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Fixed Points for (G, ϕ)-Contractions in Vector Metric Spaces Endowed with a Graph

Rakesh Batra[a], Sachin Vashistha[b], Rajesh Kumar[b]

[a]*Department of Mathematics, Hans Raj College, University of Delhi, Delhi-110007, India*
[b]*Department of Mathematics, Hindu College, University of Delhi, Delhi-110007, India*

Abstract. In this work, we will prove some fixed point results for the class of (G, ϕ)-contractions on vector metric spaces endowed with a graph. Our results extend and unify many known results for (G, ϕ)-contractions on metric spaces with a graph and for ϕ-contractions on vector metric spaces. We apply our results to obtain an existence theorem for the solution of an integral equation.

1. Introduction

In 2007, Jachymski [5] introduced the concept of G-contraction on a metric space endowed with a graph G. Further, in 2010, Bojor [2] extended the work of Jachymski for (G, ϕ)-contraction mapping on a metric space endowed with a graph G. Recently, in 2012, Petre[7] proved a fixed point theorem for ϕ-contractions on vector metric spaces. In this article, we present some fixed point results for (G, ϕ)-contractions on vector metric spaces endowed with a graph G, thereby, extend many results in the area of fixed point theory, in particular, the work of above mentioned authors.

Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ will denote the set of natural numbers, real numbers, positive real numbers and negative real numbers respectively.

2. Preliminaries

The following notations, concepts and results may be found in [1, 3, 4]. A set E equipped with a partial order "$\leq$"is called a partially ordered set. In a partially ordered set $(E, \leq)$, the notation $x < y$ means $x \leq y$ and $x \neq y$. By an order interval $[x, y]$ in E we mean, a set $\{z \in E : x \leq z \leq y\}$. We note that $[x, y] = \phi$ if $x \nleq y$. An element $z \in E$ is said to be an upper bound of a subset S of E if $x \leq z$ for all $x \in S$ and a lower bound if $z \leq x$ for all $x \in S$. A subset S of E is said to be bounded above if it has an upper bound and bounded below if it has a lower bound. Further, an element $z \in E$ is said to be a supremum of S if (i) z is an upper bound of S and (ii) for any upper bound $t \in E$ of S we have $z \leq t$. We say that z is a least upper bound of S in this case. Similarly, infimum of S can be defined as a greatest lower bound of S in E. Supremum (or infimum) of a non empty set may or may not exist, but, if it exists, it is unique. A partially ordered

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Communicated by Ljubomir Ćirić
Email addresses: rakeshbatra.30@gmail.com (Rakesh Batra), vashistha_sachin@rediffmail.com (Sachin Vashistha), rajeshhctm@rediffmail.com (Rajesh Kumar)

