



## Degenerate Abstract Volterra Equations in Locally Convex Spaces

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**Abstract.** In the paper under review, we analyze various types of degenerate abstract Volterra integro-differential equations in sequentially complete locally convex spaces. From the theory of non-degenerate equations, it is well known that the class of  $(a, k)$ -regularized  $C$ -resolvent families provides an efficient tool for dealing with abstract Volterra integro-differential equations of scalar type. Following the approach of T.-J. Xiao and J. Liang [41]-[43], we introduce the class of degenerate exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent families and discuss its basic structural properties. In the final section of paper, we will look at generation of degenerate fractional resolvent operator families associated with abstract differential operators.

### 1. Introduction and Preliminaries

The main subject considered in this paper is the following abstract Cauchy problem:

$$Bu(t) = f(t) + \int_0^t a(t-s)Au(s) ds, \quad t \in [0, \tau), \quad (1)$$

where  $0 < \tau \leq \infty$ ,  $t \mapsto f(t)$ ,  $t \in [0, \tau)$  is a continuous mapping with values in a sequentially complete locally convex space  $E$ ,  $a \in L^1_{loc}([0, \tau))$  and  $A, B$  are closed linear operators with domain and range contained in  $E$ . The reader may consult the monographs [20], [33] and the references cited there for the general theory of non-degenerate abstract Volterra equations in Banach and sequentially complete locally convex spaces, i.e., the theory of various types of resolvent (sometimes also called solution) families for (1), with  $B = I$ . The main purpose of this paper is to provide the basic information about degenerate exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent families and their applications in the study of abstract Cauchy problem (1). In our previous research studies [22]-[23], we have investigated degenerate abstract multi-term fractional differential equations with Caputo fractional derivatives, as well as their hypercyclic and topologically mixing properties.

Compared with non-degenerate case, increasingly less has been said about the well-posedness of degenerate abstract Cauchy problem (1). Concerning degenerate differential equations of first and second order,

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mention should be made (among many other important research papers not cited here) of the monographs by A. Favini, A. Yagi [11], S. G. Krein [24], R. W. Carroll, R. W. Showalter [4], I. V. Melnikova, A. I. Filinkov [31] and G. A. Sviridyuk, V. E. Fedorov [37]. The study of degenerate abstract Volterra integro-differential equations starts presumably with the papers [7]-[10] by A. Favini and H. Tanabe (for some other references on degenerate integro-differential equations, one may refer e.g. to [6]-[9], [16] and [39]), who have analyzed the well-posedness of equation (1) in the setting of Banach spaces, considering separately the so-called hyperbolic case  $\sup_{s>0, k \in \mathbb{N}} \|(B(sB + A)^{-1})^k\| < \infty$  and the parabolic case  $\sup_{\Re \lambda \geq 0} (1 + |\lambda|)^{-1} \|B(\lambda B + A)^{-1}\| < \infty$ . Recall that any of these assumptions implies that the operator  $T = BA^{-1}$  is a bounded linear operator on  $E$ , as well as that the space  $E$  has a direct decomposition representation  $E = N(T) \oplus \overline{R(T)}$  (similar assumptions have been used in [31, Sections 1.1.5-1.1.6], where the authors have considered degenerate integrated semigroups). Generally, in our approach, the resolvent set of  $A$  does not contain 0 and can be even the empty set, which clearly implies that the operator  $T$  need not be defined. Although providing only partial information about the  $C$ -wellposedness of the problem (1), it is worth noting that our method has some advantages compared with other existing because we do not use any assumption on the decomposition of the state space  $E$ .

