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Semi-Quasitriangularity of Toeplitz Operators

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Abstract. In this paper we give a necessary and sufficient condition, in terms of the coefficients of φ, in order for the Toeplitz operator T_φ to be semi-quasitriangular when φ is a trigonometric polynomial of degree two and has real coefficients.

1. Introduction

Let $B(H)$ denote the algebra of bounded linear operators on a complex separable Hilbert space H. The Hilbert space $L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane $\mathbb{C}$, has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$ $(n \in \mathbb{Z})$, and the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. If P denotes the projection operator $L^2(\mathbb{T}) \to H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operator T_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g = P(\varphi g) \quad \text{for all } g \in H^2(\mathbb{T})$$

is called the *Toeplitz operator with symbol* φ. It is familiar that the matrix representation of T_φ with respect to the basis $\{e_n : n = 0, 1, 2, \cdots\}$ is a Toeplitz matrix (λ_{ij}). In this case, $\lambda_{ij} = a_{i-j}$, where $\varphi(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ is the Fourier expansion of φ. In this paper, we concentrate a Toeplitz operator T_φ with trigonometric polynomial symbol φ of the form $\varphi(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$: its matrix representation is the following.

$$T_\varphi = \begin{pmatrix} a_0 & a_{-1} & \cdots & \cdots & a_{-N} & & \\ a_1 & a_0 & a_{-1} & & & a_{-N} & \\ \vdots & a_1 & a_0 & a_{-1} & & & \ddots \\ \vdots & & \ddots & \ddots & \ddots & & \\ a_N & & & \ddots & \ddots & \ddots & \\ & a_N & & & \ddots & \ddots & \ddots \\ & & \ddots & & & \ddots & \ddots \end{pmatrix} \tag{1}$$

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On the other hand, from the spectral property of Toeplitz operators with continuous symbols (cf. [7]), we can see that if $\varphi(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$ is a trigonometric polynomial then we have

$$\sigma_e(T_\varphi) = \varphi(\mathbb{T}) \text{ and } \operatorname{ind}(T_\varphi - \lambda) = -\operatorname{wn}(\varphi - \lambda) \text{ for each } \lambda \in \mathbb{C} \setminus \sigma_e(T_\varphi), \tag{2}$$

where $\sigma_e(\cdot)$ denotes the essential spectrum, ind $(\cdot)$ denotes the (Fredholm) index of the Fredholm operator and wn ψ denotes the winding number of ψ with respect to 0. We recall ([8, Definition 4.8]) that that an operator $T \in B(H)$ is called *quasitriangular* if there exists an increasing sequence $\{P_n\}$ of projections of finite rank in $B(H)$ that converges strongly to the identity and satisfies $\|(I - P_n)TP_n\| \to 0$. The quasitriangularity can be rewritten in terms of the "spectral picture" of the operator T, denoted $\mathcal{SP}(T)$, which consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes. By a theorem of Apostol, Foias, and Voiculescu (in brief, AFV theorem; cf. [8, Theorem 1.31]), T is quasitriangular if and only if $\mathcal{SP}(T)$ contains no hole or pseudohole with a negative Fredholm index number.

Definition 1.1. *(cf. [5]) An operator $T \in B(H)$ is called semi-quasitriangular if either T or T^* is quasitriangular.*

If φ is a trigonometric polynomial then the semi-quasitriangularity of T_φ can be determined by a geometrical character of the symbol φ.

Proposition 1.2. *If T_φ is a Toeplitz operator with trigonometric polynomial symbol φ then the following are equivalent:*

(i) T_φ is semi-quasitriangular.

(ii) $\operatorname{wn}(\varphi - \lambda)\operatorname{wn}(\varphi - \mu) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \setminus \varphi(\mathbb{T})$.

