An Integrodifferential Operator for Meromorphic Functions Associated with the Hurwitz-Lerch Zeta Function

Adel A. Attiya\textsuperscript{a}, Oh Sang Kwon\textsuperscript{b}, Park Ji Hyang\textsuperscript{c}, Nak Eun Cho\textsuperscript{d}

\textsuperscript{a}Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
\textsuperscript{b}Department of Mathematics, Kyungsung University, Busan 608-736, Republic of Korea
\textsuperscript{c}Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea
\textsuperscript{d}Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea

Abstract. In this paper, we introduce a new integrodifferential operator associated with the Hurwitz-Lerch Zeta function in the puncture open disk of the meromorphic functions. We also obtain some properties of the third-order differential subordination and superordination for this integrodifferential operator, by using certain classes of admissible functions.

1. Introduction

Let $\Sigma$ denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the punctured open unit disc $U^* = U \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The function $f(z)$ has a simple pole at $z = 0$.

We begin by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ defined by (see, for example, [18, P. 121 et seq.] and [19, P. 194 et seq.])

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

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Email addresses:aattiy@mans.edu.eg (Adel A. Attiya), oskwon@ks.ac.kr (Oh Sang Kwon), jihyang1822@naver.com (Park Ji Hyang), necho@pknu.ac.kr (Nak Eun Cho)
(b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}^- = \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, \ldots\}, s \in \mathbb{C} \text{ when } z \in \mathbb{U}, \text{ Re}(s) > 1 \text{ when } |z| = 1 ).

Several properties of \( \Phi(z, s, b) \) can be found in many papers, for example Attiya and Hakami [3], Choi et al. [8], Cho et al. [7], Ferreira and López [9], Gupta et al. [10] and Luo and Srivastava [14]. See, also Kutbi and Attiya [11, 12], Srivastava and Attiya [17], Srivastava and Gaboury [20], Srivastava et al. [21] and Owa and Attiya [16].

Analogous to the operator defined by Srivastava and Attiya [17], we define the following operator associated with the Hurwitz-Lerch Zeta function, as follows:

\[
J^*_s, b : \Sigma \rightarrow \Sigma
\]

the operator defined by:

\[
J^*_s, b f(z) = G(s, b; z) \ast f(z)
\] (1.3)

where the function \( G(s, b; z) \) defined by

\[
G(s, b; z) = \frac{b^s \Phi(z, s, b)}{z}
\] (1.4)

\((z \in \mathbb{U}^*; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C})\)

and \( \ast \) denotes the Hadamard product (or Convolution). Then we can see that

\[
J^*_s, b f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left( \frac{b}{k + b + 1} \right)^s a_k z^k
\]

\((z \in \mathbb{U}^*; f(z) \in \Sigma; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C})\).

**Remark 1.** We note that:

1. \( J^*_{0, b} f(z) = f(z), \)
2. \( J^*_1, \frac{1}{2} f(z) = \frac{1 - 2a}{a z^{a+\frac{1}{2}}} \int_0^z t^{a-\frac{1}{2}} f(t) \, dt \quad (0 < a < \frac{1}{2}), \)
3. \( J^*_1, b f(z) = \frac{b}{z^{b+1}} \int_0^z t^b f(t) \, dt, \)
4. \( J^*_{s, \frac{1}{2}} f(z) = \frac{b^s}{\Gamma(b+1)} \int_0^z t^b \left( \log \frac{1}{t} \right)^{s-1} f(t) \, dt \quad (\alpha > 0; \beta > 0), \)
5. \( J^*_1, f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^s} a_k z^k, \)
6. \( J^*_{-1, 1} f(z) = -zf'(z), \)
7. \( J^*_{-1, -2} f(z) = \frac{f(z) -zf'(z)}{2}, \)
8. \( f_{n-1} \) \( f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} (-k)^{n} a_{k} z^{k} \ (n \in \mathbb{N}) \),

9. \( f_{n} \) \( f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} (k+2)^{n} a_{k} z^{k} \ (n \in \mathbb{N}) \),

where \( f_{n} \) the operator introduced by Cho et al. [6], \( f_{s} \) the operator introduced by Lashin [13], \( f_{s,1} \) the operator introduced by Alhindi and Darus [1], \( f_{s} \) the operator defined by Urallegaddi and Somanatha [23] and \( f_{1,b} \) is the operator analogous to the generalized Bernardi operator (for Bernardi operator see [5]), when \( \text{Re}(b) > 0 \), the operator \( f_{1,b} \) introduced by Bajpai [4].

