



## On an Estimate of a Distance Function of Several Variables and Related Problems

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**Abstract.** We present new sharp estimates concerning distance function which leads to generalizations of some of our results on extremal problems in Bergman type classes.

### 1. Introduction

The intention of this note is to extend our previous results on extremal problems from [21] and [22]. We however choose another setting for this.

Namely, we study tube domains over symmetric cones and provide generalizations of our previous results. Then we provide similar extensions also in the unit ball and polydisk, but we provide these results without proofs, since proofs are completely parallel to the case of tubular domains over symmetric cones. The base of all proofs is well-known Forelli-Rudin type estimate for Bergman kernel and Bergman representation formula together with chain of transparent arguments combined with classical inequalities of functions theory. All these are valid in analytic spaces in the tubular domains over symmetric cones, unit ball and unit polydisk.

We shortly remind the history of this problem to readers.

After the appearance of [27] various papers appeared where arguments which can be seen in [27] were changed, extended and modified in various directions (see [3], [21], [22]).

In particular in mentioned papers various new results on distances for analytic function spaces in higher dimension (unit ball and polydisk) were obtained. Then by authors many new sharp results for large scales of analytic mixed norm spaces in higher dimension were proved.

Later, several new sharp results for harmonic functions of several variables in the unit ball and upper-halfplane of Euclidean space were also obtained (see, for example, [3] and references there).

We mention separately [18] and [25] where the case of higher dimension was considered in special cases of analytic spaces on subframe and new similar sharp results in the context of bounded strictly pseudoconvex domains with smooth boundary were also provided.

The classical Bergman representation formula in various forms and in various domains serves as a base in all these papers in proofs of main results.

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We would like to note also, recently, Wen Xu (see [26]) repeating arguments of Ruhan Zhao in the unit ball obtained results on distances from Bloch functions to some Möbius invariant function spaces in one and higher dimension in a relatively direct way.

The motivation of this problem related with distance function is to find a concrete formula which will help to calculate this function more concretely via the well-known Bergman kernel.

The goal of this note is to develop further some ideas from our recent mentioned papers and present a new sharp theorem in tube domain over symmetric cones and other domains.

We note in case of upper halfplane of complex plane  $\mathbb{C}$  which is a tube domain in one dimension such results already were obtained previously by first author (see [18]).

We formulate our problem in detail first in basic case of the unit disk and provide some known results. These type results will be provided in higher dimension in next section.

Let  $\mathbf{U}$  be, as usual, the unit disk on the complex plane,  $dm_2(z)$  be the normalized Lebesgue measure on  $\mathbf{U}$  so that  $m_2(\mathbf{U}) = 1$  and  $dm(\xi)$  be the Lebesgue measure on the circle  $\mathbf{T} = \{\xi : |\xi| = 1\}$ . Let further  $H(\mathbf{U})$  be the space of all analytic functions on the unit disk  $\mathbf{U}$ .

For  $f \in H(\mathbf{U})$  and  $f(z) = \sum_k a_k z^k$ , define the fractional derivative of the function  $f$  as usual in the following manner

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \quad \alpha \in \mathbb{R}.$$

We will write  $Df(z)$  if  $\alpha = 1$ . Obviously, for all  $\alpha \in \mathbb{R}$ ,  $D^\alpha f \in H(\mathbf{U})$  if  $f \in H(\mathbf{U})$ .

For  $a \in \mathbf{U}$ , let  $g(z, a) = \log\left(\frac{1}{|\varphi_a(z)|}\right)$  be the Green's function for  $\mathbf{U}$  with pole at  $a$ , where  $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ . For  $0 < p < \infty$ ,  $-2 < q < \infty$ ,  $0 < s < \infty$ ,  $-1 < q + s < \infty$ , we say that  $f \in F(p, q, s)$ , if  $f \in H(\mathbf{U})$  and

$$\|f\|_{F(p,q,s)}^p = \sup_{a \in \mathbf{D}} \int_{\mathbf{U}} |Df(z)|^p (1 - |z|^2)^q g(z, a)^s dm_2(z) < \infty.$$

As we know [27], if  $0 < p < \infty$ ,  $-2 < q < \infty$ ,  $0 < s < \infty$ ,  $-1 < q + s < \infty$ ,  $f \in F(p, q, s)$  if and only if

$$\sup_{a \in \mathbf{D}} \int_{\mathbf{U}} |Df(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|)^s dm_2(z) < \infty.$$

It is known (see [27]) that  $F(2, 0, 1) = BMOA$ .

We recall that the weighted Bloch class  $\mathcal{B}^\alpha(\mathbf{U})$ ,  $\alpha > 0$ , is the collection of the analytic functions on the unit disk satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbf{U}} |Df(z)|(1 - |z|^2)^\alpha < \infty.$$

Space  $\mathcal{B}^\alpha(\mathbf{U})$  is a Banach space with the norm  $\|f\|_{\mathcal{B}^\alpha}$ . Note  $\mathcal{B}^1(\mathbf{U}) = \mathcal{B}(\mathbf{U})$  is a classical Bloch class (see [11], [14] and the references there).

