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Some Properties of $\delta(2,2)$ Chen Ideal Submanifolds

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Abstract. In this paper we consider $\delta(2,2)$ Chen ideal submanifolds M^4 in Euclidean spaces $\mathbb{E}^6$, and investigate when such submanifolds are conformally flat, or of constant curvature, or Einstein.

1. Preliminaries

Let M^4 be a four-dimensional submanifold in the Euclidean space $\mathbb{E}^6$, $p \in M^4$, $T_p(M^4)$ be the tangent space of M^4 at p, and $\{E_1, E_2, E_3, E_4\}$ be a fixed orthonormal basis in $T_p(M^4)$. Besides, let $\{\xi_1, \xi_2\}$ be an orthonormal basis in the normal space $T_p(M^4)^\perp$. By the *equation of Gauss*, the $(0,4)$ *Riemann–Christoffel curvature tensor* of a submanifold M^4 in $\mathbb{E}^6$ is given by $R(X,Y,Z,W) = \tilde{g}(h(Y,Z), h(X,W)) - \tilde{g}(h(X,Z), h(Y,W))$, whereby $\tilde{g}$ is the Riemannian metric of the ambient Euclidean space and h is the second fundamental form of M^4 in $\mathbb{E}^6$, X, Y, Z, W being the tangent vector fields on M^4. The $(0,2)$ *Ricci curvature tensor* of M^4 is defined by $S(X,Y) = \sum_i R(X, E_i, E_i, Y)$. The *scalar curvature* of a Riemannian manifold M^n is defined by $\tau = \sum_{i<j} K(E_i \wedge E_j)$ whereby $K(E_i \wedge E_j) = R(E_i, E_j, E_j, E_i)$ is the *sectional curvature* for the plane section $\pi = E_i \wedge E_j$, $(i \neq j)$.

In [1], B.-Y. Chen introduced the $\delta(2)$ curvature which is a Riemannian scalar invariant of the manifold M^n. Later B.-Y. Chen introduced many other new scalar Riemannian invariants, which together with $\delta(2)$ are called the delta-curvatures, $\delta(n_1, n_2, ..., n_k)$, of Chen; (cfr. [2],[3],[4]). And, for all submanifolds M^n of Euclidean spaces $\mathbb{E}^{n+m}$, or of arbitrary Riemannian ambient spaces $\tilde{M}^{n+m}$, B.-Y. Chen established optimal pointwise inequalities between these intrinsic delta-curvatures of M^n and the squared mean curvature H^2, and some number determined by the curvature of the ambient space, and obtained the corresponding equality cases. In that respect, submanifolds M^n of $\tilde{M}^{n+m}$, $(n \geq 2, m \geq 1)$ which at each of their points do realise equality in the corresponding optimal inequality of B.-Y. Chen, are called *Chen ideal submanifolds* (cfr. [5],[6],[7]).

The special case of B.-Y. Chen's Theorem ([4]), Theorem 13.3, for $n_1 = n_2 = 2$, $k = 2$ is the following.

Theorem. *For any submanifold M^4 in $\mathbb{E}^6$, $\delta(2,2) \leq c(2,2)H^2$, and the equality holds at a point p, if and only if, with respect to some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ around p on M^4 in $\mathbb{E}^6$ the shape operators of M^4 in $\mathbb{E}^6$ are given by*

$$A_{\xi_\alpha} = \begin{pmatrix} b_\alpha & c_\alpha & 0 & 0 \\ c_\alpha & d_\alpha & 0 & 0 \\ 0 & 0 & e_\alpha & f_\alpha \\ 0 & 0 & f_\alpha & g_\alpha \end{pmatrix} \qquad (\alpha = 1, 2), \tag{$*$}$$

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whereby $b_\alpha + d_\alpha = e_\alpha + g_\alpha = \mu_\alpha : M^4 \to \mathbb{R}$.

The submanifolds M^4 of $\mathbb{E}^6$ for which the above Chen's inequality at all points of M^4 actually is an equality are called $\delta(2,2)$ *Chen ideal submanifolds*. ([8]). Thus, M^4 in $\mathbb{E}^6$ is a $\delta(2,2)$ Chen ideal submanifold if and only if there exists some suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ ($i = 1, \ldots, 4$; $\alpha = 1, 2$) around p on M^4 in $\mathbb{E}^6$ such that the shape operators of M^4 in $\mathbb{E}^6$ satisfy $(*)$.

