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Reducibility of Certain Kampé De Fériet Function with an Application to Generating relations for products of two Laguerre polynomials

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Abstract. It has been an interesting and natural research subject to consider the reducibility of some extensively generalized special functions. In this regard, Kampé de Fériet function has been attracted by many mathematicians. The authors [7] also established many interesting cases of the reducibility of Kampé de Fériet function by employing generalizations of the two results for the terminating ${}_2F_1(2)$ hypergeometric identities due to Kim et al. In this sequel, we first aim at presenting several interesting cases of the reducibility of Kampé de Fériet function by using generalizations of classical Kummer's summation theorem due to Lavoie et al. We next show how one can use the above-given result to obtain eleven new generating relations for products of two Laguerre polynomials in a single-form result. We also consider many interesting and potentially useful specials cases of our main results.

1. Introduction and Preliminaries

The vast popularity and immense usefulness of the hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ ($p, q \in \mathbb{N}_0$) in one variable have inspired and stimulated a large number of research workers to investigate hypergeometric functions of two or more variables. Serious and significant study of the functions of two variables was initiated by Appell [1] who presented the so-called Appell functions F_1, F_2, F_3 and F_4 which are natural generalizations of the Gaussian hypergeometric function and whose confluent forms were studied by Humbert [24, 25]. A complete list of these functions can be seen in the standard text of Erdélyi *et al*. [8]. Also, later on, the four Appell functions F_1, F_2, F_3 and F_4 and their confluent forms were further generalized by Kampé de Fériet [1], who introduced a more general hypergeometric function of two variables. The notation defined and introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [4, 5]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda

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[26, p. 423, Eq.(26)]. For this, let (h_H) denote the sequence of parameters $(h_1, h_2, \ldots, h_H)$ and, for $n \in \mathbb{N}_0$, define the Pochhammer symbol

$$((h_H))_n := (h_1)_n \cdots (h_H)_n,$$

where, when $n = 0$, the product is understood to reduce to unity. Therefore, the most convenient generalization of the Kampé de Fériet is defined as follows:

$$\begin{aligned} &F^{H:A;B}_{G:C;D}\left[\begin{matrix} (h_H) : (a_A) ; (b_B); \\ (g_G) : (c_C) ; (d_D); \end{matrix} x, y\right] \\ &\quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((h_H))_{m+n}\, ((a_A))_m\, ((b_B))_n}{((g_G))_{m+n}\, ((c_C))_m\, ((d_D))_n} \frac{x^m}{m!}\frac{y^n}{n!}. \end{aligned} \tag{1}$$

The symbol (h) is a convenient contraction for the sequence of the parameters $h_1, h_2, \ldots, h_H$ and the Pochhammer symbol $(h)_n$ is the same as defined (for $\lambda \in \mathbb{C}$) by (see [23, p. 2 and p. 5]):

$$\begin{aligned} (\lambda)_n : &= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \tag{2}$$

and $\Gamma(\lambda)$ is the familiar Gamma function. For details about the convergence for this function, we refer to [24].

It has been an interesting and natural research subject to consider the reducibility of some extensively generalized special functions. In this regard, Kampé de Fériet function has been attracted by various authors [6, 7, 10–12, 15]. The authors [7] also established many interesting cases of the reducibility of Kampé de Fériet function by employing generalizations of the two results for the terminating ${}_2F_1(2)$ hypergeometric identities due to Kim *et al.* [14].