For  $k > s$ ,  $0 < p, q \leq \infty$ , the weighted analytic Besov space  $\mathcal{B}_s^{q,p}(\mathbf{U})$  is the class of analytic functions satisfying (see [11])

$$\|f\|_{\mathcal{B}_s^{q,p}}^q = \int_0^1 \left( \int_{\mathbf{T}} |D^k f(r\xi)|^p dm(\xi) \right)^{\frac{q}{p}} (1-r)^{(k-s)q-1} dr < \infty.$$

Quasinorm  $\|f\|_{\mathcal{B}_s^{q,p}}$  does not depend on  $k$ . If  $\min(p, q) \geq 1$ , the class  $\mathcal{B}_s^{q,p}(\mathbf{U})$  is a Banach space under the norm  $\|f\|_{\mathcal{B}_s^{q,p}}$ . If  $\min(p, q) < 1$ , then we have a quasinormed class.

The well-known so called "duality" approach to extremal problems in theory of analytic functions leads to the following general formula

$$dist_Y(g, X) = \sup_{l \in X^\perp, \|l\| \leq 1} |l(g)| = \inf_{\varphi \in X} \|g - \varphi\|_Y,$$

where  $g \in Y$ ,  $X$  is subspace of a normed space  $Y$ ,  $Y \in H(\mathbf{U})$  and  $X^\perp$  is the orthogonal complement of  $X$  in  $Y^*$ , the dual space of  $Y$  and  $l$  is a linear functional on  $Y$ .

Various extremal problems in  $H^p$  Hardy classes in  $\mathbf{U}$  based on duality approach we mentioned were discussed in [8, Chapter 8]. In particular for a function  $K \in L^q(\mathbf{T})$  the following equality holds (see [8]),  $1 \leq p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\text{dist}_{L^q}(K, H^q) = \inf_{g \in H^q, K \in L^q} \|K - g\|_{H^q} = \sup_{f \in H^p, \|f\|_{H^p} \leq 1} \frac{1}{2\pi} \left| \int_{\mathbf{T}} f(\xi)K(\xi)dm(\xi) \right|.$$

It is well known that if  $p > 1$  then the inf-dual extremal problem in the analytic  $H^p$  Hardy classes has a solution, it is unique if an extremal function exists (see [8]).

Note also that extremal problems for  $H^p$  spaces in multiply connected domains were studied before in [1], [12].

Various new results on extremal problems in  $A^p$  Bergman class and in its subspaces were obtained recently by many authors (see [10] and the references there).

In this paper we will provide direct proofs for estimation of  $\text{dist}_Y(f, X) = \inf_{g \in X} \|f - g\|_Y$ ,  $X \subset Y$ ,  $X, Y \subset H(\mathbf{U})$ ,  $f \in Y$ , not only in the unit disk, but also in higher dimension.

Let further  $\Omega_{\alpha, \varepsilon}^k = \{z \in \mathbf{U} : |D^k f(z)|(1 - |z|^2)^\alpha \geq \varepsilon\}$ ,  $\alpha \geq 0$ ,  $\varepsilon > 0$ ,  $\Omega_{\alpha, \varepsilon}^0 = \Omega_{\alpha, \varepsilon}$ .

Applying famous Fefferman duality theorem, P. Jones proved the following

**Theorem A.** (see [27]) Let  $f \in \mathcal{B}$ . Then the following are equivalent:

- (a)  $d_1 = \text{dist}_{\mathcal{B}}(f, \text{BMOA})$ ;
- (b)  $d_2 = \inf\{\varepsilon > 0 : \chi_{\Omega_{1, \varepsilon}^1}(f)(z) \frac{dm_2(z)}{1 - |z|^2} \text{ is a Carleson measure}\}$ ,  
where  $\chi$  denotes the characteristic function of the mentioned set.

In the following theorem, see [21], we showed that in Zhao’s theorems (see [27]) Möebius invariant Bloch classes can be replaced by Bloch classes with general weights.

**Theorem B.** (see [21]) Let  $1 \leq p < \infty$ ,  $\alpha > 0$ ,  $0 < s \leq 1$ ,  $\alpha \geq \frac{q+2}{p}$ ,  $q > \alpha(p - 1) - s - 1$ ,  $q > s - 2 + \alpha(p - 1)$  and  $f \in \mathcal{B}^\alpha$ . Then the following are equivalent:

- (a)  $d_1 = \text{dist}_{\mathcal{B}^\alpha}(f, F(p, q, s))$ ;
- (b)  $d_2 = \inf\{\varepsilon > 0 : \chi_{\Omega_{\alpha, \varepsilon}^1}(z) \frac{dm_2(z)}{(1 - |z|^2)^{\alpha p - q - s}} \text{ is an } s - \text{Carleson measure}\}$ .

In the following theorem, in [21], we calculated distances from a weighted Bloch class to Bergman spaces for  $q \leq 1$ . Let  $\widetilde{\mathcal{B}}^{-t} = D^{-1}\mathcal{B}^{-t} = \{f \in H(\mathbf{U}) : D^{-1}f \in \mathcal{B}^{-t}\}$ ,  $t < 0$ . It is well-known that  $\mathcal{B}_s^{q, q}(\mathbf{U}) \subset \widetilde{\mathcal{B}}^{-t}(\mathbf{U})$ ,  $t = s - \frac{1}{q}$ ,  $t < 0$ ,  $s < 0$  (see [11]).

