and $m \in \mathbb{N}_0$, then*

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z) = \frac{1}{(\alpha+\beta)_m}(\alpha zD + \beta)^m\, \mathcal{H}_{\alpha,\beta+m}^{\gamma,k}(f)(z) \quad (D := \frac{d}{dz}), \tag{2.5}$$

where $(\alpha zD + \beta)^m = (\alpha zD + \beta) \circ (\alpha zD + \beta) \circ ... \circ (\alpha zD + \beta)$ to $m-$ times and $\circ$ denotes the composition $(I \circ J)(f)(z) = I(J(f(z)))$.

Example 2.1. *Putting $\gamma = k = 1$, $\alpha = 0$ and $f(z) = \frac{z}{1-z}$ in Lemma 2.2, we have the following property of the generalized Mittag-Leffler function in the unit disk $\mathbb{U}$,*

$$E_{0,\beta}^{m+1,1}(f)(z) = \frac{1}{\Gamma(\beta)} + \frac{1}{(m+1)!\,\Gamma(\beta)}(zD+1)^m\frac{z}{1-z} \quad (z \in \mathbb{U};\ m \in \mathbb{N}_0).$$

3. Differential Subordination with $\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)$

We require the following lemma due to Miller and Mocanu [8], see also [9, P. 132].

Lemma 3.1. *Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\,\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either $h(z)$ is convex, or $Q(z)$ is starlike . In addition, assume that* $\operatorname{Re}\left\{\dfrac{z\,h'(z)}{Q(z)}\right\} > 0$.

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta[p(z)] + z\,p'(z)\,\phi[p(z)] \prec \theta[q(z)] + zq'(z)\,\phi[q(z)] = h(z), \tag{3.1}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (3.1).

Now, we will prove the following theorem.

Theorem 3.1. *Let* $\frac{\mathcal{H}_{\alpha,\beta}^{\gamma+i,k}(f)(z)}{z} \neq 0$ $(i = 0, 1)$ *and*

$$\frac{z\left(\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)\right)'}{\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)} \prec q(z) + \frac{z\,(q(z))'}{q(z)+\frac{\gamma}{k}} \quad (f \in A;\ z \in \mathbb{U})$$

where $q(z)$ is univalent in $\mathbb{U}$ with $q(0) = 1$, which satisfies the following conditions:

$$\operatorname{Re}\left(q(z) + \frac{\gamma}{k}\right) > 0 \text{ and } \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)+\frac{\gamma}{k}}\right) > 0.$$

Then

$$\frac{z\left(\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)\right)'}{\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)} \prec q(z) \tag{3.2}$$

and $q(z)$ is the best dominant of (3.2).

Proof. We choose $p(z) = \frac{z\left(\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)\right)'}{\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)}$, then (2.2) becomes

$$\left(p(z) + \frac{\gamma}{k}\right)\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z) = \left(\frac{\gamma+k}{k}\right)\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z). \tag{3.3}$$

Then, from the identity (3.3), we have

$$\frac{z\left(\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)\right)'}{\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)} = \left(p(z) + \frac{z\,p'(z)}{p(z)+\frac{\gamma}{k}}\right), \tag{3.4}$$

therefore, (3.4) becomes

$$p(z) + \frac{z\,p'(z)}{p(z)+\frac{\gamma}{k}} \prec q(z) + \frac{z\,(q(z))'}{q(z)+\frac{\gamma}{k}}, \quad (z \in \mathbb{U}). \tag{3.5}$$

where $q(z)$ is defined in Theorem 3.1.

Let us choose the functions $\theta(w) = w$ and $\phi(w) = \dfrac{1}{w + \frac{\gamma}{k}}$. Then $\theta(w)$ and $\phi(w)$ are analytic with domain $\mathbb{D} = \mathbb{C} \setminus \{-\frac{\gamma}{k}\}$ which contains $q(\mathbb{U})$ and $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$.

Also, we define the function $Q(z)$ by

$$Q(z) = zq'(z)\,\phi(q(z)),$$

since

$$h(z) = \theta[q(z)] + Q(z) = q(z) + \frac{z\,(q(z))'}{q(z) + \frac{\gamma}{k}},$$

furthermore,

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z) + \frac{\gamma}{k}},$$

therefore, $Q(z)$ is starlike function in $\mathbb{U}$, and

$$\begin{aligned} \operatorname{Re}\left\{\frac{z\,h'(z)}{Q(z)}\right\} &= \operatorname{Re}\left\{\frac{1}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} \\ &= \operatorname{Re}\left(q(z) + \frac{\gamma}{k}\right) + \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0. \end{aligned}$$

Also, the condition $\dfrac{\mathcal{H}^{\gamma+1,k}_{\alpha,\beta}(f)(z)}{z} \neq 0$, gives that the function $p(z)$ is analytic in $\mathbb{U}$, $p(0) = q(0) = 1$ and $-\frac{\gamma}{k} \notin p(\mathbb{U})$, therefore $p(\mathbb{U}) \subset \mathbb{D}$. By Lemma 3.1, we deduce $\frac{z\left(\mathcal{H}^{\gamma,k}_{\alpha,\beta}(f)(z)\right)'}{\mathcal{H}^{\gamma,k}_{\alpha,\beta}(f)(z)} \prec q(z)$, and $q(z)$ is the best dominant of (3.2). □

By using the technique which is used in Theorem 3.1 and the recurrence relation (2.3), we have the following theorem.

