



Optimality Conditions via Scalarization for Approximate Quasi Efficiency in Multiobjective Optimization

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Abstract. Approximate problems that scalarize and approximate a given multiobjective optimization problem (MOP) became an important and interesting area of research, given that, in general, are simpler and have weaker existence requirements than the original problem. Recently, necessary conditions for approximation of several types of efficiency for MOPs have been obtained through the use of an alternative theorem. In this paper, we use these results in order to extend them to sufficient conditions for approximate quasi (weak, proper) efficiency. For this, we use two scalarization techniques of Tchebycheff type. All the provided results are established without convexity assumptions.

1. Introduction

Multiobjective optimization is an interesting research field with many applications, concerning economy, engineering and medicine. During the last years, a lot of studies have been reported about multiobjective optimization in many papers and books. From the large amount of publications, we refer to [1, 5, 6, 24], where the most theoretical aspects of multiobjective optimization are treated.

During the recent years, much attention was paid to approximate efficient solutions of a multiobjective optimization problem. The reasons of interest in approximate efficiency in an MOP are the following actualities: First, the mathematical models are simplified versions of the practical problems, and therefore are not a complete copy of the main problem. Second, numerical algorithms (as heuristic and interactive algorithms) may provide approximate solutions. Moreover, the efficient set of an MOP might be empty in a non-compact instance, while approximate efficient set might be nonempty under very weaker requirements. The concept of approximate solution was introduced by Kutateladze [20]. Thereafter, White [30] studied six concepts of an approximate solution for MOPs, and Loridan [23] investigated some properties of one of these concepts. Thereafter, many researchers considered some properties of and the relationships between these notions [14, 15, 32]. Moreover, extensions of approximate efficient solutions of an MOP can be seen in [3, 8, 9, 13, 15]. Furthermore, some necessary and sufficient conditions for ε -(weak) efficiency can be found in [4, 7, 9, 11, 12, 19, 26]. Also, applications of approximate efficiency in radiotherapy treatment planning considered in [27, 28]. Shao and Ehrgott [27, 28] showed that approximate efficient solutions can be obtained very faster than efficient solutions of a given MOP.

Regarding the above discussion, investigating the notion of approximate efficiency in MOPs is typically interesting both in theoretical and practical points of view. However, like efficient solutions, there are

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also ε -efficient solutions with undesired properties, and the trade-off among criteria might be unbounded. Therefore, we need to filter out the bad ε -efficient and keep the so called ε -properly efficient solutions.

Li and Wang [21] introduced the concept of ε -proper efficiency, an extension of Geoffrion proper efficiency [10], and via scalarization derived necessary conditions for ε -properly efficient solutions of a nonconvex MOP. Thereafter, using the weighted sum method, Liu [22] obtained a necessary and sufficient condition for ε -properly efficient solutions of a convex MOP. More recently, Ghaznavi-ghosoni¹⁾ and Khorram [11], Ghaznavi-ghosoni et al. [12], Rastegar and Khorram [26] and Khaledian et al. [19] derived necessary and sufficient conditions for ε -proper efficient solutions of a general (with no convexity assumption) multiobjective optimization problem, via scalarization.

Recently, Beldiman et al. [2] considered approximate quasi (weak, proper) efficiency in multiobjective optimization and derived necessary conditions for these kinds of approximate solutions by using an alternative theorem. Their concepts, generalize definitions of (weak, proper) efficiency as well as ε -(weak, proper) efficiency. Also, Panaitescu and Dogaru [25] derived two necessary conditions for ε -quasi proper efficient points as well as two sufficient conditions for ε -quasi efficient solutions of a general MOP. Moreover, Ying [31] introduced the notion of ε -quasi weakly saddle points, and obtained necessary and sufficient for the existence of ε -quasi weakly saddle points.

Motivated by the $(\varepsilon, \bar{\varepsilon})$ -quasi (weak, proper) efficiency defined in [2], in this paper, we derive some optimality conditions for these concepts, via scalarization. More precisely, utilizing the augmented weighted Tchebycheff method [29], and the modified weighted Tchebycheff method [17, 18], we complete and extend the results obtained in [2], and get necessary and sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi (weakly, properly) efficient points of an MOP. Since approximate quasi (weak, proper) efficiency extends the notion of approximate (weak, proper) efficiency, the obtained results extend some theorems in [7, 11, 12, 21]. It is worth mentioning that the obtained results are general and we do not assume no convexity assumption.

The outline of the paper is as follows: In Section 2, we provide some preliminaries, Section 3 contains the main results of the paper, and Section 4 is devoted to the conclusions.