**Theorem C.** (see [21]) Let  $0 < q \leq 1$ ,  $s < 0$ ,  $t \leq s - \frac{1}{q}$ ,  $\beta > \frac{1 - sq}{q} - 2$  and  $\beta > -1 - t$ . Let  $f \in \widetilde{\mathcal{B}}^{-t}$ . Then the following are equivalent:

- (a)  $l_1 = \text{dist}_{\widetilde{\mathcal{B}}^{-t}}(f, \mathcal{B}_s^{q, q})$ ;
- (b)  $l_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{U}} \left( \int_{\Omega_{\varepsilon, -t}(f)} \frac{(1 - |w|)^{\beta + t}}{|1 - \bar{z}w|^{2 + \beta}} dm_2(w) \right)^q (1 - |z|)^{-sq - 1} dm_2(z) < \infty\}$ .

The following theorem is a version of Theorem C for the case  $q > 1$ .

**Theorem D.** (see [21]) Let  $q > 1$ ,  $s < 0$ ,  $t \leq s - \frac{1}{q}$ ,  $\beta > \frac{-1 - sq}{q}$  and  $\beta > -1 - t$ . Let  $f \in \widetilde{\mathcal{B}}^{-t}$ . Then the following are equivalent:

- (a)  $l_1 = \text{dist}_{\widetilde{\mathcal{B}}^{-t}}(f, \mathcal{B}_s^{q, q})$ ;
- (b)  $\widehat{l}_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{U}} \left( \int_{\Omega_{\varepsilon, -t}(f)} \frac{(1 - |w|)^{\beta + t}}{|1 - \bar{z}w|^{2 + \beta}} dm_2(w) \right)^q (1 - |z|)^{-sq - 1} dm_2(z) < \infty\}$ .

Throughout the paper, we write  $C$  (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation  $A \asymp B$  means that there is a positive constant  $C$ , such that  $\frac{B}{C} \leq A \leq CB$ . We will write for two expressions  $A \lesssim B$  if there is a positive constant  $C$  such that  $A < CB$ .

2. Formulations and Proofs of Main Results

In this section we formulate and prove main results of this paper. We refer the reader for various basic notations concerning ball, polydisk and tube domain our previous mentioned papers on extremal problems (see [20], [21], [22]). Note a combination of Minkowski and Young inequalities leads to direct generalizations of some of our previously known results (see [21]) in two directions, simultaneously: to larger scales of values of parameters and to higher dimension (we consider in this paper the case of polydisk, the unit ball and the tubular domains). Note also, the case of higher dimensions were provided before in [22] only for particular values of parameters and without proofs. Moreover, complete analogues of results of this paper with similar proofs are valid also for spaces of harmonic and  $n$ -harmonic functions, based also on our previously used methods in this direction, (see [17], [20] and for particular values of parameters see [3]). We do not discuss this topic related with harmonic functions here, leaving it to our future work.

The following result can be seen in our previous paper without proof (see [21]) and it serves as model for formulations and proofs of our main results.

Actually, we are dealing now with simplest case  $\delta = 1$  in Bergman  $A_\alpha^1$  class in the unit ball (see [21]). For  $\alpha > 0$ , let

$$A_\alpha^\infty(\mathbf{B}^n) = \left\{ f \in H(\mathbf{B}^n) : \sup_{r < 1} \left( \int_{\mathbf{S}^n} |f(r\xi)| d\sigma(\xi) \right) (1-r)^\alpha < \infty \right\}, \mathbf{S}^n \text{ is a unit sphere,} \tag{1}$$

or we denote it by  $H_\alpha^1(\mathbf{B}^n)$ . Let also  $A_\alpha^p(\mathbf{B}^n) = \left\{ f \in H(\mathbf{B}^n) : \int_{\mathbf{B}^n} |f(z)|^p (1-|z|)^{p\alpha-1} dV(z) < \infty \right\}$ ,  $0 < p < \infty$  be Bergman space where  $H(\mathbf{B}^n)$  is a space of all analytic functions in the unit ball  $\mathbf{B}^n$ . Similarly we define such spaces in tube  $\mathbf{T}_\Omega$  and polydisk  $\mathbf{U}^n$  (see [19], [21], [22] for definitions). These are spaces  $H_s^\delta(\mathbf{T}_\Omega)$  (analogue of  $A_\alpha^\infty$ ),  $H_s^p(\mathbf{U}^n)$ ,  $A_s^p(\mathbf{T}_\Omega)$  and  $A_\alpha^p(\mathbf{U}^n)$ . For example, in polydisk, for  $0 < p \leq \infty$ ,  $s > 0$ , we have

$H_s^p(\mathbf{U}^n) = \left\{ f \in H(\mathbf{U}^n) : \sup_{r_j < 1} \left( \left( \int_{\mathbf{U}^n} |f(r_j \xi)|^p dm_2(\xi) \right)^{\frac{1}{p}} \prod_{j=1}^n (1-r_j)^s \right) < \infty \right\}$ , where  $H(\mathbf{U}^n)$  is a space of all analytic functions in polydisk.