Proof. Since, evidently, $\mathcal{SP}(T_\varphi)$ has no pseudoholes it follows from the AFV theorem that T_φ is semi-quasitriangular if and only if $\operatorname{ind}(T_\varphi - \lambda)\operatorname{ind}(T_\varphi - \mu) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \setminus \sigma_e(T_\varphi)$. Thus the desired equivalence follows from the second equality in (2). □

We would remark that the semi-quasitriangularity is related to the spectral mapping theorem for the Weyl spectrum (the *Weyl spectrum* of $T \in B(H)$ means the complement, in $\mathbb{C}$, of the set of all complex numbers λ which $T - \lambda$ is Fredholm of index zero.) In fact, from [4, Theorem 5], we have that the semi-quasitriangularity of T_φ is equivalent to the condition that the spectral mapping theorem holds for $\omega(T_\varphi)$, the Weyl spectrum of T_φ:

$$p\,\omega(T_\varphi) = \omega\, p(T_\varphi) \quad \text{for each polynomial } p.$$

Thus this equivalence says that if T_φ is semi-quasitriangular then to find the Weyl spectrum of $p(T_\varphi)$, it suffices to determine the following set:

$$p\Big(\varphi(\mathbb{T}) \cup \big\{\lambda \in \mathbb{C} \setminus \varphi(\mathbb{T}) \;:\; \operatorname{wn}(\varphi - \lambda) \neq 0\big\}\Big).$$

On the other hand we say that "Weyl's theorem holds" for $T \in B(H)$ when the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues of finite multiplicity (cf. [4]). Then if φ is a trigonometric polynomial and if f is an analytic function defined on some open set containing $\sigma(T_\varphi)$ then it follows from Proposition 1.2 and [2, Lemma 3.1; Theorem 3.7] that Weyl's theorem holds for $f(T_\varphi)$ whenever T_φ is semi-quasitriangular. In fact, if T_φ is a Toeplitz operator with quasicontinuous symbol then T_φ is semi-quasitriangular if and only if Weyl's theorem holds for $f(T_\varphi)$ (cf. [5]).

The following notion was introduced by W.Y. Lee [6] in an operator theory seminar at Seoul National University:

Definition 1.3. *A trigonometric polynomial φ is said to be pure if holes of $\varphi(\mathbb{T})$ have all non-negative (or all non-positive) winding numbers.*

From Proposition 1.2, we can see that if $\varphi(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$ and if T_φ is a Toeplitz operator with symbol φ then

$$T_\varphi \text{ is semi-quasitriangular} \iff \varphi \text{ is pure.} \tag{3}$$

In [2], the following problem was raised.

Problem 1.4. *If φ is a trigonometric polynomial, find necessary and sufficient conditions, in terms of the coefficients of φ, in order for T_φ to be semi-quasitriangular.*

In this paper, we give a solution to the above problem in the case that φ is a trigonometric polynomial of degree two and has real coefficients:

$$\varphi(\theta) = a_{-2}e^{-2i\theta} + a_{-1}e^{-i\theta} + a_1 e^{i\theta} + a_2 e^{2i\theta} \quad (a_1, a_2, a_{-1}, a_{-2} \in \mathbb{R}).$$

Since the semi-quasitriangularity of the Toeplitz operator T_φ does not depend on the translation, we may assume that $a_0 = 0$. For brevity, in the sequel, we use the following notations: If $\varphi(\theta) = a_{-2}e^{-2i\theta} + a_{-1}e^{-i\theta} + a_1 e^{i\theta} + a_2 e^{2i\theta}$ $(a_1, a_2, a_{-1}, a_{-2} \in \mathbb{R})$, define

$$L := |a_1 + a_{-1}|, \;\; M := |a_2 + a_{-2}|, \;\; N := |a_1 - a_{-1}|, \;\; P := |a_2 - a_{-2}|;$$
$$\operatorname{sgn}\varphi := \begin{cases} 1 & \text{if } (a_2^2 - a_{-2}^2)(a_1^2 - a_{-1}^2) \geq 0 \\ -1 & \text{if } (a_2^2 - a_{-2}^2)(a_1^2 - a_{-1}^2) < 0. \end{cases}$$

Then our main result can be stated as follows:

Theorem 1.5. *If*

$$\varphi(\theta) = a_{-2}e^{-2i\theta} + a_{-1}e^{-i\theta} + a_1 e^{i\theta} + a_2 e^{2i\theta} \quad (a_1, a_2, a_{-1}, a_{-2} \in \mathbb{R}), \tag{4}$$

then T_φ is semi-quasitriangular if and only if

$$\begin{cases} L = 0 \text{ or } N \geq 2P & \text{if } L \geq 4M, \\ PL \geq \frac{(1-sgn\,\varphi)}{4} M\Big(\sqrt{N^2 + 32P^2} - N\Big) & \text{if } L < 4M,\; N \geq 2P, \\ PL \leq M\Big(2P + (sgn\,\varphi)N\Big) & \text{if } L < 4M,\; N < 2P. \end{cases}$$

This paper consists of three sections. In §2, we consider the case that $|a_2| = |a_{-2}|$. In §3, we give a proof of Theorem 1.5.