2. Preliminaries

Consider the following multiobjective optimization problem (MOP):

$$\begin{aligned} \min f(\mathbf{x}) \\ \text{s.t. } \mathbf{x} \in X, \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is a nonempty set, and $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))^T : X \rightarrow \mathbb{R}^m$ is a vector-valued function. The image of X under f , called the set of attainable (or image) outcomes, is denoted by $\mathcal{Y} := f(X) \subset \mathbb{R}^m$. With regard to vector inequalities, the following convention will be applied: for $\mathbf{y}^1, \mathbf{y}^2 \in \mathbb{R}^m$, $\mathbf{y}^1 \leq \mathbf{y}^2$ if and only if $y_i^1 \leq y_i^2$, for all $i = 1, \dots, m$ and $\mathbf{y}^1 < \mathbf{y}^2$ if and only if $\mathbf{y}^1 \leq \mathbf{y}^2$ and $\mathbf{y}^1 \neq \mathbf{y}^2$. Moreover, $\mathbf{y}^1 < \mathbf{y}^2$ if and only if $y_i^1 < y_i^2$, for each $i = 1, \dots, m$. The Pareto cone is defined as $\mathbb{R}_{\geq}^m = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} \geq 0\}$.

We recall, the set of efficient (Pareto optimal) and weakly efficient (weakly Pareto optimal) points of MOP (1) are defined, respectively, as:

$$X_E = \{\hat{\mathbf{x}} \in X : \nexists \mathbf{x} \in X \text{ s.t. } f(\mathbf{x}) \leq f(\hat{\mathbf{x}})\},$$

$$X_{WE} = \{\hat{\mathbf{x}} \in X : \nexists \mathbf{x} \in X \text{ s.t. } f(\mathbf{x}) < f(\hat{\mathbf{x}})\}.$$

Definition 2.1. [5] Consider MOP (1).

- (i) The point $\mathbf{y}^I = (y_1^I, \dots, y_m^I)$, in which $y_i^I := \min_{\mathbf{x} \in X} f_i(\mathbf{x})$, $i = 1, \dots, m$, is called the ideal point.
- (ii) The point $\mathbf{y}^U := \mathbf{y}^I - \alpha$, in which $\alpha \in \mathbb{R}_{>}^m$, is said an utopia point.

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Definition 2.2. [2] Let $\varepsilon \in \mathbb{R}_{\geq}^m$. A feasible solution $\hat{\mathbf{x}} \in \mathcal{X}$ is called:

- (i) ε -quasi weakly efficient (ε -quasi weakly Pareto optimal) solution of MOP (1), if there is no other $\mathbf{x} \in \mathcal{X}$ such that $f(\mathbf{x}) < f(\hat{\mathbf{x}}) - \varepsilon\|\mathbf{x} - \hat{\mathbf{x}}\|$,
- (ii) ε -quasi efficient (ε -quasi Pareto optimal) solution of MOP (1), if there is no other $\mathbf{x} \in \mathcal{X}$ such that $f(\mathbf{x}) \leq f(\hat{\mathbf{x}}) - \varepsilon\|\mathbf{x} - \hat{\mathbf{x}}\|$,
- (iii) ε -quasi properly efficient solution of MOP (1), if it is ε -quasi efficient and there exists $M > 0$ such that for any $i \in \{1, 2, \dots, m\}$ and $\mathbf{x} \in \mathcal{X}$ gratifying $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}}) - \varepsilon_i\|\mathbf{x} - \hat{\mathbf{x}}\|$, there exists an index $j \in \{1, 2, \dots, m\}$ with $f_j(\hat{\mathbf{x}}) < f_j(\mathbf{x}) + \varepsilon_j\|\mathbf{x} - \hat{\mathbf{x}}\|$ such that

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x}) - \varepsilon_i\|\mathbf{x} - \hat{\mathbf{x}}\|}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}}) + \varepsilon_j\|\mathbf{x} - \hat{\mathbf{x}}\|} \leq M.$$

Following Beldiman et al. [2], $(\varepsilon, \bar{\varepsilon})$ -quasi (weakly, properly) efficient solutions of MOP (1) are defined as follows:

Definition 2.3. [2] Consider MOP (1). Let $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{\geq}^m \times \mathbb{R}_{\geq}^m$. A feasible solution $\hat{\mathbf{x}} \in \mathcal{X}$ is called:

- (i) $(\varepsilon, \bar{\varepsilon})$ -quasi weakly efficient, if there is no other $\mathbf{x} \in \mathcal{X}$ such that $f(\mathbf{x}) < f(\hat{\mathbf{x}}) - \varepsilon\|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}$,
- (ii) $(\varepsilon, \bar{\varepsilon})$ -quasi efficient, if there is no other $\mathbf{x} \in \mathcal{X}$ such that $f(\mathbf{x}) \leq f(\hat{\mathbf{x}}) - \varepsilon\|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}$,
- (iii) $(\varepsilon, \bar{\varepsilon})$ -quasi properly efficient, if it is $(\varepsilon, \bar{\varepsilon})$ -quasi efficient and there exists $M > 0$ such that for any $i \in \{1, 2, \dots, m\}$ and $\mathbf{x} \in \mathcal{X}$ satisfying $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}}) - \varepsilon_i\|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i$, there exists an index $j \in \{1, 2, \dots, m\}$ such that $f_j(\hat{\mathbf{x}}) < f_j(\mathbf{x}) + \varepsilon_j\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_j$ and

