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A Note on Integral Non-Commuting Graphs

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Abstract. The non-commuting graph $\Gamma(G)$ of group G is a graph with the vertex set $G - Z(G)$ and two distinct vertices x and y are adjacent whenever $xy \neq yx$. The aim of this paper is to study integral regular non-commuting graphs of valency at most 16.

1. Introduction

In graph theory, the techniques of graph spectral is used to estimate the algebraic properties of a graph from its structure with significant role in computer science, biology, chemistry, etc. The spectrum of a graph is based on the adjacency matrix of graph and it is strongly dependent on the form of this matrix. A number of possible disadvantage can be derived by using only the spectrum of a graph. For example, some information about expansion and randomness of a graph can be derived from the second largest eigenvalue of a graph. One of the main applications of graph spectra in chemistry is the application in Hückel molecular orbital theory for the determination of energies of molecular orbitals of π-electrons.

On the other hand, by computing the smallest eigenvalue, we can get data about independence number and chromatic number. A graph with exactly two eigenvalues is the complete graph K_n or a regular graph has exactly three eigenvalues if and only if it is a strongly regular graph. Further, some groups can be uniquely specified by the spectrum of their Cayley graphs, see for example [2, 4–6, 17, 24].

The energy $\varepsilon(\Gamma)$ of the graph Γ was introduced by Gutman in 1978 as the sum of the absolute values of the eigenvalues of Γ, see [20–22]. The stability of a molecule can also be estimated by the number of zero eigenvalues of a graph, namely the nullity of a graph. Nowadays, computing the spectrum of a graph is an interesting field for mathematicians, see for example [13–15, 18, 19, 28].

The non-commuting graph $\Gamma(G)$ of group G was first considered by Paul Erdős to answer a question on the size of the cliques of a graph in 1975, see [27]. For background materials about non-commuting graphs, we encourage the reader to references [1, 3, 12, 25, 26, 29]. In this article, we prove that there is no $k-$regular non-commuting graphs where k is an odd number. We also prove that there is no $2^s q-$regular non-commuting graph, where q is a prime number greater than 2. On the other hand, we characterized all $k-$regular integral non-commuting graphs where $1 \leq k \leq 16$. Here, in the next section, we give necessary definitions and some preliminary results and section three contains some new results on regular non-commuting graphs.

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2. Definitions and Preliminaries

Our notation is standard and mainly taken from standard books of graph and algebraic graph theory such as [7, 8, 11, 30]. All graphs considered in this paper are simple and connected. All considered groups are non-abelian groups. The vertex and edge sets of graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively.

There are a number of constructions of graphs from groups or semigroups in the literature. Let G be a non-abelian group with center $Z(G)$. The non-commuting graph $\Gamma(G)$ is a simple and undirected graph with the vertex set $G/Z(G)$ and two vertices $x, y \in G/Z(G)$ are adjacent whenever $xy \neq yx$.

The characteristic polynomial $\chi_\lambda(\Gamma)$ of graph Γ is defined as

$$\chi_\lambda(\Gamma) = |\lambda I - A|,$$

where A denotes to the adjacency matrix of Γ. The eigenvalues of graph Γ are the roots of the characteristic polynomial and form the spectrum of this graph.

3. Main Results

The aim of this section is to study the regular non-commuting graphs. First, we prove that there is no k-regular non-commuting graph where k is odd. The following theorem is implicitly contained in [1].

Proposition 3.1. *Let G be a finite non-abelian group such that $\Gamma(G)$ is a regular graph. Then G is nilpotent of class at most 3 and $G = P \times A$, where A is an abelian group, P is a p-group (p is a prime) and furthermore $\Gamma(P)$ is a regular graph.*

Theorem 3.2. *Let G be a finite group, where $\Gamma(G)$ is k-regular, then k is even.*

Proof. Suppose that k is odd. Then, for any non-central element $x \in G$,

$$k = |G| - |C_G(x)| = |C_G(x)|(|G : C_G(x)| - 1)$$

from which we deduce that $|C_G(x)|$ is odd and that $|G/Z(G)|$ is even. Since $|C_G(x)|$ is odd for all non-central elements $x \in G$, all non-central elements of G have odd order as does $Z(G)$. This contradicts the fact that $|G/Z(G)|$ is even.