In the present paper we consider $\delta(2,2)$ *Chen ideal submanifolds* $M^4 \subset \mathbb{E}^6$, and give necessary and sufficient conditions under which such submanifolds are conformally flat, or of constant curvature, or Einstein one.

In the sequel, it is convenient to introduce the following vectors in a fixed Euclidean space $\mathbb{E}^2$:

$B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$, $E = (e_1, e_2)$, $F = (f_1, f_2)$, $G = (g_1, g_2)$, $M = (\mu_1, \mu_2)$.

Then $B + D = E + G = M$, and a submanifold $M^4 \subset \mathbb{E}^6$ can be characterised by vectors B, C, D, E, F, G ($M = B + D = E + G$).

The Riemann–Christoffel curvature tensor of the $\delta(2,2)$ *Chen ideal submanifolds* $M^4 \subset \mathbb{E}^6$ is obtained by inserting the shape operator $(*)$ in the equation of Gauss. Up to the algebraic symmetries of the $(0,4)$ curvature tensor R, all non–zero components R_{ijkl} ($i, j, k, l = 1, 2, 3, 4$) of R of the considered submanifolds are the following:

$R_1 = R_{1212} = ||C||^2 - \langle B, D\rangle$, $R_2 = R_{1313} = -\langle B, E\rangle$, $R_3 = R_{1314} = -\langle B, F\rangle$,

$R_4 = R_{1323} = -\langle C, E\rangle$, $R_5 = R_{1324} = -\langle C, F\rangle = R_{1423}$,

$R_6 = R_{1414} = -\langle B, G\rangle$, $R_7 = R_{1424} = -\langle C, G\rangle$, $R_8 = R_{2323} = -\langle D, E\rangle$,

$R_9 = R_{2324} = -\langle D, F\rangle$, $R_{10} = R_{2424} = -\langle D, G\rangle$, $R_{11} = R_{3434} = ||F||^2 - \langle E, G\rangle$.

All other components of the curvature tensor R_{ijkl} ($i, j, k, l = 1, 2, 3, 4$) vanish, or differ of the previous ones mostly in sign.

Next, let S_{ij} ($i, j = 1, 2, 3, 4$) be the Ricci tensor of a $\delta(2,2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$, and τ be its scalar curvature. Then

$S_{11} = ||M||^2 - ||C||^2 - ||D||^2$, $S_{22} = ||M||^2 - ||B||^2 - ||C||^2$,

$S_{33} = ||M||^2 - ||G||^2 - ||F||^2$, $S_{44} = ||M||^2 - ||E||^2 - ||F||^2$,

$S_{12} = \langle M, C\rangle$, $S_{34} = \langle M, F\rangle$,

$S_{ij} = S_{ji}$ ($i, j = 1, 2, 3, 4$), while all other components of S vanish. Besides,

$$\tau = \mathrm{tr[S]} = \mathrm{S}_{11} + \mathrm{S}_{22} + \mathrm{S}_{33} + \mathrm{S}_{44} = 4||\mathrm{M}||^2 - ||\mathrm{B}||^2 - ||\mathrm{D}||^2 - ||\mathrm{E}||^2 - ||\mathrm{G}||^2 - \\ - 2||\mathrm{C}||^2 - 2||F||^2.$$

The *Weyl conformal curvature tensor* W_{ijkl} of such submanifold M^4 in $\mathbb{E}^6$ is defined by

$$W_{ijkl} = R_{ijkl} - \frac{1}{2}(S_{il}\delta_{jk} + S_{jk}\delta_{il} - S_{ik}\delta_{jl} - S_{jl}\delta_{ik}) + \frac{\tau}{6}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$$

($i, j, k, l = 1, 2, 3, 4$).