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x}) - \varepsilon_i\|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}}) + \varepsilon_j\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_j} \leq M.$$

We denote the set of all $(\varepsilon, \bar{\varepsilon})$ -quasi weakly efficient, $(\varepsilon, \bar{\varepsilon})$ -quasi efficient and $(\varepsilon, \bar{\varepsilon})$ -quasi properly efficient solutions of an MOP by $\mathcal{X}_{\varepsilon, \bar{\varepsilon}WE}$, $\mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$ and $\mathcal{X}_{\varepsilon, \bar{\varepsilon}PE}$, respectively.

Remark 2.4. Referring to Definition 2.3, it is obvious that

- (i) $\mathcal{X}_{\varepsilon, \bar{\varepsilon}PE} \subseteq \mathcal{X}_{\varepsilon, \bar{\varepsilon}E} \subseteq \mathcal{X}_{\varepsilon, \bar{\varepsilon}WE}$.
- (ii) For the case $\varepsilon = \bar{\varepsilon} = 0$, the definitions of $(\varepsilon, \bar{\varepsilon})$ -quasi (weak, proper) efficiency coincide with the definitions of (weak, proper) efficiency, respectively. Therefore, $(\varepsilon, \bar{\varepsilon}) \neq 0$ would be of interest

A useful technique to solve an MOP, is to formulate a single objective program (SOP) corresponding to the given MOP. Let $g : \mathcal{X} \rightarrow \mathbb{R}$ and consider a SOP as follows:

$$\begin{aligned} & \min g(\mathbf{x}) \\ & \text{s.t. } \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{2}$$

Definition 2.5. [16] Let $\delta \in \mathbb{R}_{\geq}$. A point $\hat{\mathbf{x}} \in \mathcal{X}$ is called to be

- (i) an optimal solution of SOP (2), if $g(\hat{\mathbf{x}}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$,
- (ii) a δ -optimal solution of SOP (2), if $g(\hat{\mathbf{x}}) - \delta \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$,
- (iii) a δ -quasi optimal solution of SOP (2), if $g(\hat{\mathbf{x}}) - \delta\|\mathbf{x} - \hat{\mathbf{x}}\| \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$.

Remark 2.6. Referring to Definition 2.5, it is obvious that optimality $\Rightarrow \delta$ – optimality and optimality $\Rightarrow \delta$ – quasi optimality, but the inverse inclusions are not necessarily true.

Definition 2.7. [2] Let $(\delta, \bar{\delta}) \in \mathbb{R}_{\geq} \times \mathbb{R}_{\geq}$. A feasible solution $\hat{\mathbf{x}} \in \mathcal{X}$ is said

- (i) a $(\delta, \bar{\delta})$ –quasi optimal solution of SOP (2), if $g(\hat{\mathbf{x}}) \leq g(\mathbf{x}) + \delta\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\delta}$ for all $\mathbf{x} \in \mathcal{X}$,
- (ii) a strictly $(\delta, \bar{\delta})$ –quasi optimal solution of SOP (2), if $g(\hat{\mathbf{x}}) < g(\mathbf{x}) + \delta\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\delta}$ for all $\mathbf{x} \in \mathcal{X}$.

Remark 2.8. It can be seen that for $\delta = \bar{\delta} = 0$, in (i), we have the optimality definition, for $\bar{\delta} = 0$ we reach to δ –quasi optimality and for $\delta = 0$ we get $\bar{\delta}$ –optimality concept.

3. Main Results

The main results, in this section, are necessary and sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ –quasi (weak, proper) efficient solutions of MOP (1), using the following scalarization problems.

- (a) The augmented weighted Tchebycheff scalarization problem (SOP(3)) [29]:

$$\min_{i=1, \dots, m} \max \lambda_i(f_i(\mathbf{x}) - y_i^U) + \sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U) \tag{3}$$

s.t. $\mathbf{x} \in \mathcal{X}$,

where $I = \{1, \dots, m\}$, \mathbf{y}^U is an utopia point, $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_{\geq}^m$ and $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{R}_{\geq}^m$.

- (b) The modified weighted Tchebycheff scalarization problem (SOP(4)) [17]:

$$\min_{i=1, \dots, m} \max \lambda_i[(f_i(\mathbf{x}) - y_i^U) + \sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U)] \tag{4}$$

s.t. $\mathbf{x} \in \mathcal{X}$,

where $I = \{1, \dots, m\}$, \mathbf{y}^U is an utopia point, $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_{\geq}^m$ and $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{R}_{\geq}^m$.