set $(E, \leq)$ is a lattice if each pair of elements $x, y \in E$ has a supremum and an infimum in E. We use the notations $x \vee y$ and $x \wedge y$ to denote $\sup\{x, y\}$ and $\inf\{x, y\}$ respectively. A real linear space E together with an order relation "$\leq$"which is compatible with the algebraic structure of E via the properties (i) for each $x, y, z \in E$ we have $x \leq y \Rightarrow x + z \leq y + z$ and (ii) for each $x, y \in E$ and $t \in \mathbb{R}^+$ we have $x \leq y \Rightarrow tx \leq ty$ is called an ordered linear space. The set $E^+ = \{x \in E : 0 \leq x\}$ is called the positive cone of an ordered linear space $(E, \leq)$. An ordered linear space E for which $(E, \leq)$ is a lattice is called a Riesz space or linear lattice. For detail study about Riesz spaces one may refer to [1]. The space $\mathbb{R}^n$ with usual order defined by $x = (x_1, x_2, ..., x_n) \leq y = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$ whenever $x_i \leq y_i$ for each $i = 1, 2, ..., n$ is a Riesz space [1]. Here $x \vee y = (\max\{x_1 y_1\}, \max\{x_2, y_2\}, ..., \max\{x_n, y_n\})$ and $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_n, y_n\})$. Both the vector space $C(X)$ of all continuous real functions and the vector space $C_b(X)$ of all bounded continuous real functions on the topological space X are Riesz spaces when the ordering is defined pointwise. That is, $f \leq g$ whenever $f(x) \leq g(x)$ for each $x \in X$. The lattice operations are: $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$. For any sequence (x_n) in a Riesz space E, $x_n \downarrow x$ means x_n is a decreasing sequence and $\inf\{x_n\} = x$ and for any sequence (x_n) in a Riesz space E, $x_n \uparrow x$ means x_n is an increasing sequence and $\sup\{x_n\} = x$. For any two decreasing sequences (x_n) and (y_n) in a Riesz space E, following properties are satisfied. (i) $x_n \downarrow x$ and $y_m \downarrow y$ imply $x_n + y_m \downarrow x + y$, (ii) $x_n \downarrow x$ implies $tx_n \downarrow tx$ for all $t \in \mathbb{R}^+$ and $tx_n \uparrow tx$ for all $t \in \mathbb{R}^-$ and (iii) $x_n \downarrow x$ and $y_m \downarrow y$ imply $x_n \vee y_m \downarrow x \vee y$ and $x_n \wedge y_m \downarrow x \wedge y$. Now we present some more definitions and examples useful for our main results and that may be found in [1, 3, 4]. Let E denote a Riesz space and $|x| := x \vee (-x)$ for all $x \in E$. A sequence $\{x_n\}$ in a Riesz space E is said to be an order convergent (or o-convergent) to x (we write $x_n \overset{o}{\to} x$), if there exists a sequence $\{y_n\}$ in E satisfying $y_n \downarrow 0$ and $|x_n - x| \leq y_n$ for all $n \in \mathbb{N}$. Here are some simple properties of order convergence. (i) A sequence $\{x_n\}$ in a Riesz space has at most one order limit, (ii) if $x_n \overset{o}{\to} x$ and $y_n \overset{o}{\to} y$ then $x_n + y_n \overset{o}{\to} x + y$, (iii) $\alpha x_n \overset{o}{\to} \alpha x$ for all $\alpha \in \mathbb{R}$, (iv) $|x_n| \overset{o}{\to} |x|$, (v) $x_n \vee y_n \overset{o}{\to} x \vee y$ and $x_n \wedge y_n \overset{o}{\to} x \wedge y$ and (vi) if $x_n \leq y_n$ for all $n \geq n_0$ then $x \leq y$. Let E and F be any two Riesz spaces. A function $f : E \to F$ is order continuous (or o-continuous) if $x_n \overset{o}{\to} x$ in E implies $f(x_n) \overset{o}{\to} f(y)$ in F. A sequence $\{x_n\}$ in a Riesz space is said to be an order Cauchy (or o-Cauchy), if there exists a sequence $\{y_n\}$ in E such that $y_n \downarrow 0$ and $|x_n - x_{n+p}| \leq y_n$ for all $n, p \in \mathbb{N}$. A Riesz space E is called o-complete if every o-Cauchy sequence in E is o-convergent in E. Let X be a nonempty set and E be a Riesz space. A function $d : X \times X \to E$ is said to be an E-metric or a vector metric on X if (i) $d(x, y) = 0$ if and only if $x = y$ (ii) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. Also the triplet (X, d, E) is said to be a vector metric space or an E-metric space. Vector metric spaces generalize the notion of metric spaces and for arbitrary elements x, y, z, w of a vector metric space, the following properties hold:(i) $0 \leq d(x, y)$ (ii) $d(x, y) = d(y, x)$ (iii) $|d(x, z) - d(y, z)| \leq d(x, y)$ (iv) $|d(x, z) - d(y, w)| \leq |d(x, y) - d(z, w)|$. A Riesz space E is a vector metric space with respect to $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This Vector metric is called an absolute valued metric on E. $\mathbb{R}^2$ is a Riesz space with respect to coordinatewise ordering of its elements. It is a vector metric space with respect to the vector metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $d((x, y), (z, w)) = (\alpha|x - z|, \beta|y - w|)$, where $\alpha, \beta \in \mathbb{R}^+$. $\mathbb{R}$ is a vector metric space with respect to the vector metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined by $d(x, y) = (\alpha|x - y|, \beta|x - y|)$, where $\alpha, \beta \in \mathbb{R}^+ \cup \{0\}$ with $\alpha + \beta \in \mathbb{R}^+$. Let (X, d, E) be a vector metric space. A sequence $\{x_n\}$ in X is said to be E-convergent or vectorially convergent to some $x \in X$, written as $x_n \xrightarrow{d,E} x$, if there exists a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all $n \in \mathbb{N}$.

Lemma 2.1. *Let (X, d, E) be a vector metric space and $x_n \xrightarrow{d,E} x$. Then*

(i) the limit x is unique,

(ii) any subsequence of $\{x_n\}$ is vectorial convergent to x and

(iii) if $y_n \xrightarrow{d,E} y$, then, $d(x_n, y_n) \overset{o}{\to} d(x, y)$.

Let (X, d, E) be a vector metric space. A sequence $\{x_n\}$ in X is said to be an E-Cauchy if there exists a sequence $\{a_n\}$ in E with $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ for all $n, p \in \mathbb{N}$. A vector metric space (X, d, E) is said to be E-complete if every E-Cauchy sequence in X is E-convergent to a limit in X. A subset Y of X is said to be E-closed if for any sequence $\{y_n\}$ in Y which is E-convergent to some $y \in X$, we have $y \in Y$.

Remark 2.2. *If $E = \mathbb{R}$ then the concepts of E-convergence and of E-Cauchy sequence are same as that of metric convergence and Cauchy sequence respectively. Further, if $X = E$ and d is the absolute valued metric, then, the concepts of E-convergence and o-convergence are the same.*

Let (X, d, E) and (Y, ρ, F) be vector metric spaces. A function $f : X \to Y$ is said to be vectorial continuous (or E-continuous) at $x \in X$ if for every sequence $\{x_n\}$ in X with $x_n \xrightarrow{d,E} x$ we have $f(x_n) \xrightarrow{\rho,F} f(x)$. Further, f is said to be vectorial continuous on X if f is vectorial continuous at every $x \in X$.