In [26], a list of several interesting reducibility of Kampé de Fériet function is recorded, one of which is given as follows:

$$\begin{aligned} &F^{D:0;0}_{G:1;1}\left[\begin{matrix} (d) : \text{——} ; \text{——} ; \\ (g) : p ; p ; \end{matrix} -x, x\right] \\ &\quad = {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right), \left(\frac{1}{2}d\right)+\frac{1}{2}; \\ \left(\frac{1}{2}g\right), \left(\frac{1}{2}g\right)+\frac{1}{2}, p, \frac{1}{2}p, \frac{1}{2}p+\frac{1}{2}; \end{matrix} -4^{D-G-1}x^2\right]. \end{aligned} \tag{3}$$

The result (3) is derived with the help of the following classical summation theorem due to Kummer (see, *e.g.*, [2, 3, 20, 22, 23]):

$${}_2F_1\left[\begin{matrix} a, b; \\ 1+a-b; \end{matrix} -1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma\left(1+\frac{1}{2}a-b\right)}. \tag{4}$$

Recently a good deal of progress has been made in generalizing and extending the classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ (see, *e.g.*, [13], [16] - [19], [21], [27]).

Motivated essentially by the result (3), we first give eleven identities in the form of a single result, which will be given in Theorem 1.

2. Reducibility of Kampé De Fériet Function

We establish a general formula for the reducibility of Kampé de Fériet function which is expressed in a single form containing eleven results asserted by the following theorem.

Theorem 1. *The following reducibility of Kampé de Fériet function holds true:*

$$F^{D:0;0}_{G:1;1}\left[\begin{matrix}(d) & : & \text{———} ; & \text{———} ; \\ (g) & : & p+i\,; & p\;\; ; \end{matrix}\; -x,\, x\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p)\,\Gamma(p+i)}{\Gamma\left(p+\frac{1}{2}(i+|i|)\right)} \tag{5}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2)^n\,((\frac{1}{2}d))_n\,((\frac{1}{2}d)+\frac{1}{2})_n}{n!\,((\frac{1}{2}g))_n\,((\frac{1}{2}g)+\frac{1}{2})_n\,(\frac{1}{2})_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|)+\frac{1}{2})_n}$$

$$\times \left\{\frac{\mathcal{A}'_i\,(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\,(p+\frac{1}{2}i)_n} + \frac{\mathcal{B}'_i\,(1-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i-\frac{1}{2})\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i])\,(p+\frac{1}{2}i-\frac{1}{2})_n}\right\}$$

$$+\frac{(d)}{(g)}\,2x\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p)\,\Gamma(p+i)}{\Gamma\left(1+p+\frac{1}{2}(i+|i|)\right)}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2)^n\,((\frac{1}{2}d)+\frac{1}{2})_n\,((\frac{1}{2}d+\frac{1}{2})+\frac{1}{2})_n}{n!\,((\frac{1}{2}g+\frac{1}{2}))_n\,((\frac{1}{2}g+\frac{1}{2})+\frac{1}{2})_n\,(\frac{3}{2})_n\,(\frac{1}{2}+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(1+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n}$$

$$\times \left\{\frac{\mathcal{A}''_i\,(1-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(\frac{1}{2}+\frac{1}{2}i+p)\,\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])\,(\frac{1}{2}+\frac{1}{2}i+p)_n} + \frac{\mathcal{B}''_i\,(\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i]-\frac{1}{2})\,(p+\frac{1}{2}i)_n}\right\},$$

where $i = 0, \pm 1, \ldots, \pm 5$. Here, as usual, $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$ and its absolute value is denoted by $|x|$. The coefficients $\mathcal{A}'_i$ and $\mathcal{B}'_i$ can be obtained from the Table of $\mathcal{A}_i$ and $\mathcal{B}_i$ by simply substituting a and b with $-2n$ and $1-p-2n$, respectively, while the coefficients $\mathcal{A}''_i$ and $\mathcal{B}''_i$ can be obtained from the Table of $\mathcal{A}_i$ and $\mathcal{B}_i$ by substituting a and b with $-2n-1$ and $-p-2n$, respectively.