Proposition 3.3. *If $G = P \times A$, then for every $x = (\alpha, \beta) \in G$ where $\alpha \in P$ and $\beta \in A$, we have*

$$d_{\Gamma(G)}(x) = d_{\Gamma(P)}(\alpha)|A|.$$

Proof. It is easy to see that

$$\begin{aligned}
d_{\Gamma(G)}(x) &= |G| - |C_G(x)| \\
&= |P||A| - \frac{|P||A|}{|(\alpha^P, \beta^A)|} \\
&= |P||A| - |C_P(\alpha)||C_A(\beta)| \\
&= |P||A| - |C_P(\alpha)||A| \\
&= (|P| - |C_P(\alpha)|)|A| \\
&= d_{\Gamma(P)}(\alpha)|A|.
\end{aligned}$$

Theorem 3.4. *Let G be a non-abelian finite group and assume that $\Gamma(G)$ is 2^s − regular. Then G is a 2-group.*

Proof. We have

$$2^s = |C_G(x)|(|G : C_G(x)| - 1).$$

Thus, every element of $G/Z(G)$ is a 2-element. Hence G is a 2-group.

In continuing, we determine all $2^s q$-regular non-commuting graphs where q is a prime number. To do this, let G be a finite group where $\Gamma(G)$ is 6-regular. By notations of Proposition 3. 3, the following cases hold:

- $a = 1$, hence $p^{n-i}(p^i - 1) = 6$. Thus, $p = 2$ or $p = 3$. If $p = 2$, then $i = 2$ and $n = 3$. This implies that $G \cong D_8$ or $G \cong Q_8$, both of them are contradictions, since G is 4-regular. If $p = 3$, then $i = 1, n = 2$ and so G is abelian, a contradiction.

- $a = 2$, then $p = 2$ or $p = 3$. If $p = 2$, then $n = i = 2$, a contradiction or $p = 3$ and so $3^{n-i}(3^i - 1) = 3$, a contradiction.

- $a = 3$, hence $p = 2$. This implies that $i = 1$, $n = 2$ and hence $G \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$, both of them are contradictions, since G is abelian. If $p = 3$, then $n = i = 1$, a contradiction.

- $a = 6$, thus $p = 2$, $i = 1$ and $n = 1$. It follows that $G \cong \mathbb{Z}_p \times \mathbb{Z}_6$ is abelian, a contradiction.

Hence, one can conclude the following Lemma.

Lemma 3.5. *Let G be a finite non-abelian group which is not a $p-$group and q be a prime number. Then, there is no $2^s q$-regular NC graph where $s = 1, 2, 3$.*

Proof. Let $d(x) = 2q$, where q is a prime number, then $p^{n-i}(p^i - 1)a = 2q$. Since G is not a $p-$group, $a \neq 1$ and three following cases hold:
Case 1. $a = 2$, hence $p^{n-i}(p^i - 1) = q$ and thus $p^{n-i} = q$ or $p^i - 1 = q$. If $p^{n-i} = q$, then $n - i = 1$ and $p^i - 1 = 1$. Hence, $p = 2, i = 1$ and $n = 2$, a contradiction, since G is not abelian. If $p^i - 1 = q$, then necessarily $n = i$, $p^n - 1 = q$ and so $|G| = 2q + 2$. Similar to the last discussion, $|Z(G)| \geq 2$ and so $|G/Z(G)| \leq 2q$. Hence $\Gamma(G)$ is a $2q-$regular graph on at most $2q$ vertices which is impossible.
Case 2. $a = q$, hence $p^{n-i}(p^i - 1)q = 2q$. This implies that $p^{n-i}(p^i - 1) = 2$ and so $p = 2$ or $p = 3$. If $p = 2$ then $n - i = 1$ and $i = 1$. Hence, $n = 2$ and G is abelian, a contradiction. If $p = 3$, then $n = i = 1$, a contradiction.
Case 3. $a = 2q$, then $p^{n-i}(p^i - 1) = 1$. It follows that $p = 2$ and $n = i = 1$, a contradiction, since G is not abelian.