We denote by \( H[a, n] \), the class of analytic functions in \( U \) in the form

\[
f(z) = a + \sum_{k=n}^{\infty} a_{k} z^{k} \quad (a \in \mathbb{C}; n \in \mathbb{N} = \{1, 2, \cdots \})
\]

and \( H = H[1, 1] \).

In our investigation we need the following definitions and theorem:

**Definition 1.1.** Let \( f(z) \) and \( F(z) \) be analytic functions. The function \( f(z) \) is said to be subordinate to \( F(z) \), written \( f(z) < F(z) \), if there exists a function \( w(z) \) analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), and such that \( f(z) = F(w(z)) \). If \( F(z) \) is univalent, then \( f(z) < F(z) \) if and only if \( f(0) = F(0) \) and \( f(U) \subset F(U) \).

**Definition 1.2.** [2, P. 441] Let \( D \) be the set of analytic functions \( q(z) \) and univalent on \( U \setminus E(q) \), where

\[
E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\},
\]

is such that \( \min |q'(\zeta)| = \rho > 0 \) for \( \zeta \in \partial U \setminus E(q) \). Further, let \( \mathcal{D}(a) = \{ q(z) \in \mathcal{D} : q(0) = a \} \) and \( \mathcal{D}_1 = \mathcal{D}(1) \).

**Definition 1.3.** [2, P. 440] Let \( \Psi : \mathbb{C}^{4} \times U \to \mathbb{C} \) and \( h(z) \) be univalent in \( U \). If \( p(z) \) is analytic in \( U \) and satisfies the third-order differential subordination:

\[
\psi(p(z), z p'(z), z^{2} p''(z), z^{3} p'''(z); z) < h(z), \quad (1.5)
\]

then \( p(z) \) is called a solution of the differential subordination. A univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination or more simply a dominant if \( p(z) < q(z) \) for all \( p(z) \) satisfying (1.5). A dominant \( \bar{q}(z) \) that satisfies \( \bar{q}(z) < q(z) \) for all dominants of (1.5) is called the best dominant of (1.5).

**Definition 1.4.** [2, P. 440] Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in \mathcal{D} \) and \( n \in \mathbb{N} \setminus \{1\} \). The class of admissible functions \( \Psi_{n}[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^{4} \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[
\psi(r, s, t, u; z) \notin \Omega
\]

whenever

\[
r = q(\zeta), \quad s = k \zeta q'(\zeta), \\
\text{Re} \left( \frac{t}{s} + 1 \right) \geq k \text{Re} \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right)
\]

and

\[
\text{Re} \left( \frac{u}{s} \right) \geq k^{2} \text{Re} \left( \frac{\zeta^{2} q'''(\zeta)}{q'(\zeta)} \right),
\]

where \( z \in \mathbb{U}; \zeta \in \partial U \setminus E(q) \) and \( k \geq n \).
Analogous to the second order differential superordinations introduced by Miller and Mocanu [15], Tang et al. [22] defined the differential superordinations as follows:

**Definition 1.5.** [22, P. 3] Let \( \psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C} \) and the function \( h(z) \) be analytic in \( \mathbb{U} \). If the functions \( p(z) \) and \( \psi(p(z), z\, p'(z), z^2\, p''(z), z^3\, p'''(z)) \)

are univalent in \( \mathbb{U} \) and satisfy the following third-order differential superordination:

\[
h(z) < \psi(p(z), z\, p'(z), z^2\, p''(z), z^3\, p'''(z)),
\]

then \( p(z) \) is called a solution of the differential superordination. An analytic function \( q(z) \) is called a subordinant if \( q(z) < p(z) \) for \( p(z) \) satisfying (1.6). A univalent subordinant \( \tilde{q}(z) \) that satisfies \( \tilde{q}(z) < q(z) \) for all supordinants \( q(z) \) of (1.6) is said to be the best superordinant.