For the following concepts about a graph, one may refer to [5]. Let (X, d, E) be a vector metric space and $\Delta = \{(x, x) : x \in X\}$. Consider a directed graph G with the set $V(G)$ of its vertices equal to X and the set $E(G)$ of its edges as a superset of Δ. Assume that G has no parallel edges. Now we can identify G with the pair $(V(G), E(G))$. The graph G can be converted to a weighted graph by assigning to each edge a weight equal to the distance between its vertices.

Let G^{-1} denote conversion of the graph G obtained from the graph G by reversing the direction of edges. Thus we have $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. By $\tilde{G}$ we denote the undirected graph obtained from G by ignoring the direction of edges. It is convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. That is

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \tag{1}$$

By a subgraph of G we mean a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that $V(H)$ contains the vertices of all edges of $E(H)$.

Definition 2.3. *Let (X, d, E) be a vector metric space equipped with a graph G. A mapping $f : X \to X$ is orbitally E-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers $f^{k_n}x \xrightarrow{d,E} y \Rightarrow f(f^{k_n}x) \xrightarrow{d,E} fy$ as $n \to \infty$, (G, E)-continuous if given $x \in X$ and a sequence $(x_n)_{n\in N}$ with $x_n \xrightarrow{d,E} x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $fx_n \xrightarrow{d,E} fx$ and orbitally (G, E)-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers, $f^{k_n}x \xrightarrow{d,E} y$ together with $(f^{k_n}x, f^{k_{n+1}}x) \in E(G)$ implies $f(f^{k_n}x) \xrightarrow{d,E} fy$ as $n \to \infty$.*

Clearly we have the following relations.
E-Continuity $\Rightarrow$ (G, E)-continuity $\Rightarrow$ orbital (G, E)-continuity and
E-Continuity $\Rightarrow$ orbital E-continuity $\Rightarrow$ orbital (G, E)-continuity.

If x and y are vertices in a graph G then a path in G from x to y (of length $n(n \in \mathbb{N} \cup \{0\})$) is a finite sequence $(x_i)_{i=0}^n$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, 2, ..., n$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and $x \in V(G)$ then the subgraph G_x consisting of all edges and vertices that are contained in some path in G begining at x is called the component of G containing x. In this case $V(G_x) = [x]_G$ where $[x]_G$ is the equivalence class of the relation R defined on $V(G)$ by the rule yRz if there is a path in G from y to z. Clearly G_x is connected for all $x \in G$.

Definition 2.4. *[7] Let E be a Riesz space. A function $\phi : E^+ \to E^+$ is said to be an o-comparison function if (i) ϕ is increasing, that is, $x_1, x_2 \in E^+$ and $x_1 \leq x_2$ imply $\phi(x_1) \leq \phi(x_2)$, (ii) $\phi(t) < t$ for any $t > 0$, and (iii) $\phi^n(t) \xrightarrow{o} 0$ for any $t > 0$.*

Let Φ be the set of all ϕ described in Definition 2.4.

Definition 2.5. *[7] Let (X, d, E) be a vector metric space and $\phi \in \Phi$ be an o-comparison function. A function $T : X \to X$ is said to be a nonlinear ϕ-contraction if and only if $d(Tx, Ty) \leq \phi(d(x, y))$ for any $x, y \in X$.*

Definition 2.6. *[5] Let (X, d) be a metric space and G is a directed graph with $V(G) = X$ and $\Delta \subseteq E(G)$. A mapping $T : X \to X$ is said to be a G-contraction if (i) $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$ for all $x, y \in X$ and (ii) there exists a number $k \in [0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$ we have*

$$d(Tx, Ty) \leq kd(x, y) \tag{2}$$

3. Main Results

Throughout this section we assume that $X \equiv (X, d, E)$ is a vector metric space with an E-metric d and $\mathscr{G} = \{G : G \text{ is a directed graph with } V(G) = X \text{ and } \Delta \subseteq E(G)\}$. The set of all fixed points of a self map T on X will be denoted by $\mathrm{Fix}(T)$.

Definition 3.1. *Let T be a self map on a vector metric space (X, d, E). T is an E-Picard operator (abbr., EPO) if T has a unique fixed point x_* and $T^n x \xrightarrow{d,E} x_*$ for all $x \in X$.*

Definition 3.2. *Let T be a self map on a vector metric space (X, d, E). T is a weakly E-Picard operator (abbr., WEPO) if for any $x \in X$, $\lim_{n\to\infty} T^n x$ exists (it may depend on x) and is a fixed point of T.*

Following Definition 2.5 and Definition 2.6 we introduce G-contraction and (G, ϕ)-contraction in the following manner.