We define a new subset of the unit interval and then using its characteristic function we will give a sharp assertion concerning distance function.

For  $\varepsilon > 0$ ,  $f \in H(\mathbf{B}^n)$ , let

$$L_{\varepsilon, \alpha}(f) = \{ r \in (0, 1) : (1-r)^\alpha \int_{\mathbf{S}^n} |f(r\xi)| d\sigma(\xi) \geq \varepsilon \}. \tag{2}$$

**Theorem 2.1.** *Let  $f \in A_\alpha^\infty(\mathbf{B}^n)$ ,  $\alpha > 0$ . Then the following are equivalent:*

- (a)  $s_1 = \text{dist}_{A_\alpha^\infty(\mathbf{B}^n)}(f, A_\alpha^1(\mathbf{B}^n))$ ;
- (b)  $s_2 = \inf\{\varepsilon > 0 : \int_0^1 (1-r)^{-1} \chi_{L_{\varepsilon, \alpha}(f)}(r) dr < \infty\}$ .

*Proof.* First we prove  $s_1 \geq s_2$ . Let us assume that  $s_1 < s_2$ . Then we can find two numbers  $\varepsilon, \varepsilon_1$  such that  $\varepsilon > \varepsilon_1 > 0$ , and a function  $f_{\varepsilon_1} \in A_\alpha^1(\mathbf{B}^n)$ ,  $\|f - f_{\varepsilon_1}\|_{A_\alpha^\infty(\mathbf{B}^n)} \leq \varepsilon_1$ , and  $\int_0^1 (1-r)^{-1} \chi_{L_{\varepsilon, \alpha}(f)}(r) dr = \infty$ . Hence we have

$$\begin{aligned} (1-r)^\alpha \int_{\mathbf{S}^n} |f_{\varepsilon_1}(r\xi)| d\sigma(\xi) & \tag{3} \\ & \geq (1-r)^\alpha \int_{\mathbf{S}^n} |f(r\xi)| d\sigma(\xi) - \sup_{r < 1} (1-r)^\alpha \int_{\mathbf{S}^n} |f(r\xi) - f_{\varepsilon_1}(r\xi)| d\sigma(\xi) \\ & \geq (1-r)^\alpha \int_{\mathbf{S}^n} |f(r\xi)| d\sigma(\xi) - \varepsilon_1. \end{aligned}$$

Hence for any  $s \in [-1, \infty)$ ,

$$(\varepsilon - \varepsilon_1) \int_0^1 (1-r)^s \chi_{L_{\varepsilon, \alpha}(f)}(r) dr \leq C \int_0^1 \left( \int_{\mathbf{S}^n} |f_{\varepsilon_1}(r\xi)| d\sigma(\xi) \right) (1-r)^{\alpha+s} dr. \tag{4}$$

Thus we have a contradiction.

It remains to show  $s_1 \leq Cs_2$ . Let  $I = [0, 1)$ . We argue as above and obtain from the classical Bergman representation formula (see [28]).

$$\begin{aligned}
 f(\rho\zeta) = f(z) &= C(t) \int_{L_{\varepsilon,\alpha}(f)} \int_{\mathbf{S}^n} \frac{f(r\xi)(1-r)^t}{(1-r\bar{\xi}\rho\zeta)^{t+n+1}} d\sigma(\xi) dr + C(t) \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{S}^n} \frac{f(r\xi)(1-r)^t}{(1-r\bar{\xi}\rho\zeta)^{t+n+1}} d\sigma(\xi) dr \\
 &= f_1(z) + f_2(z), \text{ where } t \text{ is large enough. Then we have} \\
 (1-\rho)^\alpha \int_{\mathbf{S}^n} |f_2(\rho\zeta)| |d\sigma(\zeta)| &\leq C(1-\rho)^\alpha \int_{\mathbf{S}^n} \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{S}^n} \frac{|f(r\xi)|(1-r)^t}{|1-r\bar{\xi}\rho\zeta|^{t+n+1}} |d\sigma(\xi)| dr |d\sigma(\zeta)| \\
 &\leq C(1-\rho)^\alpha \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{S}^n} |f(r\xi)|(1-r)^t \left( \int_{\mathbf{S}^n} \frac{1}{|1-r\bar{\xi}\rho\zeta|^{t+n+1}} |d\sigma(\zeta)| \right) |d\sigma(\xi)| dr \\
 &\leq C(1-\rho)^\alpha \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{S}^n} |f(r\xi)| |d\sigma(\xi)| \frac{(1-r)^t}{(1-r\rho)^{t+1}} dr \leq C\varepsilon(1-\rho)^\alpha \int_0^1 \frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}} dr \leq C\varepsilon.
 \end{aligned} \tag{5}$$