**Definition 1.6.** [22, P. 4] Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in H[a, n] \) and \( q'(z) \neq 0 \). The class of admissible functions \( \Psi_n[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C} \) that satisfy the following admissibility condition:

\[
\psi(r, s, t, u; \zeta) \in \Omega,
\]

whenever

\[
r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \text{Re}\left(\frac{r}{s} + 1\right) \leq \frac{1}{m} \text{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right)
\]

and

\[
\text{Re}\left(\frac{u}{s}\right) \leq \frac{1}{m^2} \text{Re}\left(\frac{z^2q'''(z)}{q'(z)}\right),
\]

where \( z \in \mathbb{U}, \zeta \in \partial \mathbb{U} \) and \( m \geq n \geq 2 \).

Also, we need the following theorems in our investigations:

**Theorem 1.1.** [2, p. 449] Let \( p(z) \in H[a, n] \) with \( n \in \mathbb{N} \setminus \{1\} \). Also, let \( q(z) \in D(a) \) and satisfy the following conditions:

\[
\text{Re}\left(\frac{zq''(z)}{q'(z)}\right) > 0, \quad \left|\frac{zq'(z)}{q'(z)}\right| \leq k,
\]

where \( z \in \mathbb{U}, \zeta \in \partial \mathbb{U} \setminus E(q) \) and \( k \geq n \). If \( \Omega \) is a set in \( \mathbb{C} \), \( \psi \in \Psi_n[\Omega, q] \) and

\[
\psi(p(z), z\, p'(z), z^2\, p''(z), z^3\, p'''(z); z) \in \Omega,
\]

then

\[
p(z) < q(z).
\]
Theorem 2.1. [22, p. 4] Let \( q(z) \in H[a, n] \) and \( \psi \in \Psi''[\Omega, q] \). If
\[
\psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z) ; z)
\]
is univalent in \( U \) and \( p(z) \in \mathbb{D}(a) \) satisfy the following conditions:
\[
\text{Re} \left( \frac{zq''''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{\psi'(z)}{q'(z)} \right| \leq m, \tag{2.1}
\]
\[ (z \in U; \zeta \in \partial U; m \geq n \geq 2) \]
and
\[
\Omega \subset \left\{ \psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z) ; z) : z \in U \right\},
\]
implies that
\[
q(z) < p(z).
\]

In this paper, by using the third-order differential subordination and superordination results by Antonino and Miller [2] and Tang et al. [22], we define certain classes of admissible functions and investigate some subordination and superordination properties of meromorphic functions associated with the integro-differential operator \( \mathcal{I}_{s,b}^* \) defined by (1.3). Furthermore, new differential sandwich-type theorems are obtained.

2. Third Order Differential Subordination with \( \mathcal{I}_{s,b}^* \)

Definition 2.1. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q(z) \in \mathbb{D} \). The class of admissible functions \( \Phi_1[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C} \) that satisfy the admissibility condition:
\[
\phi(a_1, a_2, a_3, a_4; z) \not\in \Omega,
\]
whenever
\[
a_1 = q(z), \quad a_2 = \frac{k \zeta q''(\zeta) + b q(\zeta)}{b},
\]
\[
\text{Re} \left( \frac{b (a_3 - a_1)}{(a_2 - a_1) - 2b} \right) \geq k \text{ Re} \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right),
\]
\[
\text{Re} \left( \frac{b^2 (a_4 - a_1) - 3b (b + 1) (a_3 - a_1)}{(a_2 - a_1) (a_2 - a_1) - 3b^2 + 6b + 2} \right) \geq k^2 \text{ Re} \left( \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right),
\]
where \( z \in \mathbb{U}, b \in \mathbb{C} \setminus \mathbb{Z}_0^+, s \in \mathbb{C}, \zeta \in \partial \mathbb{U} \setminus \mathbb{E}(q) \) and \( k \in \mathbb{N} \setminus \{1\} \).