Definition 3.3. *Let (X, d, E) be a vector metric space and G be a directed graph with $V(G) = X$ and $\Delta \subseteq E(G)$. A mapping $T : X \to X$ is said to be a G- contraction if*

(i) for all $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G) \tag{3}$$

(ii) There exists a number $k \in [0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$,

$$d(Tx, Ty) \leq kd(x, y) \tag{4}$$

Definition 3.4. *Let (X, d, E) be a vector metric space, $\phi \in \Phi$ be an o-comparison function and $G \in \mathscr{G}$ be given. A mapping $T : X \to X$ is said to be a (G, ϕ)-contraction if*

(i) for all $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G) \tag{5}$$

(ii) for all $x, y \in X$ with $(x, y) \in E(G)$,

$$d(Tx, Ty) \leq \phi(d(x, y)) \tag{6}$$

Remark 3.5. *Let $G \in \mathscr{G}$ be arbitrary. Then every G-contraction on (X, d, E) is a (G, ϕ)-contraction for ϕ given by $\phi(a) = ka$ for all $a \in E^+$. Here $k \in [0, 1)$ is as in Definition 3.3.*

Remark 3.6. *It follows from (5) that $(T(V(G)), (T \times T)(E(G)))$ is a subgraph of G where $(T \times T)(x, y) = (Tx, Ty)$ for all $x, y \in X$.*

Example 3.7. *Any constant function $T : X \to X$ is a (G, ϕ)-contraction for every $\phi \in \Phi$ and $G \in \mathscr{G}$. This follows because $E(G)$ contains all loops.*

Example 3.8. *Let $\phi \in \Phi$ be arbitrary. Then every ϕ-contraction is an (G_0, ϕ)-contraction for the complete graph G_0 given by $V(G_0) = X$ and $E(G_0) = X \times X$.*

Example 3.9. *Let $\preceq$ be a partial order on X. Define the graph G_1 by $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. Then $G_1 \in \mathscr{G}$ and for any $\phi \in \Phi$, a self map $T : X \to X$ is a (G_1, ϕ)-contraction if it satisfies*

(i) T is non decreasing w.r.t. $\preceq$ and

(ii) for all $x, y \in X$ with $(x, y) \in E(G_1)$,

$$d(Tx, Ty) \leq \phi(d(x, y)) \tag{7}$$

We say that T is an order ϕ-contraction if T satisfies (ii) in Example 3.9. That is, if (7) is satisfied for all $x, y \in X$ with $x \preceq y$.

Remark 3.10. *Conditions (i) and (ii) in Definition 3.4 are independent. For example, identity mapping on any vector metric space (X, d, E) endowed with a graph G preserves edges but (6) is not satisfied for any $k \in [0, 1)$ if there is at least one $(x, y) \in E(G) - \Delta$. Further, a mapping $T : E \to E$ given by $Tx = -\frac{1}{2}x$ for all $x \in E$ is an order ϕ-contraction for $\phi(a) = \frac{1}{2}a$ for all $a \in E^+$ and with respect to absolute valued metric on any Riesz space E but T is not increasing if E has at least two elements x and y with $x < y$.*

Remark 3.11. *Let G_d be the graph given by $V(G_d) = X$ and $E(G_d) = \Delta$. Then (3) and (4) are satisfied for every mapping $T : X \to X$. Thus every $T : X \to X$ is a (G_d, ϕ)-contraction for every $\phi \in \Phi$. Consequently, given $\phi \in \Phi$, there is no self mapping on X which is not a (G, ϕ)-contraction for all $G \in \mathscr{G}$. But for a fixed $G \in \mathscr{G}$ it is possible to find a $\phi \in \Phi$ and a mapping $T : X \to X$ such that T is a (G, ϕ)-contraction but not a G-contraction.*

Example 3.12. *Let $S_n = \frac{n(n+1)}{2}$, $n \in \mathbb{N} \cup \{0\}$ and $X = \{S_n : n \in \mathbb{N} \cup \{0\}\}$. Let $E = \mathbb{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ by $TS_0 = S_0$ and $TS_n = S_{n-1}$ for all $n \in \mathbb{N}$. Take $\phi(t) = S_n$ if $S_n < t \leq S_{n+1}$, $n \in \mathbb{N} \cup \{0\}$ and $\phi(S_0) = \phi(0) = 0$. Then ϕ becomes an o-comparison function on E^+. Let G be a graph given by $V(G) = X$ and $E(G) = \{(S_n, S_n) : n \in \mathbb{N} \cup \{0\}\} \cup \{(S_0, S_n) : n \in \mathbb{N}\}$. It is easy to see that T preserves edges. We show that T satisfies (6) but not (2). Clearly $(x, y) \in E(G)$ with $Tx \neq Ty$ if and only if $x = S_0$ and $y = S_n$ for some $n > 1$. Further for $n > 1$ we have $d(TS_0, TS_n)/\phi(d(S_0, S_n)) = S_{n-1}/\phi(Sn) = S_{n-1}/S_{n-1} = 1$. Thus T is a (G, ϕ)-contraction. Now for $n > 1$ we have $d(TS_0, TS_n)/d(S_0, S_n) = (S_{n-1} - S_0)/(S_n - S_0) = S_{n-1}/S_n = (n-1)/(n+1)$ which tends to 1 as $n \to \infty$. Thus T does not satisfy (2). Hence T is a (G, ϕ)-contraction which is not a G-contraction.*

Example 3.13. *Let $X = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ and $E = \mathbb{R}^2$ with componentwise ordering. Let $d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|)$ be a vector metric on X. Let*

$$T(x, y) = \begin{cases} (1/4, 1/4) & \text{if } (x, y) \neq (1, 1) \\ (1/8, 1/8) & \text{if } (x, y) = (1, 1) \end{cases}$$

T is not a ϕ-contraction for any $\phi \in \Phi$ as it is not an E-continous mapping. As discussed in Remark 3.11, T is a (G_d, ϕ)-contraction for every $\phi \in \Phi$.