By the straightforward calculations we find that the main components of the tensor W read:

$W_1 = W_{1212} = ||C||^2 - \langle B, D\rangle + \frac{1}{2}(S_{11} + S_{22}) - \frac{\tau}{6}$,

$W_2 = W_{1313} = -\langle B, E\rangle + \frac{1}{2}(S_{11} + S_{33}) - \frac{\tau}{6}$,

$W_3 = W_{1314} = -\langle B, F\rangle + \frac{1}{2}S_{34}$, $W_4 = W_{1323} = -\langle C, E\rangle + \frac{1}{2}S_{12}$,

$W_5 = W_{1324} = -\langle C, F\rangle$, $W_6 = W_{1414} = -\langle B, G\rangle + \frac{1}{2}(S_{11} + S_{44}) - \frac{\tau}{6}$,

$W_7 = W_{1424} = -\langle C, G\rangle + \frac{1}{2}S_{12}$, $W_8 = W_{2323} = -\langle D, E\rangle + \frac{1}{2}(S_{22} + S_{33}) - \frac{\tau}{6}$,

$W_9 = W_{2324} = -\langle D, F\rangle + \frac{1}{2}S_{34}$, $W_{10} = W_{2424} = -\langle D, G\rangle + \frac{1}{2}(S_{22} + S_{44}) - \frac{\tau}{6}$,

$W_{11} = W_{3434} = \|F\|^2 - \langle E, G\rangle + \frac{1}{2}(S_{33} + S_{44}) - \frac{\tau}{6}$.

All other components of the Weyl tensor W_{ijkl} $(i, j, k, l = 1, 2, 3, 4)$ vanish, or differ from the previous ones mostly in sign.

For the $\delta(2, 2)$ Chen ideal submanifold M^4 in $\mathbb{E}^6$, the condition to be *conformally flat*, $W_{ijkl} = 0$, becomes:

$$\|C\|^2 - \langle B, D\rangle + \frac{1}{2}(S_{11} + S_{22}) = \frac{\tau}{6}, \tag{1}$$

$$-\langle B, E\rangle + \frac{1}{2}(S_{11} + S_{33}) = \frac{\tau}{6}, \tag{2}$$

$$-\langle B, F\rangle + \frac{1}{2}\langle M, F\rangle = 0, \tag{3}$$

$$-\langle C, E\rangle + \frac{1}{2}\langle M, C\rangle = 0, \tag{4}$$

$$-\langle C, F\rangle = 0, \tag{5}$$

$$-\langle B, G\rangle + \frac{1}{2}(S_{11} + S_{44}) = \frac{\tau}{6}, \tag{6}$$

$$-\langle C, G\rangle + \frac{1}{2}\langle M, C\rangle = 0, \tag{7}$$

$$-\langle D, E\rangle + \frac{1}{2}(S_{22} + S_{33}) = \frac{\tau}{6}, \tag{8}$$

$$-\langle D, F\rangle + \frac{1}{2}\langle M, F\rangle = 0, \tag{9}$$

$$-\langle D, G\rangle + \frac{1}{2}(S_{22} + S_{44}) = \frac{\tau}{6}, \tag{10}$$

$$\|F\|^2 - \langle E, G\rangle + \frac{1}{2}(S_{33} + S_{44}) = \frac{\tau}{6}. \tag{11}$$

2. Main Results

First, assume that a $\delta(2, 2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$ is *conformally flat*, i.e. the all equations (1)–(11) hold true. Since

$$\langle B, D\rangle = \frac{1}{2}(\|M\|^2 - \|B\|^2 - \|D\|^2), \quad \langle E, G\rangle = \frac{1}{2}(\|M\|^2 - \|E\|^2 - \|G\|^2),$$

by adding equations (1) and (11) we find

$$\|M\|^2 = \|B\|^2 + \|D\|^2 + \|E\|^2 + \|G\|^2 + 2\|C\|^2 + 2\|F\|^2. \tag{12}$$

Substituting (12) in the expression for τ, we find that

$$\tau = 3\|M\|^2. \tag{13}$$

Next, since

$\|M\|^2 = \frac{1}{2}\|M\|^2 + \frac{1}{2}\|M\|^2 = \frac{1}{2}\|B + D\|^2 + \frac{1}{2}\|E + G\|^2 \le \|B\|^2 + \|D\|^2 + \|E\|^2 + \|G\|^2$

by equation (12) we immediately have that $C = F = 0$. Hence, $S_{12} = S_{34} = 0$,