Remark 3.1. It should be noted that, unlike the formulation (3), the bounds on trade-offs generated by formulation (4) are independent of parameters $\lambda_i, \forall i \in I$. For more details about bounds on trade-offs see [17, 18].

3.1. Case of $(\varepsilon, \bar{\varepsilon})$ –quasi weak efficiency

Recently, Beldiman et al. [2] provided a necessary condition for approximate quasi weakly efficient solutions of an MOP (see Theorem 1 in [2]). In the following theorem, via SOPs (3) and(4), we give sufficient conditions for generating $(\varepsilon, \bar{\varepsilon})$ –quasi weakly efficient solutions of MOP (1).

Theorem 3.2. Consider MOP (1). Let $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{\geq}^m \times \mathbb{R}_{\geq}^m$, $\lambda \in \mathbb{R}_{>}^m$, and $\rho \in \mathbb{R}_{\geq}^m$.

- (i) If $\hat{\mathbf{x}} \in \mathcal{X}$ is a $(\delta, \bar{\delta})$ –quasi optimal solution of SOP (3) with $\delta \leq \min_{i=1, \dots, m} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$ and $\bar{\delta} \leq \min_{i=1, \dots, m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}} \text{WE}$.
- (ii) If $\hat{\mathbf{x}} \in \mathcal{X}$ is a $(\delta, \bar{\delta})$ –quasi optimal solution of SOP (4) with $\delta \leq \min_{i=1, \dots, m} \lambda_i(\varepsilon_i + \sum_{t \in I} \rho_t \varepsilon_t)$ and $\bar{\delta} \leq \min_{i=1, \dots, m} \lambda_i(\bar{\varepsilon}_i + \sum_{t \in I} \rho_t \bar{\varepsilon}_t)$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}} \text{WE}$.

Proof. (i) By contradiction, suppose that $\hat{\mathbf{x}} \notin \mathcal{X}_{\varepsilon, \bar{\varepsilon}} \text{WE}$. Therefore, there exists $\mathbf{x} \in \mathcal{X}, \mathbf{x} \neq \hat{\mathbf{x}}$ with $f(\mathbf{x}) < f(\hat{\mathbf{x}}) - \varepsilon\|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}$. Hence,

$$\sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U) + \varepsilon_t\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t \leq \sum_{t \in I} \rho_t(f_t(\hat{\mathbf{x}}) - y_t^U),$$

and

$$\max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U + \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) < \max_{i=1,\dots,m} \lambda_i(f_i(\hat{\mathbf{x}}) - y_i^U).$$

As a result,

$$\max_{i=1,\dots,m} \lambda_i(f_i(\hat{\mathbf{x}}) - y_i^U) + \sum_{t \in I} \rho_t(f_t(\hat{\mathbf{x}}) - y_t^U) >$$

$$\max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U + \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) + \sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U + \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t) \geq$$

$$\max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U) + \|\mathbf{x} - \hat{\mathbf{x}}\| \left(\min_{i=1,\dots,m} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t \right) + \left(\min_{i=1,\dots,m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t \right) + \sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U) \geq$$

$$\max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U) + \sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U) + \delta \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\delta},$$

which is a contradiction.

(ii) The proof is similar to Part (i). \square

The following example shows that the weakly efficient set of an MOP might be empty, while the approximate quasi weakly efficient set is nonempty. In this example, the feasible set is not closed.

Example 3.3. Consider a biobjective optimization problem with feasible set

$$\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 > 2, x_1, x_2 \geq 0\}$$

and objective function $f(x_1, x_2) = (x_1, x_2)$. The image (or criterion) space is depicted in Figure 1.

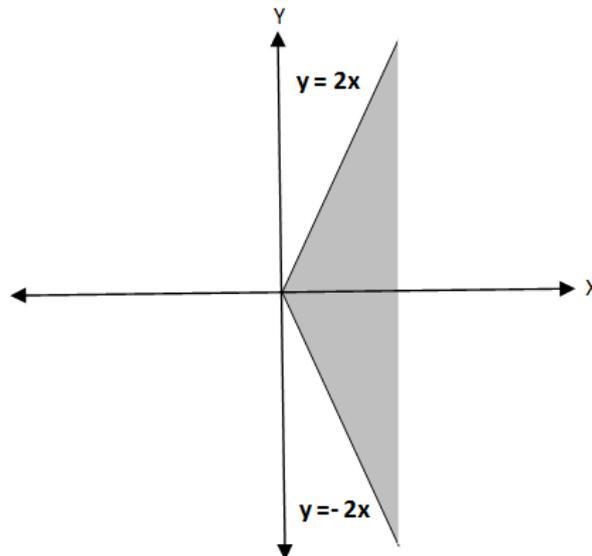


Figure 1: The image space in Example 3.3

Clearly, there are no weakly efficient solutions for this biobjective optimization problem. Now, assume that $\varepsilon = \bar{\varepsilon} = (0.1, 0.1)$ and $\mathbf{y}^U = (-1, -1)$. We choose the augmented Tchebycheff scalarization problem (3) with the parameters $(\lambda_1, \lambda_2) = (1, 1)$ and $(\rho_1, \rho_1) = (0, 0)$. By Theorem 3.2 we compute the parameters δ and $\bar{\delta}$ as $\delta = 0.1$ and $\bar{\delta} = 0.1$. Therefore, the related scalarized problem is as follows:

$$\min_{\mathbf{x} \in \mathcal{X}} \max_{i=1,2} \{x_1 + 1, x_2 + 1\}. \tag{5}$$

It is easy to see that $\hat{\mathbf{x}} = (1.1, 1.1)$ is a $(\delta, \bar{\delta})$ -quasi optimal solution for the SOP (5). Therefore, by Theorem 3.2, $\hat{\mathbf{x}}$ is an $(\varepsilon, \bar{\varepsilon})$ -quasi weakly efficient solution for the given biobjective optimization problem.

Now, related to $(\varepsilon, \bar{\varepsilon})$ -quasi weak efficiency, we provide necessary conditions.

Theorem 3.4. Let $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}WE}$ for some given $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{\geq}^m \times \mathbb{R}_{\geq}^m$. Then,

- (i) There exist $\rho \in \mathbb{R}_{\geq}^m$ and $\lambda \in \mathbb{R}_{\geq}^m$ such that $\hat{\mathbf{x}}$ is a $(\delta, \bar{\delta})$ -quasi optimal solution for SOP (3) with $\delta = \max_{i=1, \dots, m} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$ and $\bar{\delta} = \max_{i=1, \dots, m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$.
- (ii) There exist $\rho \in \mathbb{R}_{\geq}^m$ and $\lambda \in \mathbb{R}_{\geq}^m$ such that $\hat{\mathbf{x}}$ is a $(\delta, \bar{\delta})$ -quasi optimal solution for SOP (4) with $\delta = \max_{i=1, \dots, m} \lambda_i (\varepsilon_i + \sum_{t \in I} \rho_t \varepsilon_t)$ and $\bar{\delta} = \max_{i=1, \dots, m} \lambda_i (\bar{\varepsilon}_i + \sum_{t \in I} \rho_t \bar{\varepsilon}_t)$.

Proof. The proofs are consequences of [2, Theorem 1] by putting $\rho = 0$ and $\lambda_i = [f_i(\hat{\mathbf{x}}) - y_i^U]^{-1}$, $i \in I$. \square

3.2. Case of $(\varepsilon, \bar{\varepsilon})$ -quasi efficiency

In this subsection, we will study the connections between $(\delta, \bar{\delta})$ -quasi optimal solutions of scalarized Problems (3) and (4) and $(\varepsilon, \bar{\varepsilon})$ -quasi efficient solutions of MOP (1).

Theorem 3.5. Consider MOP (1) and let $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{\geq}^m \times \mathbb{R}_{\geq}^m$.

- (i) If $\hat{\mathbf{x}} \in \mathcal{X}$ is a $(\delta, \bar{\delta})$ -quasi optimal solution of SOP (3) with $\lambda \in \mathbb{R}_{\geq}^m$ and $\rho \in \mathbb{R}_{\geq}^m$, such that $\delta \leq \min_{i=1, \dots, m} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$ and $\bar{\delta} \leq \min_{i=1, \dots, m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$.
- (ii) If $\hat{\mathbf{x}} \in \mathcal{X}$ is a $(\delta, \bar{\delta})$ -quasi optimal solution of SOP (4) with $\lambda \in \mathbb{R}_{\geq}^m$ and $\rho \in \mathbb{R}_{\geq}^m$, such that $\delta \leq \min_{i=1, \dots, m} \lambda_i (\varepsilon_i + \sum_{t \in I} \rho_t \varepsilon_t)$ and $\bar{\delta} \leq \min_{i=1, \dots, m} \lambda_i (\bar{\varepsilon}_i + \sum_{t \in I} \rho_t \bar{\varepsilon}_t)$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$.