Definition 3.14. *Two sequences $\{x_n\}$ and $\{y_n\}$ in a vector metric space (X, d, E) are equivalent if $d(x_n, y_n) \xrightarrow{\circ} 0$ as $n \to \infty$.*

Proposition 3.15. *Let (X, d, E) be a vector metric space equipped with a graph G. If a mapping $T : X \to X$ is such that (5) (resp. (6)) holds, then (5) (resp. (6)) is also satisfied for G^{-1} and $\tilde{G}$. Hence if T is a (G, ϕ)-contraction then T is both a (G^{-1}, ϕ)-contraction and a $(\tilde{G}, \phi)$-contraction.*

Proof. This is an obvious consequence of symmetry of d and (1). □

Lemma 3.16. *Let $T : X \to X$ be a (G, ϕ)-contraction on a vector metric space (X, d, E) equipped with a graph G. For $x \in X$ and $y \in [x]_{\tilde{G}}$, we have $d(T^n x, T^n y) \xrightarrow{\circ} 0$ as $n \to \infty$.*

Proof. Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there exists a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x to y. That is, $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for all $i = 1, 2, \ldots, N$. By Proposition 3.15, T is a $(\tilde{G}, \phi)$-contraction. So inductively $(T^n x_{i-1}, T^n x_i) \in E(\tilde{G})$ for all $n \in \mathbb{N}$, $i = 1, 2, \ldots, N$ and

$$d\left(T^n x_{i-1}, T^n x_i\right) \leq \phi^n\left(d(x_{i-1}, x_i)\right) \tag{8}$$

for all $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$. If $x_{i-1} = x_i$ for some $i = 1, 2, ..., N$, then $d(T^n x_{i-1}, T^n x_i) = 0$ for all $n \in \mathbb{N}$. Consider the case when $x_{i-1} \neq x_i$ for all $i = 1, 2, \ldots, N$. Letting $n \to \infty$ in (8) we get $d(T^n x_{i-1}, T^n x_i) \xrightarrow{\circ} 0$ as $n \to \infty$ for all $i = 1, 2, ..., N$. By triangular inequality we get $d(T^n x, T^n y) \leq \sum_{i=1}^N d(T^n x_{i-1}, T^n x_i) \xrightarrow{\circ} 0$ as $n \to \infty$. □

Theorem 3.17. *The following statements are equivalent in a vector metric space (X, d, E) equipped with a graph G.*

(i) G is weakly connected.

(ii) For any (G, ϕ)-contraction $T : X \to X$ and $x, y \in X$, the sequences $\{T^n x\}$ and $\{T^n y\}$ are E-Cauchy and equivalent.

(iii) For any (G, ϕ)-contraction $T : X \to X$, $Card(Fix\,T) \leq 1$.

Proof. (i)⇒ (ii):

Let G be weakly connected. Let $T : X \to X$ be a (G, ϕ)-contraction and $x, y \in X$. Then $X = [x]_{\tilde{G}}$. Take $y = Tx \in [x]_{\tilde{G}}$ in Lemma 3.16. Then $d(T^n x, T^{n+1}x) \xrightarrow{o} 0$ as $n \to \infty$. So $d(T^n x, T^m x) \leq \sum_{i=1}^{n-m} d(T^{m+i-1}, T^{m+i}) \xrightarrow{o} 0$ as $n \to \infty$. Thus $(T^n x)$ is E-Cauchy. By Lemma 3.16, $(T^n x)$ and $(T^n y)$ are equivalent. So $(T^n y)$ is also E-Cauchy.

(ii) ⇒ *(iii)* Let $T : X \to X$ be a (G, ϕ)-contraction and $x, y \in Fix(T)$. By *(ii)*, $(T^n x)$ and $(T^n y)$ are E-Cauchy and equivalent. This gives $x = y$.

(iii) ⇒ *(i)* Let G be not weakly connected. Then $\tilde{G}$ is disconnected. Let $x_0 \in X$. Then both $[x_0]_{\tilde{G}}$ and $X \backslash [x_0]_{\tilde{G}}$ are non empty. Choose $y_0 \in X \backslash [x_0]_{\tilde{G}}$. Define

$$T(x) = \begin{cases} x_0 & \text{if } x \in [x_0] \\ y_0 & \text{if } x \in X \backslash [x_0]_{\tilde{G}} \end{cases}$$

Then $Fix(T) = \{x_0, y_0\}$. We now show that T is a (G, ϕ)-contraction. Let $(x, y) \in E(G)$ be arbitrary. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$. So $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X \backslash [x_0]_{\tilde{G}}$. In both cases we have $Tx = Ty$. This shows that $(Tx, Ty) \in E(G)$ because $\triangle \subseteq E(G)$. Consequently, (5) and (6) are satisfied. Thus T is a (G, ϕ)-contraction having two fixed points which violates *(iii)*. Hence G must be weakly connected. □

Corollary 3.18. *Let (X, d, E) be a complete vector metric space. Then the following statements are equivalent.*

(i) G is weakly connected.

(ii) for any (G, ϕ)-contraction $T : X \to X$, there exists an $x^* \in X$ such that $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.