Theorem 2.1. Let \( \phi \in \Phi_1[\Omega, q] \). If \( f(z) \in \Sigma \) and \( q(z) \in \mathbb{D}_1 \) satisfy the following conditions:
\[
\text{Re} \left( \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| \frac{\zeta\left(\mathcal{I}_{s-1,b}^* f(z) - \mathcal{I}_{s,b}^* f(z)\right)}{q'(\zeta)} \right| \leq \frac{k}{|b|} \tag{2.1}
\]
and
\[
\left\{ \phi(z) \mathcal{I}_{s,b}^* f(z), z \mathcal{I}_{s-1,b}^* f(z), z \mathcal{I}_{s-2,b}^* f(z), z \mathcal{I}_{s-3,b}^* f(z) ; z \right\} \in \Omega,
\]
then
\[
z \mathcal{I}_{s,b}^* f(z) < q(z). \tag{2.3}
\]
Proof. Let us define the analytic function \( p(z) \) as:
\[
p(z) = zJ^*_{s,b}f(z) \quad (z \in \mathbb{U}).
\] (2.4)
Using the definition of \( J^*_{s,b}f(z) \), we can prove that
\[
z\left(J^*_{s,b}f(z)\right)' = bJ^*_{s-1,b}f(z) - (b + 1)J^*_{s,b}f(z),
\] (2.5)
then we get
\[
zJ^*_{s-1,b}f(z) = \frac{zp'(z) + bp(z)}{b},
\] (2.6)
which implies
\[
zJ^*_{s-2,b}f(z) = \frac{z^2p''(z) + (2b + 1)zp'(z) + b^2p(z)}{b^2}.\] (2.7)
Also, we can see that
\[
zJ^*_{s-3,b}f(z) = \frac{z^3p'''(z) + 3(b + 1)z^2p''(z) + (3b^2 + 3b + 1)zp'(z) + b^3p(z)}{b^3}.\] (2.8)
Let us define the parameters \( a_1, a_2, a_3 \) and \( a_4 \) as:
\[
a_1 = r, \quad a_2 = \frac{s + br}{b}, \quad a_3 = \frac{t + (1 + 2b)s + b^2r}{b^2}
\] and
\[
a_4 = \frac{u + 3(b + 1)t + (3b^2 + 3b + 1)s + b^3r}{b^3}.
\] (2.9)
Now, we define the transformation
\[
\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}
\]
\[
\psi(r, s, t, u; z) = \phi(a_1, a_2, a_3, a_4; z).
\] (2.10)
By using the relations from (2.4) to (2.8), we have
\[
\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi \left(zJ^*_{s,b}f(z), zJ^*_{s-1,b}f(z), zJ^*_{s-2,b}f(z), zJ^*_{s-3,b}f(z); z\right).
\] (2.11)
Therefore, we can rewrite (2.2) as
\[
\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.
\]
Then the proof is completed by showing that the admissibility condition for \( \phi \in \Phi_\Gamma[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition (1.3), since
\[
\frac{t}{s} + 1 = \frac{b(a_3 - a_1)}{a_2 - a_1} - 2b
\] (2.11)
and
\[
\frac{u}{s} = \frac{b^2(a_4 - a_1) - 3b(b + 1)(a_3 - a_1)}{(a_2 - a_1)} + 3b^2 + 6b + 2.
\]
we also note that
\[ |zp'(z)| \leq \frac{bz\left(\int_{\zeta} f(z) - \int_{\zeta} f(z)\right)}{q'(\zeta)} \leq k. \]

Therefore, \( \psi \in \Psi_2[\Omega, q] \) and hence by Theorem 1.1, \( p(z) < q(z). \]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case the class \( \Phi_T[h(U), q] \) is written as \( \Phi_T[h, q] \).

The following theorem is a directly consequence of Theorem 2.1.