Throughout this paper, we shall always assume that  $E$  is an infinite-dimensional sequentially complete locally convex space over the field  $\mathbb{C}$ , SCLCS for short. The abbreviation  $\otimes$  stands for the fundamental system of seminorms which induces the topology on  $E$ . By  $L(E)$  and  $E^*$  we denote the space which consists of all continuous linear mappings from  $E$  into  $E$ , and the dual space of  $E$ , respectively. Let  $\mathcal{B}$  be the family of bounded subsets of  $E$  and let  $p_B(T) := \sup_{x \in B} p(Tx)$ ,  $p \in \otimes$ ,  $B \in \mathcal{B}$ ,  $T \in L(E)$ . Then  $p_B(\cdot)$  is a seminorm on  $L(E)$  and the system  $(p_B)_{(p, B) \in \otimes \times \mathcal{B}}$  induces the Hausdorff locally convex topology on  $L(E)$ . In the second and third section of paper,  $A$  and  $B$  denote closed linear operators acting on  $E$ , and  $C \in L(E)$  denotes an injective operator satisfying  $CA \subseteq AC$ ; the convolution like mapping  $*$  is given by  $f * g(t) := \int_0^t f(t-s)g(s) ds$ , and the  $n$ -th convolution power of the function  $a(t)$  is denoted by  $a^{*n}(t)$ . The domain, resolvent set, range and kernel space of  $A$  are denoted by  $D(A)$ ,  $\rho(A)$ ,  $R(A)$  and  $N(A)$ , respectively. Since no confusion seems likely, we will identify  $A$  with its graph. Suppose that  $F$  is a linear subspace of  $E$ . Then the part of  $A$  in  $F$ , denoted by  $A|_F$ , is a linear operator defined by  $D(A|_F) := \{x \in D(A) \cap F : Ax \in F\}$  and  $A|_F x := Ax$ ,  $x \in D(A|_F)$ . Put  $A^* := \{(x^*, y^*) \in E^* \times E^* : x^*(Ax) = y^*(x) \text{ for all } x \in D(A)\}$ . If  $A$  is densely defined, then  $A^*$  is a closed linear operator on  $E^*$ , known also as the adjoint operator of  $A$ . Before proceeding further, it should be noted that our results seem to be new even in the context of Banach spaces. If so, then the norm of an element  $x \in E$  will be denoted by  $\|x\|$ .

Given  $s \in \mathbb{R}$  in advance, set  $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$ . The Gamma function is denoted by  $\Gamma(\cdot)$  and the principal branch is always used to take the powers. Set  $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$  ( $\zeta > 0$ ,  $t > 0$ ) and, by common consent,  $0^\zeta := 0$ . Denote by  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  the Fourier transform on  $\mathbb{R}^n$  and its inverse transform, respectively.

The following condition will be used occasionally:

(P1):  $k(t)$  is Laplace transformable, i.e., it is locally integrable on  $[0, \infty)$  and there exists  $\beta \in \mathbb{R}$  such that

$$\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b \rightarrow \infty} \int_0^b e^{-\lambda t} k(t) dt := \int_0^\infty e^{-\lambda t} k(t) dt \text{ exists for all } \lambda \in \mathbb{C} \text{ with } \Re \lambda > \beta. \text{ Put } \text{abs}(k) := \inf\{\Re \lambda : \tilde{k}(\lambda) \text{ exists}\}, \text{ and denote by } \mathcal{L}^{-1} \text{ the inverse Laplace transform.}$$

In Theorem 2.8 and Remark 2.9, we shall employ the condition

(P2):  $k(t)$  satisfies (P1) and  $\tilde{k}(\lambda) \neq 0$ ,  $\Re \lambda > \beta$  for some  $\beta \geq \text{abs}(k)$ .

Let  $\omega \in \mathbb{R}$ . Following [42, Definition 1.1.3], it will be said that a function  $h : (\omega, \infty) \rightarrow E$  belongs to the class  $LT - E$  if there exists a function  $f \in C([0, \infty) : E)$  such that for each  $p \in \otimes$  there exists  $M_p > 0$  satisfying  $p(f(t)) \leq M_p e^{\omega t}$ ,  $t \geq 0$  and  $h(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$ ,  $\lambda > \omega$ ; then the function  $\lambda \mapsto h(\lambda)$ ,  $\lambda > \omega$  can be analytically extended to the right half plane  $\{\lambda \in \mathbb{C} : \Re \lambda > \omega\}$ . We refer the reader to [1]-[2], [42, Chapter 1] and [20, Section 1.2] for further information concerning vector-valued Laplace transform. In the sequel, we shall use the following uniqueness type theorem for the Laplace transform.

**Lemma 1.1.** Let  $\omega \geq 0$ , and let  $f_1, f_2 \in C([0, \infty) : E)$  satisfy that for each  $p \in \otimes$  there exists  $M'_p > 0$  such that  $p(f_1(t)) + p(f_2(t)) \leq M'_p e^{\omega t}$ ,  $t \geq 0$ . Suppose that a continuous function  $g : [0, \infty) \rightarrow \mathbb{C}$  satisfies (P1) and that  $A$  is a closed linear operator on  $E$  satisfying that for  $\lambda > a$ ,

$$\int_0^\infty e^{-\lambda t} f_1(t) dt \in D(A),$$

and

$$A \int_0^\infty e^{-\lambda t} f_1(t) dt = \int_0^\infty e^{-\lambda t} f_2(t) dt + \int_0^\infty e^{-\lambda t} g(t)x dt, \quad \lambda > a,$$

for some  $x \in E$ . Then, for every  $t \geq 0$ , one has  $f_1(t) \in D(A)$  and  $Af_1(t) = f_2(t) + g(t)x$ .