Remark 3.19. *Example 3.2 of [5] justifies the fact that we may not improve Corollary 3.18 by adding in (ii) that x^* is a fixed point of T. Mapping T in Example 3.2 of [5] is obviously an order F-contraction for $F(\alpha) = \ln \alpha$ for all $\alpha > 0$.*

Theorem 3.20. *Let $T : X \to X$ be a (G, ϕ)-contraction such that $Tx_0 \in [x_0]_{\tilde{G}}$ for some $x_0 \in X$. Let $\tilde{G}_{x_0}$ be component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is T-invariant and $T|_{[x_0]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0}, \phi)$-contraction. Moreover, if $x, y \in [x_0]_{\tilde{G}}$ then the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are E-Cauchy and equivalent.*

Proof. Let $x \in [x_0]_{\tilde{G}}$ be arbitrary. Then there exists a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to x. That is, $x_N = x$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for all $i = 1, 2, \ldots, N$. By Proposition 3.15, every (G, ϕ)-contraction is a $(\tilde{G}, \phi)$-contraction. So T is a $(\tilde{G}, \phi)$-contraction. This implies $(Tx_{i-1}, Tx_i) \in E(\tilde{G})$ for all $i = 1, 2, \ldots, N$. Consequently $(Tx_i)_{i=0}^N$ is a path in $\tilde{G}$ from Tx_0 to Tx. Thus $Tx \in [Tx_0]_{\tilde{G}}$. But $Tx_0 \in [x_0]_{\tilde{G}}$. So $[Tx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$. Hence $Tx \in [x_0]_{\tilde{G}}$. This proves that $[x_0]_{\tilde{G}}$ is T-invariant.

Now let $(x, y) \in E(\tilde{G}_{x_0})$ be arbitrary. This means there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to y such that $x_{N-1} = x$. Repeating the argument from the first part of the proof we infer that $(Tx_i)_{i=0}^N$ is a path in $\tilde{G}$ from Tx_0 to Ty. Since $Tx_0 \in [x_0]_{\tilde{G}}$, therefore we have a path $(y_i)_{i=0}^M$ in $\tilde{G}$ from x_0 to Tx_0. It follows that $(y_0, y_1, \ldots, y_M, Tx_1, Tx_2, \ldots, Tx_N)$ is a path in $\tilde{G}$ from x_0 to Ty. In particular $(Tx_{N-1}, Tx_N) \in E(\tilde{G}_{x_0})$. That is $(Tx, Ty) \in E(\tilde{G}_{x_0})$. Since $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$ and T is a (G, ϕ)-contraction, therefore (6) holds for the graph $\tilde{G}_{x_0}$ as well. Thus $T|_{[x_0]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0}, \phi)$-contraction.

Finally Theorem 3.17 and connectedness of $\tilde{G}_{x_0}$ imply that $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are E-Cauchy and equivalent for all $x, y \in [x_0]_{\tilde{G}}$. □

Theorem 3.21. *Let (X, d, E) be a complete vector metric space, $T : X \to X$ be a (G, ϕ)-contraction, $X_T = \{x \in X : (x, Tx) \in E(G)\}$ and (X, d, E, G) satisfy the following property.*

For any $x \in X$ with the sequence $T^n x \xrightarrow{d,E} x^*$ and $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{T^{k_n} x\}_{n\in\mathbb{N}}$ of $\{T^n x\}_{n\in\mathbb{N}}$ satisfying $(T^{k_n} x, x^*) \in E(G)$ for all $n \in \mathbb{N}$. (9)

Then

(i) Card Fix T=Card$\{[x]_{\tilde{G}} : x \in X_T\}$.

(ii) Fix $T \neq \phi$ if and only if $X_T \neq \phi$.

(iii) T has a unique fixed point if and only if there exists a point $x_0 \in X_T$ such that $X_T \subseteq [x_0]_{\tilde{G}}$.

(iv) For any $x \in X_T$, $T|_{[x]_{\tilde{G}}}$ is an EPO.

(v) If $X_T \neq \phi$ and G is weakly connected then T is an EPO.

(vi) If $X' = \cup\{[x]_{\tilde{G}} : x \in X_T\}$ then $T|_{X'}$ is a WEPO.

(vii) If $T \subseteq E(G)$ then T is a WEPO.

Proof. Let us first prove (iv) and (v). Let $x \in X_T$ be arbitrary. Then $(x, Tx) \in E(G)$. This implies that $Tx \in [x]_{\tilde{G}}$. So by Theorem 3.20, for any $y \in X$ sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are E-Cauchy and equivalent. Since (X, d, E) is complete, therefore, there exists an $x^* \in X$ such that $T^n x \xrightarrow{d,E} x^*$ and $T^n y \xrightarrow{d,E} x^*$. Since $(x, Tx) \in E(G)$, (5) yields for all $n \in \mathbb{N}$

$$(T^n x, T^{n+1} x) \in E(G) \tag{10}$$

By (9), it follows that there exists a subsequence $(T^{k_n} x)$ of $(T^n x)$ such that $(T^{k_n} x, x^*) \in E(G)$ for all $n \in \mathbb{N}$. By (10), $(x, Tx, T^2 x, ..., T^{k_1} x, x^*)$ is a path in G (and hence in $\tilde{G}$) from x to x^*. So $x^* \in [x]_{\tilde{G}}$. Now $d(T^{k_n+1} x, Tx^*) \leq d(T^{k_n} x, x^*)$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get $d(x^*, Tx^*) = 0$. That is $Tx^* = x^*$. Hence $T|[x]_{\tilde{G}}$ is an EPO.