Proof. The proof of our first main result (5) is quite straight forward. So we give the outline of its proof. For this, denoting the left-hand side of (5) by $\mathcal{S}$ and expressing the Kampé de Fériet function in double series, we have

$$\mathcal{S} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{((d))_{m+n}}{((g))_{m+n}}\,\frac{(-1)^m\,x^{m+n}}{(p+i)_m\,(p)_n\,m!\,n!}.$$

Replacing n by $n-m$, using a well-known double series manipulation (see, *e.g.*, [20, p. 56]), using the elementary identities for Pochhammer's symbol (see, *e.g.*, [23, p. 5]), and expressing the inner sum as a ${}_2F_1$, we get

$$\mathcal{S} = \sum_{n=0}^{\infty} \frac{((d))_n}{((g))_n}\,\frac{x^n}{(p)_n\,n!}\;{}_2F_1\left[\begin{matrix}-n,\; 1-p-n\,; \\ p+i\,;\end{matrix}\; -1\right].$$

Separate the final summation into even and odd powers of x, and evaluate both ${}_2F_1$ with the help of the following generalization of classical Kummer's summation theorem (4) due to Lavoie *et al.* [18]:

$${}_2F_1\left[\begin{matrix}a,\; b\,; \\ 1+a-b+i\,;\end{matrix}\; -1\right] = \frac{2^{-a}\,\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\,\Gamma(1+a-b+i)}{\Gamma\left(1-b+\frac{1}{2}(i+|i|\right)}$$
$$\times \left\{\frac{\mathcal{A}_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\left[\frac{i+1}{2}\right]\right)\Gamma\left(1+\frac{1}{2}a-b+\frac{1}{2}i\right)} \right. \tag{6}$$
$$\left. + \frac{\mathcal{B}_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i\right)}\right\},$$

Table 1: Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
5	$-4(6+a-b)^2+2b(6+a-b)$ $+b^2+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)$ $-b^2-34(6+a-b)-b+62$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)$ $-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)$ $-b^2+16(a-b-4)-b+12$

where $i = 0, \pm 1, \ldots, \pm 5$. Here, as usual, $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$ and its absolute value is denoted by $|x|$. The coefficients $\mathcal{A}_i(a,b) := \mathcal{A}_i$ and $\mathcal{B}_i(a,b) := \mathcal{B}_i$ are given in the following table.

After some algebra, we arrive at the right-hand side of our general formula (5). This completes the proof of (5). $\square$

3. Special Cases and Applications

In this section, first we consider a few special cases of (5) and next we give an interesting application of the results in Theorem 1. For this, the special case of (5) when $i = 0$ is easily seen to yield (3). The special cases of (5) when $i = \pm 1$ are given as follows:

$$
\begin{aligned}
&F^{D:0;0}_{G:1;1}\left[\begin{matrix}(d)\ :\ \text{——};\ \text{——};\\ (g)\ :\ p+1\ ;\ \ p\ \ ;\end{matrix}\ -x,\, x\right]\\
&= {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right),\left(\frac{1}{2}d\right)+\frac{1}{2};\\ \left(\frac{1}{2}g\right),\left(\frac{1}{2}g\right)+\frac{1}{2},\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,;\end{matrix}\ -4^{D-G-1}\,x^2\right]
\end{aligned}
\tag{7}
$$

$$+\frac{(d)}{(g)}\frac{x}{p(p+1)}$$
$$\times {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right)+\frac{1}{2},\ \left(\frac{1}{2}d++\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\ p+1,\ \frac{1}{2}p+1,\ \frac{1}{2}p+\frac{3}{2}\,;\end{matrix}\ -4^{D-G-1}\,x^2\right]$$

and

$$F^{D:0;0}_{G:1;1}\left[\begin{matrix}(d)\ :\ \text{——}\,;\ \text{——}\,; \\ (g)\ :\ p-1\,;\ \ p\ \ ;\end{matrix}\ -x,\,x\right]$$
$$= {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right),\ \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ p-1,\ \frac{1}{2}p,\ \frac{1}{2}p+\frac{1}{2}\,;\end{matrix}\ -4^{D-G-1}\,x^2\right] \tag{8}$$
$$-\frac{(d)}{(g)}\frac{x}{p(p-1)}$$
$$\times {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d+\frac{1}{2}\right),\ \left(\frac{1}{2}d+\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g+\frac{1}{2}\right),\ \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,;\end{matrix}\ -4^{D-G-1}\,x^2\right].$$