$S_{11} = \|M\|^2 - \|D\|^2, \quad S_{22} = \|M\|^2 - \|B\|^2, \quad S_{33} = \|M\|^2 - \|G\|^2,$

$S_{44} = \|M\|^2 - \|E\|^2, \quad \|M\|^2 = \|B\|^2 + \|D\|^2 + \|E\|^2 + \|G\|^2.$

By equation (2) we immediately find that $\|B - E\|^2 = 0$, thus $B = E$. Similarly, by equation (6) we get $\|B - G\|^2 = 0$, i.e. $B = G$. Therefore, $B = E = G = M/2$, and consequently $D = M/2$. Hence, $B = D = E = G = M/2$.

Conversely, if $C = F = 0$ and $B = D = E = G = M/2$, where $M \in \mathbb{E}^2$ is arbitrary, then

$$S_{11} = S_{22} = S_{33} = S_{44} = \tfrac{3}{4}\|M\|^2, \quad \tau = 3\|M\|^2, \quad S_{12} = S_{34} = 0,$$

and a straightforward calculation shows that $W_i = 0$ ($i = 1, \dots, 11$). So, M^4 is conformally flat submanifold. Hence we get

Theorem 2.1. *A $\delta(2,2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$ is conformally flat if and only if $C = F = 0$ and $B = D = E = G = M/2$, where $M \in \mathbb{E}^2$ is arbitrary.*

If $M^4 \subset \mathbb{E}^6$ is conformally flat, then the corresponding shape operators read $A_{\xi_i} = m_i I_4$ ($i = 1, 2$), where I_4 is the unit matrix of order 4 and $m_1, m_2 \in \mathbb{R}$ are arbitrary scalars.

Corollary 2.1. *A $\delta(2,2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$ is conformally flat if and only if it is either an open portion of a totally geodesic 4–space $\mathbb{E}^4 \subset \mathbb{E}^6$ or an open portion of an ordinary hypersphere in a hyperplane of $\mathbb{E}^6$.*

Next, we consider another property of the $\delta(2,2)$ Chen ideal submanifolds. By a straightforward calculations it is easy to check that if such a submanifold $M^4 \subset \mathbb{E}^6$ is conformally flat, then it satisfies the relation

$$R_{ijkl} = \frac{\tau}{12}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \quad (i, j, k, l = 1, 2, 3, 4),$$

thus, it is of *constant curvature*. Since the converse is obviously true, we get the following Corollary.

Corollary 2.2. *A $\delta(2,2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$ is conformally flat if and only if it is of constant curvature.*

Finally, we discuss when such a submanifold $M^4 \subset \mathbb{E}^6$ is an *Einstein submanifold*, i.e. it satisfies $G_{ij} = S_{ij} - \frac{\tau}{4}\delta_{ij} = 0$ ($i, j = 1, 2, 3, 4$). The above condition means that $\langle M, C\rangle = \langle M, F\rangle = 0$, i.e. $M \perp F, C$, and

$$S_{11} = \|M\|^2 - \|C\|^2 - \|D\|^2 = \tfrac{\tau}{4}, \quad S_{22} = \|M\|^2 - \|C\|^2 - \|B\|^2 = \tfrac{\tau}{4},$$
$$S_{33} = \|M\|^2 - \|F\|^2 - \|G\|^2 = \tfrac{\tau}{4}, \quad S_{44} = \|M\|^2 - \|E\|^2 - \|F\|^2 = \tfrac{\tau}{4}.$$

Hence we have the following.

Proposition 2.1. *A $\delta(2,2)$ Chen ideal submanifold $M^4 \subset \mathbb{E}^6$ is Einstein if and only if $F, C \perp M$ and $\|B\| = \|D\|$, $\|E\| = \|G\|$, $\|B\|^2 + \|C\|^2 = \|E\|^2 + \|F\|^2$ holds true.*

Example 2.1. If $B = D = E = G = M/2$ ($M \in \mathbb{E}^2$), and $C, F \perp M$, $\|C\| = \|F\|$, then M^4 is an Einstein submanifold.

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