2. The Case that $|a_2| = |a_{-2}|$

In [2], it is shown that the cases that $|a_2| = |a_{-2}|$ are extremal among all possibilities for hyponormality of T_φ. In this section we consider the semi-quasitriangularity of T_φ with symbol φ defined as in (4) when $|a_2| = |a_{-2}|$.

We begin with:

Lemma 2.1. *Let $\varphi(\theta) = \big(x(\theta), y(\theta)\big)$ $(0 \le \theta \le 2\alpha)$ be a continuous curve with $\varphi(0) = \varphi(2\alpha)$. Suppose $\varphi(\theta)$ satisfies the following properties:*

(i) $x(\theta)$ is strictly increasing (or strictly decreasing) in $(0, \alpha)$ and is symmetric with respect to the line $\theta = \alpha$.

(ii) $y(0) = y(\alpha) = y(2\alpha) = 0$ and $y(\theta)$ is non-constant and anti-symmetric with respect to the line $\theta = \alpha$.

Then we have that φ is pure if and only if $y(\theta)$ is non-negative or non-positive in $(0, \alpha)$.

Proof. Observe that the curve of $\varphi(\theta)$ is symmetric with respect to the line $y = 0$. On the other hand, φ is not pure if and only if φ has at least two holes of winding numbers with different signs. But by the conditions (i) and (ii), φ has at least two holes of winding numbers with different signs if and only if $y(\theta)$ has at least two values with different signs in $(0, \alpha)$. □

We then have:

Theorem 2.2. *Let φ be defined as in (4). Then we have:*

(i) If $a_2 = a_{-2}$, then T_φ is semi-quasitriangular.

(ii) If $a_2 = -a_{-2}$, then T_φ is semi-quasitriangular if and only if

$$\text{either } \ a_1 + a_{-1} = 0 \quad \text{or} \ \ 2|a_2 - a_{-2}| \le |a_1 - a_{-1}|.$$

(iii) If $a_1 = -a_{-1}$, then T_φ is semi-quasitriangular.

Proof. Suppose that $\varphi(\theta) = a_{-2}e^{-2i\theta} + a_{-1}e^{-i\theta} + a_1 e^{i\theta} + a_2 e^{2i\theta}$. In view of (3), it suffices to consider the purity of φ.

(i) Let $a_2 = a_{-2}$. Then

$$\varphi(\theta) = 4a_2\cos^2\theta + (a_1 + a_{-1})\cos\theta - 2a_2 + i(a_1 - a_{-1})\sin\theta \quad (0 \le \theta \le 2\pi).$$

If $a_1 = \pm a_{-1}$ then φ represents a segment or a parabola, so that $\varphi(\mathbb{T})$ has no holes. If $a_1 \ne \pm a_{-1}$, then a straightforward calculation shows that φ is simple in $(0, 2\pi)$. Thus φ has just one hole. But then this case gives that φ has at most one hole; therefore φ is pure.

(ii) Let $a_2 = -a_{-2}$. Then $\varphi(\theta) = (a_1 + a_{-1})\cos\theta + i\big((a_1 - a_{-1})\sin\theta + (a_2 - a_{-2})\sin 2\theta\big)$. If $a_1 = a_{-1}$ then by Lemma 2.1, φ is not pure. Thus we assume that $a_1 \ne a_{-1}$. Now we put

$$\varphi(\theta) = \big(x(\theta), y(\theta)\big) = \left((a_1 + a_{-1})\cos\theta,\ (a_1 - a_{-1})\left(\sin\theta + \frac{a_2 - a_{-2}}{a_1 - a_{-1}}\sin 2\theta\right)\right).$$