Further if G is weakly connected and $x \in X_T$ then $X = [x]_{\tilde{G}}$. So T is an EPO.

Now (vi) is a consequence of (iv). To prove (vii) observe that $T \subseteq E(G)$ implies that $X = X_T$ which gives $X' = X$ and hence by (iv), T becomes a WEPO on X.

To prove (i), consider the mapping $\pi : Fix\,T \to C$ given by $\pi(x) = [x]_{\tilde{G}}$ for all $x \in Fix\,T$ where $C = \{[x]_{\tilde{G}} : x \in X_T\}$. It suffices to show that π is a bijection. Let $x \in X_T$ be arbitrary. By (iv), $T|_{[x]_{\tilde{G}}}$ is an EPO. Let $x^* = \lim_{n\to\infty} T^n x$. Then $x^* \in Fix\,T \cap [x]_{\tilde{G}}$ and $\pi x^* = [x^*]_{\tilde{G}} = [x]_{\tilde{G}}$. So π is surjective. Now let $x_1, x_2 \in Fix\,T$ be arbitrary with $[x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$. Then $x_2 \in [x_1]_{\tilde{G}}$. By (iv), $\lim_{n\to\infty} T^n x_2 \in Fix\,T \cap [x_1]_{\tilde{G}} = \{x_1\}$. But $T^n x_2 = x_2$ for all $n \in \mathbb{N}$. Thus we get $x_1 = x_2$. Hence π is a bijection.

Finally (ii) and (iii) follows by (i). □

Corollary 3.22. *Let (X, d, E) be a complete vector metric space and (X, d, E, G) satisfy the property (9). Then the following statements become equivalent.*

(i) G is weakly connected.

(ii) Every (G, ϕ)-contraction $T : X \to X$ such that $(Tx_0, x_0) \in X$ for some $x_0 \in X$, is an EPO.

(iii) For any (G, ϕ)-contraction $T : X \to X$, card Fix $T \leq 1$.

Proof. $(i) \Rightarrow (ii)$ follows from Theorem 3.21 (v).

$(ii) \Rightarrow (iii)$:

Let $T : X \to X$ be a (G, ϕ)-contraction. If $X_T = \phi$ then so is *Fix T* as *Fix* $T \subseteq X_T$. In case $X_T \neq \phi$ then by (ii) *Fix T* is singleton. In both cases *card Fix* $T \leq 1$.

$(iii) \Rightarrow (i)$:

Follows by Theorem 3.17. □

Theorem 3.23. *Let (X, d, E) be complete and $T : X \to X$ be an orbitally (G, E)-continuous (G, ϕ)-contraction. Let $X_T = \{x \in X : (x, Tx) \in E(G)\}$. Then the following statements hold:*

(i) Fix $T \neq \phi$ if and only if $X_T \neq \phi$.

(ii) For any $x \in X_T$ and $y \in [x]_{\tilde{G}}$, the sequence $(T^n y)_{n\in\mathbb{N}}$ converges to a fixed point of T and $\lim_{n\to\infty} T^n y$ does not depend on y.

(iii) If $X_T \neq \phi$ and G is weakly connected, then T is an EPO.

(iv) If $T \subseteq E(G)$ then T is a WEPO.

Proof. We begin with (ii). Let $x \in X$ satisfy $(x, Tx) \in E(G)$ and $y \in [x]_{\tilde{G}}$. By Theorem 3.20, sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ vectorially converge to the same point x_*. Moreover $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Since T is orbitally (G, E)-continuous we get $T(T^n x) \xrightarrow{d,E} Tx_*$. This yields $Tx_* = x_*$ since, simultaneously, $T(T^n x) = T^{n+1}x \xrightarrow{d,E} x_*$. Thus we proved (ii) and '$\Leftarrow$'of (i). '$\Rightarrow$'of (i) follows by the assumption that $E(G) \supseteq \triangle$. (iv) is an immediate consequence of (ii) since $T \subseteq E(G)$ means $X_T = X$. To Prove (iii) observe that if $x_0 \in X_T$ then $[x_0]_{\tilde{G}} = X$ so (ii) yields T is an EPO. □

Continuity condition on T can be strengthened by the following version of Theorem 3.23.

Theorem 3.24. *Let (X, d, E) be complete and $T : X \to X$ be an orbitally E-continuous (G, ϕ)-contraction. Let $X_T = \{x \in X : (x, Tx) \in E(G)\}$. Then the following statements hold:*

(i) $FixT \neq \phi$ if and only if there exists an $x_0 \in X$ with $Tx_0 \in [x_0]_{\tilde{G}}$.

(ii) If $x \in X$ is such that $Tx \in [x]_{\tilde{G}}$, then for any $y \in [x]_{\tilde{G}}$ the sequence$(T^n y)_{n\in\mathbb{N}}$ converges to a fixed point of T and $lim_{n\to\infty}T^n y$ does not depend on y.