**Theorem 2.2.** Let \( \phi \in \Phi_T[h, q] \). If \( f(z) \in \Sigma \) and \( q(z) \in D_1 \) satisfy the following conditions:
\[
\Re \left( \frac{\zeta q''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z\left(\int_{\zeta} f(z) - \int_{\zeta} f(z)\right)}{q'(z)} \right| \leq \frac{k}{|b|} \tag{2.12}
\]
and
\[
\phi(z_{\zeta, b} f(z), z_{\zeta, b} f(z), z_{\zeta, b} f(z), z_{\zeta, b} f(z) ; z) \leq h(z), \tag{2.13}
\]
then
\[
z_{\zeta, b} f(z) < q(z).
\]

The next corollary is an extension of Theorem 2.1 to the case where the behavior of \( q(z) \) on \( \partial U \) is not known.

**Corollary 2.1.** Let \( \Omega \subset \mathbb{C} \) and let \( q(z) \) be univalent in \( U \), \( q(0) = 1 \). Let \( \phi \in \Phi_T[\Omega, q] \) for some \( \rho \in (0, 1) \) where \( q_p(z) = q(\rho z) \). If \( f(z) \in \Sigma \) satisfies
\[
\Re \left( \frac{\zeta q''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z\left(\int_{\zeta} f(z) - \int_{\zeta} f(z)\right)}{q'(z)} \right| \leq \frac{k}{|b|} \tag{2.14}
\]
and
\[
\phi(z_{\zeta, b} f(z), z_{\zeta, b} f(z), z_{\zeta, b} f(z), z_{\zeta, b} f(z) ; z) \in \Omega, \tag{2.15}
\]
then
\[
z_{\zeta, b} f(z) < q(z),
\]
where \( z \in U \) and \( \zeta \in \partial U \setminus E(q_\rho) \).

**Proof.** By using Theorem 2.1, we have \( f_{\zeta, b} f(z) < q_\rho(z) \). Then we obtain the result from \( q_\rho(z) < q(z) \).}

**Corollary 2.2.** Let \( \Omega \subset \mathbb{C} \) and let \( q(z) \) be univalent in \( U \), \( q(0) = 1 \). Let \( \phi \in \Phi_T[h, q] \) for some \( \rho \in (0, 1) \) where \( q_p(z) = q(\rho z) \). If \( f(z) \in \Sigma \) satisfies
\[
\Re \left( \frac{\zeta q''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z\left(\int_{\zeta} f(z) - \int_{\zeta} f(z)\right)}{q'(z)} \right| \leq \frac{k}{|b|} \tag{2.16}
\]
Corollary 2.3. Let $\phi = \frac{zq'(z) + bq(z)}{b}, \frac{z^2q''(z) + (2b + 1)zq'(z) + b^2q(z)}{b^2}$, then

$$\phi(zJ_{s+1}f(z), zJ_{s-1}f(z), zJ_{s-2}f(z), zJ_{s-3}f(z); z) < h(z),$$

(2.17)

then

$$zJ_{s+1}f(z) < q(z),$$

where $z \in \mathbb{U}$ and $\zeta \in \partial \mathbb{U} \setminus E(q)$. 

**Theorem 2.3.** Let $h(z)$ be univalent in $\mathbb{U}$. Let $\phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$. Suppose that the differential equation:

$$\phi \left( q(z), \frac{zq'(z) + bq(z)}{b}, \frac{z^2q''(z) + (2b + 1)zq'(z) + b^2q(z)}{b^2}, \frac{z^3q'''(z) + 3(b + 1)z^2q''(z) + 3b^2 + 3b + 1)zq'(z) + b^3q(z)}{b^3} ; z \right) = h(z),$$

(2.18)

has a solution $q(z)$ with $q(0) = 1$ which satisfies (2.1). If $f(z) \in \Sigma$ satisfies (2.17) and

$$\phi(zJ_{s+1}f(z), zJ_{s-1}f(z), zJ_{s-2}f(z), zJ_{s-3}f(z); z)$$

is analytic in $\mathbb{U}$, then

$$zJ_{s+1}f(z) < q(z)$$

(2.19)

and $q(z)$ is the best dominant of (2.19).