Let now $d(x) = 4q$, similar to the last discussion, the following cases hold:
Case 1. $a = 2$, hence $p^{n-i}(p^i - 1) = 2q$. Thus $p = q = 3$ and $n - i = i = 1$, a contradiction or $p = 2, n - i = 1$ and $2^i - 1 = q$. Then $|G| = p^n a = 2^{i+2} = 4(q + 1)$. It follows that $\Gamma(G)$ is a $4q-$regular graph on at most $|G/Z(G)| \leq 4(q + 1) - 8 = 4q - 4$ vertices which is impossible.
Case 2. $a = 4$, thus $p^{n-i}(p^i - 1) = q$ implies that $p = q = 2$ and $n - i = i = 1$, a contradiction or $(p^i - 1) = q$ and $n = i$. In this case, $|G| = p^n a = 4(q + 1)$, since $|Z(P)| \geq p$, we have $|V(\Gamma(G))| = |G/Z(G)| \leq 4q + 4 - 4p < 4q$, a contradiction.
Case 3. $a = q$, hence $p^{n-i}(p^i - 1) = 4$. Thus $p = 5$ and $n = i = 1$, a contradiction or $p = 2, n - i = 2$ and $i = 1$. In this case $|P| = 8$ and so we have a 4-regular graph with at most $|V(\Gamma)| = |P| - |Z(P)| \leq 8 - 2 = 6$ vertices which is impossible (according to Proposition 3. 3).
Case 4. $a = 2q$, hence $p^{n-i}(p^i - 1) = 2$. Thus $p = 2$ or $p = 3$. If $p = 2$, then $n - i = 1$ and $i = 1$ which is impossible, since G is not abelian. If $p = 3$, then $n = i = 1$, a contradiction.
Case 5. $a = 4q$, therefore $p^{n-i}(p^i - 1) = 1$. This implies that $p = 2$ and $n = i = 1$, a contradiction.

Finally, suppose $d(x) = 8q$, then the following cases hold:
Case 1. $a = 2$, hence $p^{n-i}(p^i - 1) = 4q$. Thus $p = q = 5$ and $n - i = i = 1$, a contradiction or $p = 2, n - i = 2$ and $p^i = q + 1$. In this case $|G| = 4(q + 1)$ and so $\Gamma(G)$ is an $8q-$regular graph on $|G/Z(G)| \leq 4q$ vertices, a contradiction.
Case 2. $a = 4$, thus $p^{n-i}(p^i - 1) = 2q$ and so $p = q = 3$, $n - i = i = 1$, a contradiction or $p = 2$ and $(2^i - 1) = q$. In this case, $|G| = p^n a = 2(q + 1)$. Since $|Z(P)| \geq 2$, one can see that $|V(\Gamma(G))| = |G/Z(G)| \leq 4q + 4 - 4p < 4q$, a contradiction.
Case 3. $a = 8$, thus $p^{n-i}(p^i - 1) = q$ implies that $p = q = 2$ and $n - i = i = 1$, a contradiction or $(p^i - 1) = q$ and $n = i$. In this case, $|G| = p^n a = 8(q + 1)$ and since $|Z(P)| \geq p$ hence $|V(\Gamma(G))| = |G/Z(G)| \leq 8q + 8 - 8p < 8q$, a contradiction with Γ is $8q-$regular.
Case 4. $a = q$, hence $p^{n-i}(p^i - 1) = 8$. Thus $p = 2, n - i = 3$ and $i = 1$. It follows that $|G| = 16q$, since $|Z(G)| \geq 2q$, by Proposition 3.3, $\Gamma(P)$ is an 8-regular graph with $|V(\Gamma(P))| \leq 14$, a contradiction.
Case 5. $a = 2q$, hence $p^{n-i}(p^i - 1) = 4$. This implies that $p = 5$ and $n = i = 1$, a contradiction or $p = 2, n - i = 2$ and $i = 1$. It follows that $\Gamma(P)$ is an 4-regular graph with $|V(\Gamma(P))| \leq 6$ vertices, a contradiction.
Case 6. $a = 4q$, therefore $p^{n-i}(p^i - 1) = 2$. This implies that $p = 3$ and $n = i = 1$, a contradiction or $p = 2$ and

$n - i = i = 1$, a contradiction with G is not abelian.
Case 7. $a = 8q$, therefore $p^{n-i}(p^i - 1) = 1$. This implies that $p = 2$ and $n = i = 1$, a contradiction, since G is not abelian.