If $a_1 + a_{-1} = 0$ then evidently, φ is pure. If $a_1 + a_{-1} \ne 0$ then Lemma 2.1 gives that φ is pure if and only if $y(\theta) = (a_1 - a_{-1})\sin\theta\left(1 + \frac{2(a_2 - a_{-2})}{a_1 - a_{-1}}\cos\theta\right)$ is non-negative or non-positive in $(0, \pi)$. Since $\sin\theta > 0$ in $(0, \pi)$, it follows that φ is pure if and only if $|\cos\theta| = \left|\frac{a_1 - a_{-1}}{2(a_2 - a_{-2})}\right| \ge 1$, and hence $2|a_2 - a_{-2}| \le |a_1 - a_{-1}|$.

(iii) Let $a_1 = -a_{-1}$. Put

$$\varphi(\theta) = \big(x(\theta), y(\theta)\big) = \big((a_2 + a_{-2})\cos 2\theta,\ (a_1 - a_{-1})\sin\theta + (a_2 - a_{-2})\sin 2\theta\big).$$

If $a_2 = \pm a_{-2}$ then by (i) and (ii), φ is pure. If instead $a_2 \ne \pm a_{-2}$, then a straightforward calculation shows that φ is simple in $(0, \pi)$. But since $x(0) = x(\pi)$, $y(0) = y(\pi) = 0$ and $y(\theta)$ has at most one zero in $(0, \pi)$, it follows that φ is pure. □

Example 2.3. *(a) An application of Theorem 2.2 shows that the matrix T_φ is semi-quasitriangular, while T_ψ is not:*

$$T_\varphi = \begin{pmatrix} 0 & -3 & 1 & & & & \\ 1 & 0 & -3 & 1 & & & \\ -1 & 1 & 0 & -3 & 1 & & \\ & -1 & 1 & 0 & -3 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \; T_\psi = \begin{pmatrix} 0 & -2 & 1 & & & & \\ 1 & 0 & -2 & 1 & & & \\ -1 & 1 & 0 & -2 & 1 & & \\ & -1 & 1 & 0 & -2 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

(b) If U is the unilateral shift on ℓ_2 then $a\,U^2 + b\,U + c\,U^* + a\,U^{*2}$ is semi-quasitriangular for any $a, b, c \in \mathbb{R}$.

(c) If $|a_2| = |a_{-2}|$ and $\det\begin{pmatrix} a_1 & a_2 \\ a_{-1} & a_{-2}\end{pmatrix} = 0$ then $T_\varphi = a_{-2}U^{*2} + a_{-1}U^* + a_1 U + a_2 U^2$ is semi-quasitriangular because the given condition implies that if $a_2 = -a_{-2}$ then $a_1 + a_{-1} = 0$. In fact, T_φ is hyponormal (cf. [2, Theorem 1.4]).

3. Proof of Theorem 1.5

To prove the main theorem we need the following:

Lemma 3.1. *The curve of $\varphi(\theta) = (L\cos\theta + M\cos 2\theta,\ N\sin\theta \pm P\sin 2\theta)$ $(0 < \theta < \pi)$ with $L, M, P > 0$ and $N \geq 0$ has at most one crossing point.*

Proof. Write

$$x(\theta) := L\cos\theta + M\cos 2\theta \quad \text{and} \quad y(\theta) := N\sin\theta \pm P\sin 2\theta$$

and suppose that, for $0 < \theta_1 < \theta_2 < \pi$, $x(\theta_1) = x(\theta_2)$ and $y(\theta_1) = y(\theta_2)$. Then we have $\cos\theta_1 + \cos\theta_2 = -\frac{L}{2M}$, so that

$$\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) = -\frac{L}{4M}.$$

Thus noting that $\frac{\theta_1+\theta_2}{2} > \frac{\pi}{2}$, we get

$$(\sin\theta_1 - \sin\theta_2)\Big(N \pm 2P(\cos\theta_1 + \cos\theta_2)\Big) = \pm 2P\sin(\theta_1 - \theta_2),$$

which gives

$$\theta_1 = \cos^{-1}\left(-\sqrt{\frac{PL}{2(PL \mp MN)}}\right) - \cos^{-1}\left(\sqrt{\frac{L(PL \mp MN)}{8M^2P}}\right),$$
$$\theta_2 = \cos^{-1}\left(-\sqrt{\frac{PL}{2(PL \mp MN)}}\right) + \cos^{-1}\left(\sqrt{\frac{L(PL \mp MN)}{8M^2P}}\right).$$