Next we apply the result (5) to give generating relations for products of two Laguerre polynomials. The Laguerre polynomials are defined by (see [8])

$$L_n^{(a)}(x) = \frac{(a)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n\,; \\ a+1\,;\end{matrix}\ x\right]. \tag{9}$$

In a two-dimensional extension of a very general series transform due to Bailey [22], Exton [9] deduced the following interesting double generating relation for a product of two Laguerre polynomials:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p')_m\,(p)_n}\,L_m^{(p'-1)}(y)\,L_n^{(p-1)}(t)$$
$$= F^{D:0;0}_{G:1;1}\left[\begin{matrix}(d)\ :\ \text{——}\ \ ;\ \text{——}\,; \\ (g)\ :\ p'\ \ \ ;\ \ p\ \ ;\end{matrix}\ -xy,\ xt\right] \tag{10}$$

and deduced several interesting special cases including the following result:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p')_m\,(p)_n}\,L_m^{(p'-1)}(y)\,L_n^{(p-1)}(y)$$
$$= {}_{D+2}F_{G+3}\left[\begin{matrix}(d),\ \frac{1}{2}(p'+p-1),\ \frac{1}{2}(p'+p)\,; \\ (g),\ p',\ p,\ p'+p-1\,;\end{matrix}\ -4\,xy\right]. \tag{11}$$

Here, by using (10) and (5), we establish a general generating relation which includes eleven identities for product of two Laguerre polynomials asserted by the following theorem.

Theorem 2. *The following generating function holds true:*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p+i)_m\,(p)_n}\,L_m^{(p+i-1)}(y)\,L_n^{(p-1)}(y) = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma(p)\,\Gamma(p+i)}{\Gamma\left(p+\frac{1}{2}(i+|i|)\right)} \tag{12}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2\,y^2)^n\,((\frac{1}{2}d))_n\,((\frac{1}{2}d)+\frac{1}{2})_n}{n!\,((\frac{1}{2}g))_n\,((\frac{1}{2}g)+\frac{1}{2})_n\,(\frac{1}{2})_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|)+\frac{1}{2})_n}$$

$$\times \left\{ \frac{\mathcal{A}_i'\,(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\,(p+\frac{1}{2}i)_n} + \frac{\mathcal{B}_i'\,(1-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i-\frac{1}{2})\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i])\,(p+\frac{1}{2}i-\frac{1}{2})_n} \right\}$$

$$+\frac{(d)}{(g)}\,2x\,y\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma(p)\,\Gamma(p+i)}{\Gamma\left(1+p+\frac{1}{2}(i+|i|)\right)}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2\,y^2)^n\,((\frac{1}{2}d)+\frac{1}{2})_n\,((\frac{1}{2}d+\frac{1}{2})+\frac{1}{2})_n}{n!\,((\frac{1}{2}g+\frac{1}{2}))_n\,((\frac{1}{2}g+\frac{1}{2})+\frac{1}{2})_n\,(\frac{3}{2})_n\,(\frac{1}{2}+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(1+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n}$$

$$\times \left\{ \frac{\mathcal{A}_i''\,(1-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(\frac{1}{2}+\frac{1}{2}i+p)\,\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])\,(\frac{1}{2}+\frac{1}{2}i+p)_n} + \frac{\mathcal{B}_i''\,(\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i]-\frac{1}{2})\,(p+\frac{1}{2}i)_n} \right\},$$

where $i = 0, \pm 1, \ldots, \pm 5$, the coefficients $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{A}_i'$, $\mathcal{B}_i'$, $\mathcal{A}_i''$, $\mathcal{B}_i''$, and other notations are same as in (5).