**Proof.** By using Theorem 2.1 that $q(z)$ is a dominant of (2.17). Since $q(z)$ satisfies (2.18), it is also a solution of (2.17) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the case $q(z) = 1 + Mz$ ($M > 0$) and in view of the Definition 2.1, the class of admissible functions $\Phi_T[\Omega, q]$ denoted by $\Phi_T[\Omega, M]$ is defined below.

**Definition 2.2.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_T[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi \left( 1 + Me^{i\theta}, 1 + \frac{(b + k)Me^{i\theta}}{b}, 1 + \frac{L + (b^2 + k(2b + 1))Me^{i\theta}}{b^2}, \frac{N + 3(b + 1)L + (b^3 + k(3b^2 + 3b + 1))Me^{i\theta}}{b^3} ; z \right) \in \Omega,$$

(2.20)

where $z \in \mathbb{U}$, $\text{Re}(Le^{-i\theta}) \geq (k - 1)kM$ and $\text{Re}(Ne^{-i\theta}) \geq 0$ for all real $\theta$ and $k \in \mathbb{N} \setminus \{1\}$.

**Corollary 2.3.** Let $\phi \in \Phi_T[\Omega, M]$. If $f(z) \in \Sigma$ satisfies the following conditions:

$$\left| z \left( f_{s+1} - f_{s} \right) \right| \leq \frac{kM}{|b|}$$

(2.21)

and

$$\phi(zJ_{s+1}f(z), zJ_{s-1}f(z), zJ_{s-2}f(z), zJ_{s-3}f(z); z) \in \Omega,$$

(2.22)

then

$$\left| zJ_{s+1}(z) - 1 \right| < M.$$
In the case $\Omega = q(U) = \{ w : |w - 1| < M \ (M > 0) \}$, for simplification we denote by $\Phi_T[\Omega, M]$ to the class $\Phi_T[\Omega, M]$.

**Corollary 2.4.** Let $\phi \in \Phi_T[\Omega]$. If $f(z) \in \Sigma$ satisfies the condition (2.21) and
\[
\left| \phi(z_{j,b}^* f(z), z_{j,1,b}^* f(z), z_{j,b}^* f(z), z_{j,-1,b}^* f(z), z_{j,-2,b}^* f(z); z) - 1 \right| < M, \tag{2.23}
\]
then
\[
\left| z_{j,b}^* f(z) - 1 \right| < M.
\]

Putting $\phi(a_1, a_2, a_3, a_4; z) = a_2 = 1 + \frac{b + k}{M} e^{i\theta}$ in Corollary 2.4, we have the following corollary:

**Corollary 2.5.** Let $M > 0$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0$ with $\text{Re}(b) < \frac{k}{2} \ (k \in \mathbb{N}\setminus\{1\})$. If $f(z) \in \Sigma$ satisfies the condition (2.21) and
\[
\left| z_{j,b}^* f(z) - 1 \right| < M,
\]
then
\[
\left| z_{j,b}^* f(z) - 1 \right| < M.
\]

**Corollary 2.6.** Let $k \in \mathbb{N}\setminus\{1\}$, $M > 0$ and $b \in \mathbb{C} \setminus \mathbb{Z}_0$. If $f(z) \in \Sigma$ satisfies the condition
\[
\left| z \left( z_{j,b}^* f(z) - z_{j,b}^* f(z) \right) \right| < \frac{kM}{|b|}, \tag{2.24}
\]
(2.21) then
\[
\left| z_{j,b}^* f(z) - 1 \right| < M.
\]

**Proof.** Let
\[
\phi(a_1, a_2, a_3, a_4; z) = a_2 - a_1.
\]
Using Corollary 2.3 with $\Omega = h(U)$ and
\[
h(z) = \frac{kM}{|b|} z \ (z \in U).
\]
Now we show that $\phi \in \Phi_T[\Omega, M]$.