In general, we have the following theorem:

Theorem 3.6. *Suppose that G is a non-abelian finite group and $\Gamma(G)$ is $k-$regular. Then $k \neq 2^s q$ where q is an odd prime.*

Proof. Suppose that $k = 2^s q$. First assume that $G/Z(G)$ is not a 2-group. Let $x \in G$ be such that $Z(G)x$ has odd order greater than 1. We have

$$2^s q = |C_G(x)|(|G : C_G(x)| - 1)$$

and so, as $x \in C_G(x)$, $|C_G(x)| = 2^a q$ for some integer a and $Z(G)x$ has order q. In particular, $Z(G)$ is a 2-group. Let Q be a Sylow q-subgroup of G and $y \in Z(Q)$ with $y \neq 1$. Then $Z(G)y$ has odd order greater than 1 as $Z(G)$ is a 2-group. Thus $|C_G(y)| = 2^b q$ for some integer b. Since $Q \leq C_G(y)$ we see that Q has order q. Thus every element of odd order in G has order q and G has Sylow q-subgroups of order q. Especially, $|G| = 2^c q$ for some integer c. Now

$$2^c q = |G| = 2^s q + |C_G(x)| = 2^s q + 2^b q = 2^b q(2^{s-b} + 1)$$

from which we deduce $2^{s-b} + 1 = 2$ and $|G| = 2^{s+1} q$. Further $C_G(x)$ has index 2 in G. But then every conjugate of x is contained in $C_G(x)$. Hence $\langle x \rangle$ is normal in G. Let y be a 2-element which does not commute with x. Then $|C_G(y)| = 2^d$ whereas $|G| - |C_G(y)| = 2^s q = |G| - |C_G(x)|$, a contradiction. Next consider the case that $G/Z(G)$ is a 2-group. In this case q divides $|Z(G)|$. Let x be a non-central element of G. Then $|C_G(x)| = 2^b q$ for some integer b. Now

$$2^s q = |C_G(x)|(|G : C_G(x)| - 1) = 2^b q(2^c - 1)$$

so, we deduce that $2^c - 1 = 1$ and $C_G(x) = G$ which is impossible. This proves the claim.

3.1. *Which non-commuting graphs are integral?*

An integral graph is a graph with integral spectrum considered by Harary and Schwenk [23] for the first time. Cvetković et al. [6, 9, 10] determined all cubic integral graphs. Following their method, we classify all groups whose non-commuting $k-$regular graphs are integral where $1 \leq k \leq 16$. The following two lemmas are crucial in what follows.

Lemma 3.7. *[1] Let Γ be a non-commuting graph with diameter d, then $d \leq 2$.*

Lemma 3.8. *[10] Let Γ be an integral k-regular graph on n vertices with diameter d. Then*

$$n \leq \frac{k(k-1)^d - 2}{k-2}.$$

According to Lemma 3. 7, the diameter of a non-commuting graph is at most 2, then from Lemma 3. 8, it follows that the number of vertices of Γ is less than or equal to $\frac{k(k-1)^2-2}{k-2}$. Clearly, there is no integral regular non-commuting graph of odd degree. So, we should study just the regular non-commuting graph with even valency.

Theorem 3.9. *If $\Gamma(G)$ is $k-$regular integral non-commuting graph where $k \leq 16$, then $k = 4$ and $G \cong D_8, Q_8$ or $k = 8$ and $G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, SU(2), M_{16}, \mathbb{Z}_4 \ltimes \mathbb{Z}_4, \mathbb{Z}_4 \ltimes \mathbb{Z}_2 \times \mathbb{Z}_2$ or $k = 16$ and $G \cong SmallGroup(32, i)$ where*

$$i \in \{2, 4, 5, 12, 17, 22, 23, 24, 25, 26, 37, 38, 46, 47, 48, 49, 50\}.$$

Proof. According to Theorem 3. 2, k is even. Since G is non-abelian, $k \geq 3$ and so $k \in \{4, 6, 8, 10, 12, 14, 16\}$. According to Theorem 3.4, $k \notin \{6, 10, 12, 14\}$ and so $k \in \{4, 8, 16\}$. Let $\Gamma(G)$ be a 4-regular integral non-commuting graph. According to Lemma 3. 8,

$$n \leq \frac{4 \times 3^2 - 2}{4 - 2} = 17.$$

But Γ is not a complete graph and then $6 \leq n \leq 17$. Since $|Z(G)|$ divides $|G|$, we can suppose $|G/Z(G)| = t$ and so