Proof. We can derive our generating relation in a straightforward way. Indeed, if we set $t = y$ and $p' = p + i$ in (10), then, for $i = 0, \pm 1, \ldots, \pm 5$, we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p+i)_m\,(p)_n}\,L_m^{(p+i-1)}(y)\,L_n^{(p-1)}(y) \\ &\quad = F_{G:1;1}^{D:0;0}\left[\begin{array}{llll} (d) & : \text{———} & ; \text{———}; & \\ (g) & : \ p+i & ; \ p \ ; & -xy,\, xy \end{array}\right]. \end{aligned} \tag{13}$$

Replacing x by xy in (5) and applying the resulting identity to (13), we get our desired generating relation (12). This completes the proof of (12). □

We also consider some interesting special cases of Theorem 2. The special case of (12) when $i = 0$ gives the following result:

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((d))_{m+n}\,(-1)^n\,x^{m+n}}{((g))_{m+n}\,(p)_m\,(p)_n}\,L_m^{(p-1)}(y)\,L_n^{(p-1)}(y) \\ &= {}_{2D}F_{2G+3}\left[\begin{array}{c} \left(\frac{1}{2}d\right),\left(\frac{1}{2}d+\frac{1}{2}\right); \\ \left(\frac{1}{2}g\right),\left(\frac{1}{2}g+\frac{1}{2}\right),\, p,\, \frac{1}{2}p,\, \frac{1}{2}p+\frac{1}{2}\,; \end{array} -4^{D-G-1}\,x^2\,y^2\right], \end{aligned} \tag{14}$$

which is a known result due to Exton [9]. Further, in (14), if we set

(i) $D = 1$ and $G = 0$;

(ii) $D = 2$, $G = 0$, $d_1 = p$ and $d_2 = 2p$;

(iii) $D = 2$, $G = 0$, $d_1 = p$ and $d_2 = 2p - 1$,

we, respectively, obtain the following results:

$$\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(p)_{m+n}\,(-1)^n\,x^{m+n}}{(p)_m\,(p)_n}\,L_m^{(p-1)}(y)\,L_n^{(p-1)}(y) \\ &\quad = {}_0F_1\left[\begin{array}{l} -; \\ p\,; \end{array} -x^2\,y^2\right] = \Gamma(p)\,(xy)^{1-p}\,J_{p-1}(2xy), \end{aligned} \tag{15}$$

where $J_\nu(z)$ is the Bessel function of the first kind having the following connection with ${}_0F_1(\cdot)$ (see, *e.g.*, [3, p. 675]):

$$ {}_0F_1\left[\begin{matrix} \text{—} ; \\ \nu+1 ; \end{matrix} \; -\frac{z^2}{4}\right] = \Gamma(\nu+1)\left(\frac{z}{2}\right)^{-\nu} J_\nu(z); \tag{16} $$

$$ \begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(p)_{m+n}\,(2p)_{m+n}\,(-1)^n\,x^{m+n}}{(p)_m\,(p)_n} L_m^{(p-1)}(y)\,L_n^{(p-1)}(y) \\ &\qquad = {}_1F_0\left[\begin{matrix} p+\frac{1}{2} ; \\ \text{—} ; \end{matrix} \; -4\,x^2\,y^2\right] = \left(1+4\,x^2\,y^2\right)^{-p-\frac{1}{2}} ; \end{aligned} \tag{17} $$

$$ \begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(p)_{m+n}\,(2p-1)_{m+n}\,(-1)^n\,x^{m+n}}{(p)_m\,(p)_n} L_m^{(p-1)}(y)\,L_n^{(p-1)}(y) \\ &\qquad = {}_1F_0\left[\begin{matrix} p-\frac{1}{2} ; \\ \text{—} ; \end{matrix} \; -4\,x^2\,y^2\right] = \left(1+4\,x^2\,y^2\right)^{-p+\frac{1}{2}} . \end{aligned} \tag{18} $$

It is noted that the results (15) to (18) were derived by Exton [9] but the identity (18) is a corrected form.

Similarly, many other interesting results can be obtained.

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