Theorem 3.2. *Let* $\frac{\mathcal{H}^{\gamma,k}_{\alpha,\beta+i}(f)(z)}{z} \neq 0$ $(i = 0, 1)$ *and*

$$\frac{z\left(\mathcal{H}^{\gamma,k}_{\alpha,\beta}(f)(z)\right)'}{\mathcal{H}^{\gamma,k}_{\alpha,\beta}(f)(z)} \prec q(z) + \frac{z\,(q(z))'}{q(z) + \frac{\gamma}{k}}$$

where $q(z)$ is univalent in $\mathbb{U}$ with $q(0) = 1$, which satisfies the following conditions:

$$\operatorname{Re}\left(q(z) + \frac{\beta}{\alpha}\right) > 0 \text{ and } \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z) + \frac{\beta}{\alpha}}\right) > 0.$$

Then

$$\frac{z\left(\mathcal{H}^{\gamma,k}_{\alpha,\beta+1}(f)(z)\right)'}{\mathcal{H}^{\gamma,k}_{\alpha,\beta+1}(f)(z)} \prec q(z) \tag{3.6}$$

and $q(z)$ is the best dominant of (3.6).

Corollary 3.1. *Let* $\mathrm{Re}\left(\frac{\gamma}{k}\right) \geq -\delta;\ \delta \in [0,1)$. *Also, let*

$$\frac{z\left(\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)\right)'}{\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}(f)(z)} \prec h(z) \quad (z \in \mathbb{U}),$$

for all $f \in A$ *satisfies* $\frac{\mathcal{H}_{\alpha,\beta}^{\gamma+i,k}(f)(z)}{z} \neq 0$ $(i = 0,1)$, *then*

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z) \in S^*(\delta), \ \ \delta \text{ is the best possible,}$$

where $S^*(\delta)$ *is starlike function of order* δ *and*

$$h(z) = -1 + 2\delta - \frac{3-2\delta}{1-z} - \frac{1+\frac{\gamma}{k}}{1+\frac{\gamma}{k}+\left(1-2\delta-\frac{\gamma}{k}\right)z}. \tag{3.7}$$

Proof. Putting $q(z) = \dfrac{1+(1-2\alpha)z}{1-z}$, therefore under the condition $\mathrm{Re}\left(\frac{\gamma}{k}\right) \geq -\delta$, we have

$$\mathrm{Re}\left(q(z) + \frac{\gamma}{k}\right) > 0. \tag{3.8}$$

After some calculations, we have,

$$\begin{aligned} & 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)+\frac{\gamma}{k}} \\ = \quad & -1 + \frac{1}{1-z} + \frac{1+\frac{\gamma}{k}}{1+\frac{\gamma}{k}+\left(1-2\delta-\frac{\gamma}{k}\right)z}, \end{aligned}$$

therefore,

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)+\frac{\gamma}{k}}\right) > 0 \tag{3.9}$$

By using (3.8), (3.9) and applying Theorem 3.1, we complete the corollary. □

By using the technique which is used in Corollary 3.1, we have the following corollary.

Corollary 3.2. *Let* $\mathrm{Re}\left(\frac{\beta}{\alpha}\right) \geq -\delta;\ \delta \in [0,1)$. *Also, let*

$$\frac{z\left(\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)\right)'}{\mathcal{H}_{\alpha,\beta}^{\gamma,k}(f)(z)} \prec h(z) \quad (z \in \mathbb{U}),$$

for all $f \in A$ *satisfies* $\dfrac{\mathcal{H}_{\alpha,\beta+i}^{\gamma,k}(f)(z)}{z} \neq 0$ $(i = 0,1)$, *then*

$$\mathcal{H}_{\alpha,\beta+1}^{\gamma,k}(f)(z) \in S^*(\delta), \ \ \delta \text{ is the best possible,}$$

where $S^*(\delta)$ *is starlike function of order* δ *and* $h(z)$ *is defined by* (3.7).

Example 3.1. *We can show that the function $f(z)$ $(z \in \mathbb{U})$ defined by*

$$\mathcal{H}^{2,1}_{0,\beta}(f)(z) = \frac{1}{2}\left(f(z) + zf^{'}(z)\right) = z(1-\delta z)(1-z)^{2\delta-3}, \tag{3.10}$$

satisfies the hypotheses of Corollary 3.1. Also, the equation (3.10) *is a first order linear differential equation, which has a solution*

$$f(z) = \frac{z}{(1-z)^{2(1-\delta)}}$$

that is the extremal function for the class of starlike function of order δ.Therefore,

$$\mathcal{H}^{1,1}_{0,\beta}(f)(z) = f(z) \in S^{*}(\delta),$$

and δ is the best possible.

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