Proof. (i) Assume that $\hat{\mathbf{x}} \notin \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$. Then, there exists $\mathbf{x} \in \mathcal{X}$, $\mathbf{x} \neq \hat{\mathbf{x}}$ with $f(\mathbf{x}) \leq f(\hat{\mathbf{x}}) - \varepsilon \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}$. Therefore,

$$\sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U + \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t) < \sum_{t \in I} \rho_t (f_t(\hat{\mathbf{x}}) - y_t^U),$$

and

$$\lambda_i (f_i(\mathbf{x}) - y_i^U + \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) \leq \lambda_i (f_i(\hat{\mathbf{x}}) - y_i^U), \text{ for all } i \in \{1, \dots, m\}.$$

Consequently,

$$\max_{i=1, \dots, m} \lambda_i (f_i(\hat{\mathbf{x}}) - y_i^U) + \sum_{t \in I} \rho_t (f_t(\hat{\mathbf{x}}) - y_t^U) >$$

$$\max_{i=1, \dots, m} \lambda_i (f_i(\mathbf{x}) - y_i^U + \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) + \sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U + \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t) \geq$$

$$\max_{i=1, \dots, m} \lambda_i (f_i(\mathbf{x}) - y_i^U) + \|\mathbf{x} - \hat{\mathbf{x}}\| \left(\min_{i=1, \dots, m} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t \right) + \left(\min_{i=1, \dots, m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t \right) + \sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U) \geq$$

$$\max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U) + \sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U) + \delta \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\delta}.$$

This makes a contradiction. Therefore, $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$.

(ii) Suppose $\hat{\mathbf{x}} \notin \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$. Then, there exists $\mathbf{x} \in \mathcal{X}$ with $\mathbf{x} \neq \hat{\mathbf{x}}$ and $f(\mathbf{x}) \leq f(\hat{\mathbf{x}}) - \varepsilon \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}$. Hence,

$$\begin{aligned} & \max_{i=1,\dots,m} \lambda_i(f_i(\hat{\mathbf{x}}) - y_i^U + \sum_{t \in I} \rho_t(f_t(\hat{\mathbf{x}}) - y_t^U)) > \\ & \max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U + \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i + \sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U + \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon})) \geq \\ & \geq \max_{i=1,\dots,m} \lambda_i(f_i(\mathbf{x}) - y_i^U + \sum_{t \in I} \rho_t(f_t(\mathbf{x}) - y_t^U)) + \delta \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\delta}, \end{aligned}$$

which contradicts with $(\delta, \bar{\delta})$ -quasi optimality of $\hat{\mathbf{x}}$. \square

Remark 3.6. Beldiman et al. [2, Theorems 4 and 5] used SOPs (3) and (4) to obtain sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi efficiency. The following facts show that the results given in Theorem 3.5, provide better sufficient conditions than those obtained in [2]:

- (a) $\rho > 0$ and $\lambda > 0 (\geq 0)$ given in Theorem 3.5 are arbitrary, while in [2], $\lambda > 0$ is required to take a pre-specified value and its value devolves on $(\varepsilon, \bar{\varepsilon})$ -quasi efficient solution and it makes difficult using of that theorem.
- (b) In Theorem 3.5, decision maker allows to select $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{\geq}^m \times \mathbb{R}_{\geq}^m$, while in [2] all components of $(\varepsilon, \bar{\varepsilon})$ must be positive, that is $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{>}^m \times \mathbb{R}_{>}^m$.

In the following theorem, we provide some other sufficient conditions for generating $(\varepsilon, \bar{\varepsilon})$ -quasi efficient solutions of MOP (1). The proofs are similar to Theorem 3.5 and therefore are omitted.

Theorem 3.7. Given $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{\geq}^m \times \mathbb{R}_{\geq}^m$.

- (i) If $\hat{\mathbf{x}} \in \mathcal{X}$ is a strictly $(\delta, \bar{\delta})$ -quasi optimal point of SOP (3) with $\lambda \in \mathbb{R}_{>}^m$ and $\rho \in \mathbb{R}_{\geq}^m$, such that $\delta \leq \min_{i=1,\dots,m} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$ and $\bar{\delta} \leq \min_{i=1,\dots,m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$.
- (ii) If $\hat{\mathbf{x}} \in \mathcal{X}$ is a strictly $(\delta, \bar{\delta})$ -quasi optimal point of SOP (4) with $\lambda \in \mathbb{R}_{\geq}^m$ and $\rho \in \mathbb{R}_{\geq}^m$, such that $\delta \leq \min_{i=1,\dots,m} \lambda_i (\varepsilon_i + \sum_{t \in I} \rho_t \varepsilon_t)$ and $\bar{\delta} \leq \min_{i=1,\dots,m} \lambda_i (\bar{\varepsilon}_i + \sum_{t \in I} \rho_t \bar{\varepsilon}_t)$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$.

3.3. Case of $(\varepsilon, \bar{\varepsilon})$ -quasi proper efficiency

Beldiman et al. [2, Theorems 2 and 3], using SOPs (3) and (4), obtained necessary conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi properly efficient solutions. Now, in the following theorem, we complete and extend their results to sufficient conditions for approximate quasi properly efficient solutions.

Theorem 3.8. Given $(\varepsilon, \bar{\varepsilon}) \in \mathbb{R}_{>}^m \times \mathbb{R}_{>}^m$.