For  $\alpha > 0$  we have

$$\begin{aligned}
 \int_{\mathbf{B}^n} (1-\rho)^{\alpha-1} |f_1(\rho\zeta)| dV(\rho\zeta) & \\
 &\leq C \int_{\mathbf{B}^n} (1-\rho)^{\alpha-1} \int_{L_{\varepsilon,\alpha}(f)} \int_{\mathbf{S}^n} \frac{|f(r\xi)|(1-r)^t}{|1-r\bar{\xi}\rho\zeta|^{t+n+1}} |d\sigma(\xi)| dr dV(\rho\zeta) \\
 &\leq C \sup_{r < 1} \left( (1-r)^\alpha \int_{\mathbf{S}^n} |f(r\xi)| |d\sigma(\xi)| \right) \int_{L_{\varepsilon,\alpha}(f)} \frac{(1-r)^{t-\alpha}}{(1-r)^{t+1-\alpha}} dr \\
 &= C \sup_{r < 1} \left( (1-r)^\alpha \int_{\mathbf{S}^n} |f(r\xi)| |d\sigma(\xi)| \right) \int_{L_{\varepsilon,\alpha}(f)} \frac{1}{(1-r)} dr.
 \end{aligned} \tag{6}$$

Note that the implication  $\|f_1\|_{A_{\alpha}^1(\mathbf{B}^n)} < \infty$  follows directly from the known estimate for  $\alpha > 0$ ,  $f_1 \in H(\mathbf{B}^n)$

$$\int_0^1 (1-\rho)^{\alpha-1} \left( \int_{\mathbf{S}^n} |f_1(\rho\xi)| |d\sigma(\xi)| \right) d\rho \leq C \int_{\mathbf{B}^n} (1-\rho)^{\alpha-1} |f_1(\rho\xi)| dV(\rho\xi). \tag{7}$$

Hence  $\inf_{g \in A_{\alpha}^1(\mathbf{B}^n)} \|f - g\|_{A_{\alpha}^{\infty}(\mathbf{B}^n)} \leq C \|f - f_1\|_{A_{\alpha}^{\infty}(\mathbf{B}^n)} = \|f_2\|_{A_{\alpha}^{\infty}(\mathbf{B}^n)} \leq C\varepsilon$ .  
 The theorem is proved.  $\square$

Let for tube  $\mathbf{T}_{\Omega}$  over cone  $\Omega$

$$A_{\nu}^{q,p}(\mathbf{T}_{\Omega}) = \left\{ f \in H(\mathbf{T}_{\Omega}) : \int_{\Omega} \left( \int_{\mathbb{R}^n} |f(x+iy)|^q dx \right)^{\frac{p}{q}} \Delta(y)^{\nu-\frac{n}{r}} dy < \infty \right\}, \quad 1 \leq p, q < \infty, \nu > \frac{n}{r} - 1,$$

where  $\Delta$  is a determinant function in  $\mathbf{T}_{\Omega}$  (see [4]),  $H(\mathbf{T}_{\Omega})$  is a space of all analytic functions in  $\mathbf{T}_{\Omega}$ . Also, we put  $A_{\nu}^{p,p} = A_{\nu}^p$ ,  $1 \leq p < \infty$ ,  $\nu > \frac{n}{r} - 1$ .

Let now  $1 \leq p < \infty$ ,  $1 \leq q < \infty$ . Let also  $q \leq s$ . Then (see [4])  $A_{\nu}^{p,q}(\mathbf{T}_{\Omega}) \subset H_{\beta}^s(\mathbf{T}_{\Omega})$ , where  $\nu > \frac{n}{r} - 1$  and where  $\beta = \frac{\nu}{q} + \frac{n}{rp} - \frac{n}{rs}$ . For  $p = q = s$  this embedding with appropriate estimate is taking obviously a very simple form (see [4], Proposition 3.5) and the distance problem here can be easily posed again obviously in

general case and in mentioned simple case. Note for analytic and harmonic function spaces it was posed and solved in [3], [21] and [22] in various domains but not in tube domains.

We define  $\tilde{H}_s^p(T_\Omega)$ ,  $1 \leq p < \infty, s \in \mathbb{R}$  as a subset of  $H_s^p(T_\Omega)$  so that for each  $f$  function from that subset the Bergman representation formula with large enough  $\alpha_0$  index is valid. As we will see from our discussion below this assumption can be removed since this subset coincide with all space.

Since (see [4])  $A_\nu^{p,\delta}(T_\Omega) \subset H_\beta^s(T_\Omega)$ , where  $1 \leq p < \infty, 1 \leq \delta < \infty, \delta \leq s, \nu > \frac{n}{r} - 1, \beta = \frac{\nu}{\delta} + \frac{n}{r} \left( \frac{1}{p} - \frac{1}{s} \right)$ , we have:

**Theorem 2.2.** Let  $s = \frac{\nu}{\delta}, 1 \leq \delta < \infty, s \in \mathbb{R}, \nu > \delta \left( \frac{n}{r} - 1 \right)$ ,

$$L_{s,\varepsilon,\delta}(f) = \left\{ y \in \Omega : \left( \int_{\mathbb{R}^n} |f(x + iy)|^\delta dx \right)^{1/\delta} \Delta^s(y) \geq \varepsilon \right\}$$

for  $\varepsilon > 0, s > 0$ . Let  $f \in \tilde{H}_s^\delta(T_\Omega)$ . Then

$$l_1(f) \asymp l_2(f),$$

where  $l_1(f) = \text{dist}_{\tilde{H}_s^\delta(T_\Omega)}(f, A_\nu^{\delta,\delta}(T_\Omega))$  and  $l_2(f) = \inf \left\{ \varepsilon > 0 : \int_\Omega \chi_{L_{s,\varepsilon,\delta}}(f) \Delta^{-\frac{n}{r}}(y) dy < \infty \right\}, \frac{1}{\delta} + \frac{1}{\delta'} = 1, \frac{n}{r} < \delta'$ .