$$n = |G| - |Z(G)| = t|Z(G)| - |Z(G)| = (t - 1)|Z(G)|.$$

Let $n = 6$, since $(t - 1)|Z(G)| = 6$, $|Z(G)| = 1, 2, 3$ or 6. If $|Z(G)| = 6$, then $|G/Z(G)| = 2$ and so G is abelian, a contradiction. If $|Z(G)| = 3$, then $|G| = 9$ and so G is abelian, a contradiction. If $|Z(G)| = 2$, then $|G| = 8$ and $G \cong D_8$ or Q_8. By a direct computation, one can see that both $\Gamma(D_8)$ and $\Gamma(Q_8)$ are 4-regular graphs. If $|Z(G)| = 1$, then $|G| = 7$ and so G is abelian, a contradiction. Let now $n = 7$, then $|Z(G)| = 1$ or 7. If $|Z(G)| = 7$, then $|G| = 14$ and the only non-abelian group of order 14 is D_{14}. The non-commuting graph of D_{14} is not 4-regular. If $|Z(G)| = 1$, then $|G| = 8$, a contradiction, since the center of a 2−group is not trivial. In continuing, let $n = 8$. Since G is non-abelian, $|Z(G)| = 2, 4$ or 8. If $|Z(G)| = 2$, then $|G| = 10$ and so $G \cong D_{10}$. One can see that $\Gamma(D_{10})$ is not 4-regular. If $|Z(G)| = 4, 8$ then $\Gamma(G)$ is not 4-regular. In this case, by using a GAP program [16] (presented in the end of this paper), we can prove that just the non-commuting graphs $\Gamma(D_8)$ and $\Gamma(Q_8)$ are 4-regular graphs. Let now, $\Gamma(G)$ be an 8-regular integral non-commuting graph. According to Lemma 3. 8,

$$7 \leq n \leq \frac{8 \times 7^2 - 2}{8 - 2} = 65.$$

If $n = 7$, then $|Z(G)| = 1$ or 7. If $|Z(G)| = 1$, then $|G| = 8$ and there is not a group of order 8 whose non-commuting graph is 8-regular. For $|Z(G)| = 7$, $|G| = 14$ and similar to the last discussion, the only non-abelian group of order 14 is D_{14} and the degrees of vertices of $\Gamma(D_{14})$ are 7 and 12, a contradiction. This implies that $n \geq 8$. If $n = 8$, then $|G| = 9, 10, 12, 16$. From the part one, we can conclude that $|G| \neq 9, 10, 12$. Suppose $|G| = 16$ and $|Z(G)| = 8$, then $\frac{G}{Z(G)}$ is cyclic and so G is abelian, a contradiction. If $n = 10$, then $|G| = 11, 12, 15, 20$ and similar to the last discussion their non-commuting graphs are not regular. Suppose $n = 12$, then $|Z(G)| = 1, 2, 3, 4, 6, 12$ and so $|G| = 13, 14, 15, 16, 18, 24$. By these conditions, one can prove that $|G| = 16$ and $|Z(G)| = 4$. There are six non-abelian groups whose non-commuting graphs are 8-regular. They are $\mathbb{Z}_2 \times D_8$, group of the Pauli matrices $\mathbb{Z}_2 \times Q_8$, $SU(2)$, modular or Isanowa group M_{16} of order 16 and Semidirect products $\mathbb{Z}_4 \ltimes \mathbb{Z}_4$, $\mathbb{Z}_4 \ltimes \mathbb{Z}_2 \times \mathbb{Z}_2$. By continuing our method and applying GAP program, we can deduce that the above groups are the only groups whose non-commuting graphs are 8-regular. Finally, suppose $k = 16$, then

$$n \leq \frac{16 \times 15^2 - 2}{16 - 2} = 257.$$

Again, by applying GAP program, we can deduce that only for $|G| = 32$, there are some non-abelian groups whose non-commuting graphs are 16-regular. We name them as $SmallGroup(32, i)$ where

$$i \in \{2, 4, 5, 12, 17, 22, 23, 24, 25, 26, 37, 38, 46, 47, 48, 49, 50\}.$$

This completes the proof. $\square$

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A GAP program for computing the non-commuting graph of groups

```
  f := function(G)
local x, y, M, MM, i, j, s, d;
 M := []; MM := []; s := 0; d := [];
  for x in Difference(Elements(G), Elements(Center(G))) do
     for y in Difference(Elements(G), Elements(Center(G))) do
      if x * y = y * x then
     Add(M, 0);
      else
      Add(M, 1);
      fi;
    od;
    Add(MM, M); M := [];
  od;
 Print(MM);
 Print(Size(Center(G)));
 for i in MM do
  for j in i do
   s := s + j;
  od;
  Add(d, s); s := 0;
 od;
Print(d);
Print(" *************************************");
 return;
end;
```