Since the condition (2.21) is satisfied from the condition (2.24) and
\[
\left| \phi \left( 1 + M e^{i\theta} \right), 1 + \frac{(b + k) M e^{i\theta}}{b}, 1 + \frac{L + (b^2 + k(2b + 1)) M e^{i\theta}}{b^2} \right. \left. \frac{N + 3(b + 1)L + (b^3 + k(3b^2 + 3b + 1)) M e^{i\theta}}{b^3}; z \right) \frac{1}{b^3}
\]
\[
= \frac{kM}{|b|},
\]
then we have Corollary 2.6. \[\square\]
Corollary 2.7. Let \( k \in \mathbb{N}\setminus\{1\}, \) \( M > 0 \) and \( b \in \mathbb{C}\setminus\mathbb{Z}_0 \). If \( f(z) \in \Sigma \) satisfies the condition (2.21) and
\[
\left| z \left( J_{s-3,b}^* f(z) - J_{s-2,b}^* f(z) \right) \right| < 2 \left( \frac{|b| + 1|^2 + |2b + 3|}{|b|^3} \right) M,
\tag{2.25}
\]
then
\[
\left| z J_{s,b}^*(z) - 1 \right| < M.
\]

Proof. We define
\[
\phi(a_1, a_2, a_3, a_4; z) = a_4 - a_3.
\]
Using Corollary 2.3 with \( \Omega = h(U) \) and
\[
h(z) = \frac{2 \left( \frac{|b| + 1|^2 + |2b + 3|}{|b|^3} \right) M}{|b|^3} z \quad (z \in U).
\]
Now we show that \( \phi \in \Phi_{\Gamma}[\Omega, M] \).
Since
\[
\phi \left( 1 + Me^{i\theta}, 1 + \frac{(b + k) Me^{i\theta}}{b}, 1 + \frac{L + \left( b^2 + k(2b + 1) \right) Me^{i\theta}}{b^2}, \frac{N + 3(b + 1)L + \left( b^3 + k(3b^2 + 3b + 1) \right) Me^{i\theta}}{b^3} ; z \right)
= \left| \frac{N + (2b + 3) L + k(b + 1)^2 M}{b^3} e^{-i\theta} \right|
\geq \frac{\Re \left( N e^{-i\theta} \right) + |2b + 3| \Re \left( L e^{-i\theta} \right) + k |b + 1|^2 M}{|b|^3}
\geq \frac{(k - 1) k M |2b + 3| + k |b + 1|^2 M}{|b|^3}
\geq \frac{2 \left( \frac{|b| + 1|^2 + |2b + 3|}{|b|^3} \right) M}{|b|^3},
\]
we completes the proof of Corollary 2.7. \( \square \)

3. Third Order Differential Superordination with \( J_{s,b}^* \)

Definition 3.1. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q(z) \in H \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Phi_{\Gamma}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times \overline{U} \rightarrow \mathbb{C} \) that satisfy the admissibility condition:
\[
\phi(a_1, a_2, a_3, a_4; \zeta) \in \Omega,
\]
whenever
\[
a_1 = q(z), \quad a_2 = \frac{\partial q}{\partial z}(z) + b q(z) \frac{\partial q}{\partial b}, \quad m_b.
\]
Theorem 3.1. Let $\phi \in \Phi^1_\Gamma[\Omega, q]$. If $f(z) \in \Sigma$ and $zJ_{s,b}^* f(z) \in \mathbb{D}_1$ satisfy the following conditions:
\[
\text{Re} \left( \frac{zf''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z(f_{s-1,b}^* f(z) - f_{s,b}^* f(z))}{q'(z)} \right| \leq \frac{m}{|b|^s} \tag{3.1}
\]
\[
\{ \phi(zJ_{s,b}^* f(z), zJ_{s-1,b}^* f(z), zJ_{s-2,b}^* f(z), zJ_{s-3,b}^* f(z) ; z) : z \in \mathbb{U} \}
\]
is univalent, and
\[
\Omega \subset \{ \phi(zJ_{s,b}^* f(z), zJ_{s-1,b}^* f(z), zJ_{s-2,b}^* f(z), zJ_{s-3,b}^* f(z) ; z) : z \in \mathbb{U} \},
\tag{3.2}
\]
then
\[
q(z) < zJ_{s,b}^* f(z).
\]