*Proof.* The proof follows standard scheme we provided in [3], [2], [15], [21]. And the new ingredient is application of Minkowski and Youngs inequality which leads to generalizations of our previous results (case  $\delta = 1$ ).

First we prove  $l_1(f) \geq l_2(f)$  in a general form for  $A_\nu^{p,\delta}(T_\Omega)$  spaces. Let us assume  $l_1 < l_2$ . Then there are two numbers  $\varepsilon, \varepsilon_1, \varepsilon > 0, \varepsilon_1 > 0$ , such that  $f_{\varepsilon_1} \in A_\nu^{p,\delta}(T_\Omega), \varepsilon > \varepsilon_1, \|f - f_{\varepsilon_1}\|_{H_s^\delta(T_\Omega)} \leq \varepsilon_1$  and  $\int_\Omega \chi_{L_{s,\varepsilon,\delta}(f)} \Delta^{-\frac{n}{r}}(y) dy = \infty$ , we first consider general case of  $A_\nu^{p,\delta}(T_\Omega)$  spaces.

We have

$$\begin{aligned} & \left( \int_{\mathbb{R}^n} |f_{\varepsilon_1}(\tau + iy)|^\delta d\tau \right)^{\frac{1}{\delta}} \Delta^s(y) \geq \\ & \geq \Delta^s(y) \left( \int_{\mathbb{R}^n} |f(\tau + iy)|^\delta d\tau \right)^{\frac{1}{\delta}} - \sup_{y \in \Omega} \Delta^s(y) \left( \int_{\mathbb{R}^n} |f(\tau + iy) - f_{\varepsilon_1}(\tau + iy)|^\delta d\tau \right)^{\frac{1}{\delta}} \geq \\ & \geq \Delta^s(y) \left( \int_{\mathbb{R}^n} |f(\tau + iy)|^\delta d\tau \right)^{\frac{1}{\delta}} - \varepsilon_1. \end{aligned}$$

Hence we have now

$$(\varepsilon - \varepsilon_1)^p \int_\Omega (\Delta(y))^{\tilde{\alpha}} \chi_{L_{s,\varepsilon,\delta}(f)}(y) dy \leq C \int_\Omega \left( \int_{\mathbb{R}^n} |f_{\varepsilon_1}(\tau + iy)|^\delta d\tau \right)^{\frac{p}{\delta}} \Delta^{sp}(y) \Delta^{\alpha - \frac{n}{r}}(y) dy,$$

$\tilde{\alpha} = -\frac{n}{r} + \alpha$  for any  $\alpha > 0$ . We have a contradiction.

Hence for  $sp > \frac{n}{r} - 1, \delta \in [1, \infty), p \in [1, \infty), p \leq \delta, s \in \mathbb{R}$

$$\text{dist}_{H_s^\delta(T_\Omega)}(f, A_{sp}^{\delta,p}(T_\Omega)) \geq \inf \left\{ \varepsilon > 0 : \int_\Omega \chi_{L_{s,\varepsilon,\delta}(f)}(y) (\Delta(y))^{-\frac{n}{r}} dy < \infty \right\}.$$

Let us show the reverse implication.

We assume first that if  $f \in H_s^\delta(T_\Omega)$ , then for large enough  $\tilde{\alpha}, \tilde{\alpha} > \alpha_0$

$$f(z) = \int_\Omega \int_{\mathbb{R}^n} B_{\tilde{\alpha}}(z, w) f(w) \Delta^{\tilde{\alpha} - \frac{n}{r}}(I_m w) dV(w).$$

Using this we define

$$f_1(z) = \int_{L_{s,\varepsilon,\nu}} \int_{\mathbb{R}^n} B_{\tilde{\alpha}}(z, w) f(w) \Delta^{\tilde{\alpha} - \frac{n}{r}}(I_m w) dV(w),$$

$$f_2(z) = \int_{\Omega \setminus L_{s,\varepsilon,\nu}} \int_{\mathbb{R}^n} B_{\tilde{\alpha}}(z, w) f(w) \Delta^{\tilde{\alpha} - \frac{n}{r}}(I_m \omega) dV(w).$$