References

[1] A. Abdollahi, S. Akbari, H. R. Maimani, Non-commuting graph of a group, J. Algebra 298 (2006) 468–492.
[2] A. Abdollahi, M. Jazaeri, Groups all of whose undirected Cayley graphs are integral, European Journal of Combinatorics 38 (2014) 102–109.
[3] A. Abdollahi, H. Shahverdi, Characterization of the alternating group by its non-commuting graph, J. Algebra 357 (2012) 203–207.
[4] A. Abdollahi, E. Vatandoost, Which Cayley graphs are integral?, Electronic J. Comb. 16(1)R122 (2009) 1–17.
[5] O. Ahmadi, N. Alon, L. F. Blake, I. E. Shparlinski, Graphs with integral spectrum, Linear Alg. Appl. 430 (2009) 547–552.
[6] K. Balinska, D. Cvetković, Z. Rodosavljević, S. Simić, D. A. Stevanović, Survey on integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. 13 (2003) 42–65.
[7] J. A. Bondy, J. S. Murty, Graph theory with aplication, Elsevier, 1977.
[8] N. L. Biggs, Algebraic Graph Theory, Cambridge University Press, 1974.
[9] F. C. Bussemaker, D. Cvetković, There are exactly 13 connected, cubic, integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 544–576 (1976) 43–48.
[10] D. Cvetković, Cubic integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 498–541 (1975) 107–113.
[11] D. Cvetković, P. Rowlinson, S. Simić, An introduction to the theory of graph spectra, London Mathematical Society, London, 2010.
[12] M. R. Darafsheh, Groups with the same non-commuting graph, Discrete Appl. Math. 157 (2009) 833–837.
[13] M. DeVos, L. Goddyn, B. Mohar, R. Šámal, Cayley sum graphs and eigenvalues of (3,6)-fullerenes, J. Combin. Theor. Series B 99 (2009) 358–369.
[14] P. W. Fowler, P. E. John, H. Sachs, (3,6)-cages, hexagonal toroidal cages, and their spectra, Discrete mathematical chemistry, New Brunswick, NJ, 1998, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 51 Amer. Math. Soc. Providence, RI (2000) 139–174.
[15] P. W. Fowler, P. Hansen, D. Stevanović, A note on the smallest eigenvalue of fullerenes, MATCH Commun. Math. Comput. Chem. 48 (2003) 37–48.
[16] GAP, Groups, Algorithms and Programming, Lehrstuhl De fur Mathematik, RWTH, Aachen, 1992.
[17] M. Ghorbani, On the eigenvalues of normal edge-transitive Cayley graphs, Bull. Iranian Math. Soc. 41 (2015) 101–107.
[18] M. Ghorbani, N. Azimi, Characterization of split graphs with at most four distinct eigenvalues, Disc. Appl. Math. 184 (2015) 231–236.

[19] M. Ghorbani, E. Bani-Asadi, Remarks on characteristic coefficients of fullerene graphs, Appl. Math. Comput. 230 (2014) 428–435.
[20] M. Ghorbani, M. Faghani, A. R. Ashrafi, S. Heidari-Rad, A. Graovać, An upper bound for energy of matrices associated to an infinite class of fullerenes, MATCH Commun. Math. Comput. Chem. 71 (2014) 341–354.
[21] I. Gutman, The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz Graz 103 (1978) 1–22.
[22] I. Gutman, O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Berlin, 1986.
[23] F. Harary, A.J. Schwenk, Which graphs have integral spectra?, in: R. Bari, F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Mathematics, 406, Springer, Berlin (1974) 45–51.
[24] W. Klotz, T. Sander, Integral Cayley graphs over abelian groups, Electronic J. Combinatorics 17 R81 (2010) 1–13.
[25] A. R. Moghaddamfar, W. J. Shi, W. Zhou, A. R. Zokayi, On the non-commuting graph associated with a finite group, Siberian Math. J. 46 (2005) 325–332.
[26] G. L. Morgan, C. W. Parker, The diameter of the commuting graph of a finite group with trivial centre, J. Algebra 393 (2013) 41–59.
[27] B. H. Neumann, A problem of Paul Erdős on groups, J. Austral. Math. Soc. Ser. A 21 (1976) 467–472.
[28] W. C. Shiu, W. Li, W. H. Chan, On the spectra of the fullerenes that contain a nontrivial cyclic-5-cutset, Australian J. Combin. 47 (2010) 41–51.
[29] A. Talebi, On the non-commuting graph of group D_{2n}, International J. Algebra 2 (2009) 957–961.
[30] D. B. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, 1996.