□

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Suppose that $\varphi(\theta) = a_{-2}e^{-2i\theta} + a_{-1}e^{-i\theta} + a_1 e^{i\theta} + a_2 e^{2i\theta}$. In view of (3), it suffices to consider the purity of φ. We write

$$\varphi(\theta) = \big(x(\theta), y(\theta)\big) = \Big((a_1 + a_{-1})\cos\theta + (a_2 + a_{-2})\cos 2\theta,\ (a_1 - a_{-1})\sin\theta + (a_2 - a_{-2})\sin 2\theta\Big).$$

Note that since replacing $x(\theta)$ (resp. $y(\theta)$) with $-x(\theta)$ (resp. $-y(\theta)$) does not influence the purity of φ, it is sufficient to consider the following four cases for purity of φ:

$$\begin{cases} \text{Case 1:} & x(\theta) = L\cos\theta + M\cos 2\theta, \ \ y(\theta) = N\sin\theta - P\sin 2\theta \\ \text{Case 2:} & x(\theta) = L\cos\theta + M\cos 2\theta, \ \ y(\theta) = N\sin\theta + P\sin 2\theta \\ \text{Case 3:} & x(\theta) = L\cos\theta - M\cos 2\theta, \ \ y(\theta) = N\sin\theta + P\sin 2\theta \\ \text{Case 4:} & x(\theta) = L\cos\theta - M\cos 2\theta, \ \ y(\theta) = N\sin\theta - P\sin 2\theta. \end{cases} \tag{5}$$

Furthermore, since $x(\theta)$ is symmetric with respect to $\theta=\pi$ and $y(\theta)$ is anti-symmetric with respect to $\theta=\pi$, considering the above cases only for $0\le\theta\le\pi$ gives the desired information. If at least one of L, M and P is zero then the result follows from Theorem 2.2. Thus we assume that L, M and P are all non-zero. Now we split the proof of the theorem into the four cases in (5).

(i) Case 1: $\varphi(\theta)=\big(x(\theta),y(\theta)\big)=\big(L\cos\theta+M\cos 2\theta,\ N\sin\theta-P\sin 2\theta\big)$

Write

$$\begin{cases}\theta_x := \text{the local minimizer of } x(\theta) \ \text{ if } L<4M,\\ \theta_{y_1} := \text{the local maximizer of } y(\theta),\\ \theta_{y_2} := \text{the local minimizer of } y(\theta) \ \text{ if } N<2P,\\ \theta_0 := \text{the zero point of } y(\theta) \ \text{ if } N<2P.\end{cases}$$

Then a straightforward calculation shows

$$\begin{cases}\frac{\pi}{2}<\theta_x<\pi, & \cos(\theta_x)=-\frac{L}{4M},\\ \frac{\pi}{2}<\theta_{y_1}\le\frac{3\pi}{4}, & \cos(\theta_{y_1})=\frac{N-\sqrt{N^2+32P^2}}{8P},\\ 0<\theta_{y_2}\le\frac{\pi}{4}, & \cos(\theta_{y_2})=\frac{N+\sqrt{N^2+32P^2}}{8P},\\ 0<\theta_0\le\frac{\pi}{2}, & \cos(\theta_0)=\frac{N}{2P}.\end{cases}$$

Now, in view of Lemma 3.1, the curve tracing in rough of $\varphi(\theta)$ can be classified in terms of $L\ge 4M$ ($L<4M$), $N\ge 2P$ ($N<2P$), $\theta_x, \theta_{y_1}, \theta_0$ into several cases and three cases can be chosen for φ to be pure. We then have

$$\varphi \text{ is pure} \Longleftrightarrow \begin{cases}\text{(i) } L\ge 4M,\ N\ge 2P\\ \text{(ii) } L<4M,\ N\ge 2P,\ PL\ge\frac{M}{2}\left(\sqrt{N^2+32P^2}-N\right)\\ \text{(iii) } L<4M,\ N<2P,\ PL\le M(2P-N).\end{cases}$$