If  $\|f_2\|_{H_s^\delta(T_\Omega)} < C_1\varepsilon$ ,  $\|f_1\|_{A_v^\delta(T_\Omega)} < C\|f\|_{H_s^\delta(T_\Omega)}$ ,  $\frac{n}{r} < \delta$ ,  $\nu > \frac{n}{r} - 1$ ,  $s = \frac{\nu}{\delta}$ ,  $1 \leq \delta < \infty$ , then we get what we need since we have obviously the following

$$\text{dist}_{H_s^\delta(T_\Omega)}(f, A_v^\delta(T_\Omega)) = \inf_{g \in A_v^\delta} \|f - g\|_{H_s^\delta(T_\Omega)} \leq C\|f - f_1\|_{H_s^\delta(T_\Omega)} = \|f_2\|_{H_s^\delta(T_\Omega)} < \infty.$$

Now using Minkowsky and Young’s inequality and lemmas above we have now the following estimates (we use that  $A_\beta^{p,q}(T_\Omega) \subset A_\nu^{s,t}(T_\Omega)$ ,  $1 \leq p \leq s < \infty$ ,  $1 \leq q \leq t < \infty$ ,  $\beta > \frac{n}{r} - 1$ ,  $\nu > \frac{n}{r} - 1$ ,  $\frac{\beta}{q} - \frac{\nu}{t} = \frac{n}{r} \left( \frac{1}{s} - \frac{1}{p} \right)$ , see [15]):

$$\begin{aligned} \|f_2\|_{H_s^\delta(T_\Omega)} &\leq C \sup_y \Delta^s(y) \left( \int_{\mathbb{R}^n} \left( \int_{\Omega \setminus L_{s,\varepsilon,\nu}} \int_{\mathbb{R}^n} |B_{\tilde{\alpha}}(z, w)| |f(w)| \Delta^{\tilde{\alpha} - \frac{n}{r}}(I_m \omega) dV(w) \right)^\delta dx \right)^{\frac{1}{\delta}} \leq \\ &\leq C \sup_y \Delta^s(y) \int_{\Omega \setminus L_{s,\varepsilon,\nu}} \Delta^{-\tilde{\alpha}}(I_m z + I_m w) \|f(w)\|_\delta \Delta^{\tilde{\alpha} - \frac{n}{r}}(I_m w) dV(w) \leq \\ &\leq \varepsilon C \sup_y \Delta^s(y) \int_{\Omega} \Delta^{-\tilde{\alpha}}(x + y) \Delta^{\tilde{\alpha} - \frac{n}{r} - s}(y) dy \leq C\varepsilon. \end{aligned}$$

Using embedding we mentioned above and lemmas above

$$\begin{aligned} \|f_1\|_{A_v^\delta} &\leq C \int_{\Omega} \Delta^{\frac{\nu}{\delta} - \frac{n}{r}}(\tilde{y}) \left( \int_{\mathbb{R}^n} |f_1(x + i\tilde{y})|^\delta dx \right)^{\frac{1}{\delta}} d\tilde{y} \leq \\ &\leq \int_{\Omega} \Delta^{\frac{\nu}{\delta} - \frac{n}{r}}(\tilde{y}) \int_{L_{s,\varepsilon,\nu}} \Delta^{-\tilde{\alpha}}(\tilde{y} + y) \|f_y\|_\delta \Delta^{\tilde{\alpha} - \frac{n}{r}}(y) dy d\tilde{y} \leq \\ &\leq C_1 \|f\|_{H_s^\delta(T_\Omega)} \int_{L_{s,\varepsilon,\nu}} \int_{\Omega} \Delta^{\frac{\nu}{\delta} - \frac{n}{r}}(\tilde{y}) \Delta^{-\tilde{\alpha}}(\tilde{y} + y) d\tilde{y} \Delta^{\tilde{\alpha} - \frac{n}{r} - s}(y) dy \leq \\ &\leq C_2 \|f\|_{H_s^\delta(T_\Omega)}, \nu > \left( \frac{n}{r} - 1 \right) \delta, \tilde{\alpha} > \alpha_0. \end{aligned}$$

The proof of Theorem 2.2 is complete.  $\square$

**Remark 2.3.** For  $\delta = 1$  case theorem was proved before by first author in [15] in more general Siegel domains of second type. This theorem for  $\delta = 1$  was proved in [19] in tube domains.

It is easy to show that  $A_s^p$  is embedded in  $H_s^p$  for all positive  $p$  and  $s$ , so we pose dist problem. Using classical embedding between Hardy and Bergman spaces in unit ball (see [14]) we have

**Theorem 2.4.** Let  $\varepsilon > 0$ ,  $s > 0$  and

$$\widehat{L}_{s,\varepsilon,\delta}(f) = \left\{ r \in (0, 1) : \left( \int_{S^n} |f(r\xi)|^\delta d\xi \right)^{\frac{1}{\delta}} (1 - r)^s \geq \varepsilon \right\}.$$

Then

$$\text{dist}_{H_s^\delta(\mathbb{B}^n)}(f, A_v^\delta(\mathbb{B}^n)) \asymp \inf \left\{ \varepsilon > 0 : \int_0^1 \chi_{\widehat{L}_{s,\varepsilon,\delta}}(y) (1 - y)^{-1} dy < \infty \right\},$$

$1 \leq \delta < \infty$ ,  $s \in \mathbb{R}$ ,  $s = \frac{\nu}{\delta}$ ,  $\nu > 0$ .