(iii) If G is weakly connected, then T is an EPO.

(iv) If $Tx \in [x]_{\tilde{G}}$ for any $x \in X$ then T is a WEPO.

Proof. We begin with (ii). Let $x \in X$ be such that $Tx \in [x]_{\tilde{G}}$ and let $y \in [x]_{\tilde{G}}$. By Theorem 3.20, sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in N}$ vectorially converge to the same point x_*. Since T is orbitally E-continuous we get $T(T^n x) \xrightarrow{d,E} Tx_*$. This yields $Tx_* = x_*$ since, simultaneously, $T(T^n x) = T^{n+1}x \xrightarrow{d,E} x_*$. Thus we proved (ii) and '$\Leftarrow$'of (i). '$\Rightarrow$'of (i) follows by the fact that $x \in [x]_{\tilde{G}}$ for any $x \in X$. Now if G is weakly connected then $X = [x]_{\tilde{G}}$ for any $x \in X$. In particular $Tx \in [x]_{\tilde{G}}$ for any $x \in X$ and by (ii) we infer that T is an EPO. Thus (iii) holds. Finally (iv) is an immediate consequence of (ii). □

Corollary 3.25. *Let (X, d, E) be complete. Then the following statements are equivalent.*

(i) G is weakly connected.

(ii) Every orbitally E-continous (G, ϕ)-contraction is an EPO.

(iii) For every orbitally E-continous (G, ϕ)-contraction $T : X \to X$, card Fix $T \leq 1$.

Hence if $\tilde{G}$ is disconnected then there exist at least one orbitally E- continous (G, ϕ)-contraction $T : X \to X$ which has at least two fixed points.

Proof. Theorem 3.24 (iii) yields that $(i) \Rightarrow (ii)$. $(ii) \Rightarrow (iii)$ is obvious. $(iii) \Rightarrow (i)$ follows by the proof of $(iii) \Rightarrow (i)$ of Theorem 3.17. T defined there is orbitally E-continous. □

4. Applications

Theorem 4.1. *Let $I = [a, b]$ be any interval of the real line, $(B, \|.\|)$ be a partially ordered Banach space satisfying the property that for any sequence $\{b_n\}$ in B with $b_n \leq b_{n+1}$ for all $n \in \mathbb{N}$ and $b_n \to b, b \in B$, we have $b_n \leq b$. Let $E = C(I, \mathbb{R}^+)$ be the space of all continuous functions defined on I taking values in $\mathbb{R}^+$, with usual partial order and the usual operations of addition and multiplication. The space E is a Riesz space under the pointwise lattice operations. Let $X = C(I, B)$ be the space of all continuous functions defined on I with values in B and pointwise partial order. Let $d : X \times X \to E$ defined by $d(x, y)(.) = \|x(.) - y(.)\|$ for any $x, y \in X$, be a complete vector metric on X. Let $h : I \times I \times B \to B$ be continuous and $\alpha \in X$. Consider the Fredholm type integral equation*

$$x(t) = \int_I h(t, s, x(s))ds + \alpha(t), \tag{11}$$

for $t \in I$. Assume that

(i) $h(t, s, .) : B \to B$ is nondecreasing for each $t, s \in I$,

(ii) there exists an o-comparison function $\phi : E_+ \to E_+$ and a continuous function $w : I \times I \to \mathbb{R}^+$ such that $\|h(t, s, x(s)) - h(t, s, y(s))\| \leq w(t, s)\phi(d(x, y))(t)$ for each $t, s \in I$ and $x \leq y$,

(iii) $sup_{t\in I} \int_I w(t, s)ds \leq 1$,

(iv) there exists $x_0 \in X$ such that $x_0(t) \leq \int_I h(t, s, x_0(s))ds + \alpha(t)$ for all $t \in I$.

Then the integral equation (11) has a unique solution in the set $\{x \in X : x \leq x_0$ or $x \geq x_0\}$.

Proof. Define a mapping $T : X \to X$ by

$$T(x)(t) := \int_I h(t, s, x(s))ds + \alpha(t)$$

for all $t \in I$. Clearly T is a well defined mapping. Consider a graph G with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \leq y\}$. By the given condition (i), we observe that T is nondecreasing. Thus $(x, y) \in E(G)$ implies that $(Tx, Ty) \in E(G)$. Further, G satisfies the property (9) because B satisfies the same for nondecreasing sequences. Now for any $x, y \in X$ with $(x, y) \in E(G)$, we have $d(T(x), T(y))(t) = \|T(x)(t) - T(y)(t)\| \leq \int_I \|h(t, s, x(s)) - h(t, s, y(s))\|ds \leq \int_I w(t, s)\phi(d(x, y))(t)ds = \phi(d(x, y))(t) \int_I w(t, s)ds \leq \phi(d(x, y))(t)$ for all $t \in I$. Thus $d(T(x), T(y)) \leq \phi(d(x, y))$ for all $x, y \in X$ with $(x, y) \in E(G)$. So T is a (G, ϕ)-contraction. By (iv), we have $(x_0, Tx_0) \in E(G)$. Also, $[x_0]_{\tilde{G}} = \{x \in X : x \leq x_0$ or $x \geq x_0\}$. The conclusion follows by Theorem 3.21. □

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