- (i) Let $\hat{\mathbf{x}}$ be a $(\delta, \bar{\delta})$ -quasi optimal point to SOP (3) with $\delta = \max_{i \in I} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$, $\bar{\delta} = \max_{i \in I} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$, $\rho \in \mathbb{R}_{>}^m$, and y_i^U such that for any $i \in I$, $\lambda_i := [f_i(\hat{\mathbf{x}}) - y_i^U]^{-1} > 0, i = 1, \dots, m$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}PE}$.
- (ii) Let $\hat{\mathbf{x}}$ be a $(\delta, \bar{\delta})$ -quasi optimal point to SOP (4) with $\delta = \max_{i \in I} \lambda_i (\varepsilon_i + \sum_{t \in I} \rho_t \varepsilon_t)$, $\bar{\delta} = \max_{i \in I} \lambda_i (\bar{\varepsilon}_i + \sum_{t \in I} \rho_t \bar{\varepsilon}_t)$, $\rho \in \mathbb{R}_{>}^m$, and y_i^U such that for any $i \in I$, $\lambda_i := [f_i(\hat{\mathbf{x}}) - y_i^U + \sum_{t \in I} \rho_t (f_t(\hat{\mathbf{x}}) - y_t^U)]^{-1} > 0, i = 1, \dots, m$, then $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}PE}$.

Proof. (i) ε -efficiency of $\hat{\mathbf{x}}$ follows from [2, Theorem 5]. Consider $i \in I$ and $\mathbf{x} \in \mathcal{X}$ with $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}}) - \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i$. To prove, $(\varepsilon, \bar{\varepsilon})$ -quasi proper efficiency of $\hat{\mathbf{x}}$, we need to show that there exist $M > 0$ and an index $j \in \{1, 2, \dots, m\}$ such that $f_j(\hat{\mathbf{x}}) - \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_j < f_j(\mathbf{x})$ and

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x}) - \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}}) + \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_j} \leq M. \tag{6}$$

Now, we define:

$$M = \max_{k, h \in I} \left\{ \frac{\lambda_h + \sum_{t \in I} \rho_t}{\rho_k} \right\}.$$

Since $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}}$, then there exists $t \in \{1, \dots, m\}$ such that $f_t(\hat{\mathbf{x}}) - \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_t < f_t(\mathbf{x})$. Define

$$f_j(\hat{\mathbf{x}}) - \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_j - f_j(\mathbf{x}) = \min_{t \in I} (f_t(\hat{\mathbf{x}}) - \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_t - f_t(\mathbf{x})). \tag{7}$$

It is obvious that $f_j(\hat{\mathbf{x}}) - \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_j - f_j(\mathbf{x}) < 0$. We claim that the above index j is the desired one in (6). By contradiction, assume that

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x}) - \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}}) + \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_j} > M > \frac{\max_{h \in I} \{\lambda_h\} + \sum_{t \neq i} \rho_t}{\rho_i}.$$

Therefore

$$\begin{aligned} & \max_{h \in I} \{\lambda_h\} (f_j(\hat{\mathbf{x}}) - f_j(\mathbf{x}) - \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_j) + \\ & \sum_{t \neq i} \rho_t (f_j(\hat{\mathbf{x}}) - f_j(\mathbf{x}) - \varepsilon_j \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_j) + \rho_i (f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x}) - \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i) > 0. \end{aligned}$$

Thus by (7), we have

$$\lambda_k (f_k(\hat{\mathbf{x}}) - f_k(\mathbf{x}) - \varepsilon_k \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_k) + \sum_{t \in I} \rho_t (f_t(\hat{\mathbf{x}}) - f_t(\mathbf{x}) - \varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_t) > 0, \forall k \in I.$$

Therefore,

$$\begin{aligned} & \max_{i \in I} \lambda_i (f_i(\mathbf{x}) - y_i^U) + \sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U) < \\ & \max_{i \in I} \lambda_i (f_i(\hat{\mathbf{x}}) - y_i^U - \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i) + \sum_{t \in I} \rho_t (f_t(\hat{\mathbf{x}}) - y_t^U) - \|\mathbf{x} - \hat{\mathbf{x}}\| \sum_{t \in I} \rho_t \varepsilon_t - \sum_{t \in I} \rho_t \bar{\varepsilon}_t. \end{aligned}$$

Hence:

$$\begin{aligned} & \max_{i \in I} \lambda_i (f_i(\mathbf{x}) - y_i^U) + \sum_{t \in I} \rho_t (f_t(\mathbf{x}) - y_t^U) < \\ & \max_{i \in I} \lambda_i (f_i(\hat{\mathbf{x}}) - y_i^U) + \sum_{t \in I} \rho_t (f_t(\hat{\mathbf{x}}) - y_t^U) - \min_{i \in I} \lambda_i (\varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) - \sum_{t \in I} \rho_t (\varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t). \end{aligned}$$