It is easy to show that  $A_s^p$  is embedded in  $H_s^p$  for all positive  $p$  and  $s$ , so we pose dist problem. Using classical embedding between Hardy and Bergman spaces in the polydisk (see [14]) we have

**Theorem 2.5.** Let  $\varepsilon > 0$ ,  $s > 0$ ,  $\mathbf{T}^n$  is unit torus of polydisk,  $I^n = [0, 1)^n$  and

$$\tilde{L}_{s,\varepsilon,\delta}(f) = \left\{ r \in I^n : \left( \int_{\mathbf{T}^n} |f(r\xi)|^\delta d\xi \right)^{\frac{1}{\delta}} \prod_{k=1}^n (1 - r_k)^s \geq \varepsilon \right\}.$$

Then

$$\text{dist}_{H_s^\delta(\mathbf{U}^n)}(f, A_v^\delta(\mathbf{U}^n)) \asymp \inf \left\{ \varepsilon > 0 : \int_0^1 \cdots \int_0^1 \chi_{\tilde{L}_{s,\varepsilon,\delta}}(y) \prod_{j=1}^n (1 - y_j)^{-1} dy_1 \cdots dy_n < \infty \right\},$$

$$1 \leq \delta < \infty, s \in \mathbb{R}, s = \frac{\nu}{\delta}, \nu > 0.$$

**Remark 2.6.** Theorem 2.4 and 2.5 were known for  $\delta = 1$  case (see [20], [21], [22] for various types of domains).

Theorems 2.4, 2.5 repeats in proofs all arguments of proof of Theorem 2.2, and we omit proofs.

Now we have to return to assumption we made before formulation of Theorem 2.2 to show that Hardy spaces  $\tilde{H}_v^p(T_\Omega)$  and  $H_v^p(T_\Omega)$  coincide first for function with compact support.

Now, if  $f \in H_v^1(T_\Omega)$  then we can use Bergman representation formula to show it for all values of  $\nu$ . Indeed, if  $f \in H_v^1(T_\Omega)$  then we use estimate

$$\begin{aligned} & \left( \int_\Omega \left( \int_{\mathbb{R}^n} |f(x + iy)| dx \right)^q \Delta^{\alpha - \frac{n}{r}}(y) dy \right)^{\frac{1}{q}} \leq \\ & \leq C \left( \sup_{y \in \Omega} \left( \int_{\mathbb{R}^n} |f(x + iy)| dx \right)^q (\Delta^{qv}(y)) \right)^{\frac{1}{q}} \times \left( \int_\Omega \Delta^{\alpha - \frac{n}{r} - q\nu}(y) dy \right)^{\frac{1}{q}} \leq \\ & \leq C_1 \|f\|_{H_v^1(T_\Omega)}, \end{aligned}$$

if  $\alpha > \frac{n}{r} - 1$ ,  $\alpha - \frac{n}{r} - q\nu < -\frac{2n}{r} + 1$ , (similarly  $H_v^\delta(T_\Omega) \subset A_\alpha^{\delta,q}(T_\Omega)$ ,  $q > q_0$ ,  $\alpha > \frac{n}{r} - 1$ ).

Now we have to use  $H_v^1(T_\Omega) \subset A_\alpha^{1,q}(T_\Omega)$ ,  $q > q_0$ ,  $\alpha > \frac{n}{r} - 1$ . But for all  $\tilde{\alpha} > \frac{n}{r} - 1$ ,  $f \in A_\alpha^{1,q}(T_\Omega)$ ,  $1 \leq q < \infty$  (see [15])

$$f(z) = \int_{T_\Omega} B_{\tilde{\alpha}}(z, w) f(w) \Delta^{\tilde{\alpha} - \frac{n}{r}}(I_m w) dV(w), \quad z \in T_\Omega.$$

Same is true for  $H_v^k(T_\Omega)$ .

In general case (not only for  $f$  functions with compact support as we had above) the mentioned above embeddings between weighted Hardy and Bergman spaces are also true (see [5], [6]). The proof is the same as in one-dimensional case of analytic functions in the unit disk and it is based on a fact that  $L^p(\mathbb{R}^n)$  norm of  $f$  function is monotone in cone and on existence of  $r$ -lattice of cone with nice properties (see [5], [6]).

Note also in addition in [11] and [14] it was actually proved that for weighted analytic Hardy spaces  $H_\delta^s$  the Bergman representation formula in the ball and in the polydisk is valid with large enough kernel index.

**Remark 2.7.** The tubular domains over symmetric cones are the most typical examples of unbounded Siegel domains of second type. Our theorem (it is complete analogue) is valid also for the most typical example of bounded Siegel domain of second type namely unit ball in  $\mathbb{C}^n$ . We formulated it above. Note this type observation was made also for sharp trace theorems and embedding theorems for multifunctional analytic spaces which were proved before by first author in [24] and [23]. Namely results of first section of [23] are valid also for the tubular domains over symmetric cones and results of [24] are valid also for bounded pseudoconvex domains with smooth boundary in  $\mathbb{C}^n$  with very similar proofs.

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