(ii) Case 2: $\varphi(\theta)=\big(x(\theta),y(\theta)\big)=\big(L\cos\theta+M\cos 2\theta,\ N\sin\theta-P\sin 2\theta\big)$

With the notations of *Case 1*, a straightforward calculation also shows

$$\begin{cases}\frac{\pi}{2}<\theta_x<\pi, & \cos(\theta_x)=-\frac{L}{4M},\\ \frac{\pi}{4}\le\theta_{y_1}<\frac{\pi}{2}, & \cos(\theta_{y_1})=\frac{-N+\sqrt{N^2+32P^2}}{8P},\\ \frac{3\pi}{4}\le\theta_{y_2}<\pi, & \cos(\theta_{y_2})=\frac{-N-\sqrt{N^2+32P^2}}{8P},\\ \frac{\pi}{2}\le\theta_0<\pi, & \cos(\theta_0)=-\frac{N}{2P}.\end{cases}$$

Now after classifying the curve tracing in rough of $\varphi(\theta)$ into several cases in the same manner as *Case 1*, we can choose three cases for φ to be pure. We then have

$$\varphi \text{ is pure} \Longleftrightarrow \begin{cases}\text{(i) } L\ge 4M,\ N\ge 2P\\ \text{(ii) } L<4M,\ N\ge 2P\\ \text{(iii) } L<4M,\ N<2P,\ PL\le M(2P+N).\end{cases}$$

(iii) Case 3: $\varphi(\theta)=\big(x(\theta),y(\theta)\big)=\big(L\cos\theta-M\cos 2\theta,\ N\sin\theta+P\sin 2\theta\big)$

Replacing $\varphi(\theta)$ with $-\varphi(\theta-\pi)$ reduces this case to *Case 1*. Furthermore, since such a replacement represents a reflection and translation, it does not influence the purity of φ; therefore this case has the same result as *Case 1*.

(iv) Case 4: $\varphi(\theta)=\big(x(\theta),y(\theta)\big)=\big(L\cos\theta-M\cos 2\theta,\ N\sin\theta-P\sin 2\theta\big)$

Replacing $\varphi(\theta)$ with $-\varphi(\theta-\pi)$ reduces this case to *Case 2*, and thus this case has the same result as *Case 2*. This completes the proof. $\square$

Remark 3.2. *By generalized circulant we mean a (finite Toeplitz) matrix of the form*

$$\begin{pmatrix} a_0 & e^{i\omega}a_N & \dots & \dots & e^{i\omega}a_1 \\ a_1 & a_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 & e^{i\omega}a_N \\ a_N & \dots & \dots & a_1 & a_0 \end{pmatrix}$$

for some fixed $\omega \in [0, 2\pi)$. In [1], it was shown that a finite Toeplitz matrix is normal if and only if it is either a generalized circulant or a translation and rotation of a hermitian Toeplitz matrix. But this is not the case for a Toeplitz operator. In fact, if $\varphi(\theta)=\sum_{n=-N}^{N} a_n e^{in\theta}$ is a generalized circulant polynomial (i.e., $a_{-k}=e^{i\omega}a_{N-k+1}$ for every $1\le k\le N$), then a Toeplitz operator with symbol φ need not be even hyponormal (cf. [3]). But our Theorem 1.5 shows that a 2×2 real Toeplitz operator with generalized circulant polynomial symbol, i.e.,

$$\begin{pmatrix} a_0 & e^{i\omega}a_2 & e^{i\omega}a_1 & & & & \\ a_1 & a_0 & e^{i\omega}a_2 & e^{i\omega}a_1 & & & \\ a_2 & a_1 & a_0 & e^{i\omega}a_2 & e^{i\omega}a_1 & & \\ & a_2 & a_1 & a_0 & e^{i\omega}a_2 & e^{i\omega}a_1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (\omega = 0,\ \pi;\ a_0, a_1, a_2 \in \mathbb{R})$$

is semi-quasitriangular because this case implies that $L = M$ and $N = P$.

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