Now, assume we choose $y_i^U = f_i(\hat{\mathbf{x}}) - \varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\varepsilon}_i, \forall i \in I$. Therefore, $\lambda_i = (\varepsilon_i \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i)^{-1} > 0, \forall i \in I$. But, since, $\delta = \max_{i \in I} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$, and $\bar{\delta} = \max_{i \in I} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$, we have:

$$\delta \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\delta} = \max_{i=1, \dots, m} \{\lambda_i \varepsilon_i\} \|\mathbf{x} - \hat{\mathbf{x}}\| + \max_{i=1, \dots, m} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t (\varepsilon_t \|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t)$$

$$\begin{aligned}
 &= \max_{i=1, \dots, m} \lambda_i(\varepsilon_i\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) + \sum_{t \in I} \rho_t(\varepsilon_t\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t) = 1 + \sum_{t \in I} \rho_t(\varepsilon_t\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t) \\
 &= \min_{i=1, \dots, m} \lambda_i(\varepsilon_i\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_i) + \sum_{t \in I} \rho_t(\varepsilon_t\|\mathbf{x} - \hat{\mathbf{x}}\| + \bar{\varepsilon}_t).
 \end{aligned}$$

Therefore,

$$\max_{i \in I} \lambda_i(f_i(\hat{\mathbf{x}}) - y_i^U) - \delta\|\mathbf{x} - \hat{\mathbf{x}}\| - \bar{\delta} > \max_{i \in I} \lambda_i(f_i(\mathbf{x}) - y_i^U) + \sum \rho_t(f_t(\mathbf{x}) - y_t^U),$$

which is a contradiction with $(\delta, \bar{\delta})$ -quasi optimality of $\hat{\mathbf{x}}$.

(ii) The proof is similar to part (i). \square

Remark 3.9. Beldiman et al. [2, Theorems 4 and 5], under the same assumptions of Theorem 3.8, showed that $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}E}$. Therefore, in Theorem 3.8 we obtained stronger results than the ones in [2].

Taking into consideration the sufficient conditions in Theorem 3.8 and the necessary conditions in [2, Theorems 2 and 3], we have the following necessary and sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi properly efficient points of MOP (1).

Corollary 3.10. (i) Let $\delta = \max_{i \in I} \{\lambda_i \varepsilon_i\} + \sum_{t \in I} \rho_t \varepsilon_t$ and $\bar{\delta} = \max_{i \in I} \{\lambda_i \bar{\varepsilon}_i\} + \sum_{t \in I} \rho_t \bar{\varepsilon}_t$. Then, $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}PE}$ if and only if there exist $\rho \in \mathbb{R}_>^m$ and $\lambda \in \mathbb{R}_>^m$ such that $\hat{\mathbf{x}}$ is a $(\delta, \bar{\delta})$ -quasi optimal solution of SOP (3).

(ii) Let $\delta = \max_{i \in I} \lambda_i(\varepsilon_i + \sum_{t \in I} \rho_t \varepsilon_t)$ and $\bar{\delta} = \max_{i \in I} \lambda_i(\bar{\varepsilon}_i + \sum_{t \in I} \rho_t \bar{\varepsilon}_t)$. Then, $\hat{\mathbf{x}} \in \mathcal{X}_{\varepsilon, \bar{\varepsilon}PE}$ if and only if there exist $\rho \in \mathbb{R}_>^m$ and $\lambda \in \mathbb{R}_>^m$ such that $\hat{\mathbf{x}}$ is a $(\delta, \bar{\delta})$ -quasi optimal solution of SOP (4).

4. Conclusion

In this paper, we used some scalarization techniques to consider $(\varepsilon, \bar{\varepsilon})$ -quasi (weakly, properly) efficient points of a general (with no convexity assumption) multiobjective program. The relations between approximate quasi (weakly, properly) efficient solutions of an MOP and approximate quasi-optimal solutions of the associated SOP were considered. More precisely, utilizing the augmented weighted Tchebycheff and the modified weighted Tchebycheff scalarizing problems, we obtained sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi (weakly) efficient solutions of an MOP. Also, we derived two necessary conditions for approximate quasi weak efficiency. Moreover, via given scalarization methods, we completed the results in [2], and derived sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi properly efficient solutions of an MOP. Therefore, two necessary and sufficient conditions for $(\varepsilon, \bar{\varepsilon})$ -quasi proper efficiency were obtained. The results generalized corresponding ones in [7, 11, 12, 17, 21, 29].

However, in this paper, \mathbb{R}_{\geq}^m is the order cone, considering the problem in vector optimization under a general ordering cone can be a worthwhile direction for the future investigation. Also, $(\varepsilon, \bar{\varepsilon})$ -duality in multiobjective optimization framework can be worth studying.

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