Proof. Let the functions $p(z)$ and $\psi$ be defined by (2.4) and (2.9). Since $\phi \in \Phi^1_\Gamma[\Omega, q]$, Therefore (2.10) and (3.2) imply
\[
\Omega \subset \psi(p(z), z, z^2 p''(z), z^3 p'''(z); z).
\]
The admissible condition for $\phi \in \Phi^1_\Gamma[\Omega, q]$ is equivalent to the admissible condition for $\psi$ in Definition 1.6 with $n = 2$. Therefore, $\psi \in \Psi^2_\Gamma[\Omega, q]$, and by using (3.1) and Theorem 1.2, we have
\[
q(z) < p(z)
\]
which yields
\[
q(z) < zJ_{s,b}^* f(z).
\]

Therefore we completes the proof of Theorem 3.1. $\square$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto $\Omega$. In this case the class $\Phi^1_\Gamma[h(\mathbb{U}), q]$ is written as $\Phi^1_\Gamma[h, q]$.

The following theorem is a directly consequence of Theorem 2.1.

Theorem 3.2. Let $\phi \in \Phi^1_\Gamma[h, q]$. Also, let $h(z)$ be analytic in $\mathbb{U}$. If $f(z) \in \Sigma$ and $zJ_{s,b}^* f(z) \in \mathbb{D}_1$ satisfies the condition (3.1),
\[
\{ \phi(zJ_{s,b}^* f(z), zJ_{s-1,b}^* f(z), zJ_{s-2,b}^* f(z), zJ_{s-3,b}^* f(z) ; z) : z \in \mathbb{U} \}
\]
is univalent in $\mathbb{U}$, and
\[
h(z) < \phi(zJ_{s,b}^* f(z), zJ_{s-1,b}^* f(z), zJ_{s-2,b}^* f(z), zJ_{s-3,b}^* f(z) ; z),
\tag{3.3}
\]
then
\[
q(z) < zJ_{s,b}^* f(z).
\]
Theorem 3.3. Let \( h(z) \) be analytic in \( U \), also, let \( \phi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C} \) and \( \psi \) be given by (2.9). Suppose that the differential equation (2.18) has a solution \( q(z) \in D_1 \). If \( f(z) \in \Sigma \) satisfies the condition (3.1),

\[
\left\{ \phi(z f^*_p(z), z f^*_{p-1}(z), z f^*_{p-2}(z), z f^*_{p-3}(z); z) : z \in U \right\}
\]

is univalent in \( U \), and

\[
h(z) < \phi(z f^*_p(z), z f^*_{p-1}(z), z f^*_{p-2}(z), z f^*_{p-3}(z); z),
\]

then

\[
q(z) < z f^*_p(z).
\]

and \( q(z) \) is the best subordinant of (3.3).

Proof. The proof is similar to that of Theorem 2.3 and it is being omitted here. □

By combining Theorem 2.2 and Theorem 3.2 we obtain the following sandwich type result.

Corollary 3.1. Let \( h_1(z) \) and \( q_1(z) \) be analytic in \( U \). Also, let \( h_2(z) \) be univalent in \( U \), \( q_2(z) \in D_1 \) with \( q_1(0) = q_2(0) = 1 \) and \( \phi \in \Phi_{(h_1,q)} \cap \Phi_{r}(h_1,q) \). If \( f(z) \in \Sigma \), \( z f^*_p(z) \in D_1 \cap H \),

\[
\left\{ \phi(z f^*_p(z), z f^*_{p-1}(z), z f^*_{p-2}(z), z f^*_{p-3}(z); z) : z \in U \right\}
\]

is univalent in \( U \), and the conditions (2.12) and (3.1) are satisfied, Also, let

\[
h_1(z) < \phi(z f^*_p(z), z f^*_{p-1}(z), z f^*_{p-2}(z), z f^*_{p-3}(z); z) < h_2(z),
\]

then \( q_1(z) < z f^*_p(z) < q_2(z) \).

References