Let  $0 < \tau \leq \infty$  and  $a \in L^1_{loc}([0, \tau])$ . Then we say that the function  $a(t)$  is a kernel on  $[0, \tau)$  iff for each  $f \in C([0, \tau])$  the assumption  $\int_0^t a(t-s)f(s) ds = 0$ ,  $t \in [0, \tau)$  implies  $f(t) = 0$ ,  $t \in [0, \tau)$ . If  $\tau = \infty$  and  $a \neq 0$  in  $L^1_{loc}([0, \infty))$ , then the famous Titchmarsh theorem [38, Theorem VII] implies that the function  $a(t)$  is automatically a kernel on  $[0, \infty)$ ; the situation is quite different in the case that  $\tau < \infty$ , then we can apply the Titchmarsh–Foiş theorem [2, Theorem 2.1] (cf. also [17, Theorem 3.4.40]) in order to see that the function  $a(t)$  is a kernel on  $[0, \tau)$  iff  $0 \in \text{supp}(a)$ . In this place, it is worth noting that, for any function  $a(t)$  satisfying (P1), the condition  $0 \in \text{supp}(a)$  is necessary and sufficient for the equality  $\limsup_{\lambda \rightarrow \infty} \lambda^{-1} \ln |\tilde{a}(\lambda)| = 0$  to be true, or equivalently, for the convolution mapping  $\mathcal{K} : f \mapsto a * f$  to be an injective operator on  $C([0, \infty))$  with dense range in the Fréchet space  $C_*([0, \infty))$  of all continuous functions  $g : [0, \infty) \rightarrow \mathbb{C}$  such that  $g(0) = 0$ , equipped with the seminorms  $\|g\|_n := \sup_{t \in [0, n]} |g(t)|$  ( $n \in \mathbb{N}$ ); see e.g. [1, p. 106]. In the sequel, we shall assume that  $a \neq 0$  in  $L^1_{loc}([0, \infty))$  and  $k \neq 0$  in  $C([0, \infty))$ ; consequently, the functions  $a(t)$  and  $k(t)$  will be kernels on  $[0, \infty)$ . The notions of completely positive, creep and log-convex functions will be understood in the sense of [33].

Assume  $\alpha > 0$ ,  $m = [\alpha]$  and  $\beta \in \mathbb{R}$ . Recall that the Caputo fractional derivative  $\mathbf{D}_t^\alpha u$  ([3], [20]) is defined for those functions  $u \in C^{m-1}([0, \infty) : E)$  for which  $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u^{(k)}(0)g_{k+1}) \in C^m([0, \infty) : E)$ ; if this is the case, we have

$$\mathbf{D}_t^\alpha u(\cdot) = \frac{d^m}{dt^m} \left[ g_{m-\alpha} * \left( u - \sum_{k=0}^{m-1} u^{(k)}(0)g_{k+1} \right) \right].$$

The Mittag-Leffler function  $E_{\alpha, \beta}(z)$  is defined by  $E_{\alpha, \beta}(z) := \sum_{n=0}^\infty z^n / \Gamma(\alpha n + \beta)$ ,  $z \in \mathbb{C}$ . In this place, we assume that  $1/\Gamma(\alpha n + \beta) = 0$  if  $\alpha n + \beta \in -\mathbb{N}_0$ . Set, for short,  $E_\alpha(z) := E_{\alpha, 1}(z)$ ,  $z \in \mathbb{C}$ . Then, for every  $j \in \mathbb{N}$  and  $\alpha > 0$ , there exist uniquely determined real numbers  $c_{l, j, \alpha}$  ( $1 \leq l \leq j$ ) such that:

$$E_\alpha^{(j)}(z) = \sum_{l=1}^j c_{l, j, \alpha} E_{\alpha, \alpha j - (j-l)}(z), \quad z \in \mathbb{C}. \tag{2}$$

The asymptotic behaviour of the entire function  $E_{\alpha, \beta}(z)$  is given in the following auxiliary lemma.

**Lemma 1.2 ([40]).** Let  $0 < \sigma < \frac{1}{2}\pi$ . Then, for every  $z \in \mathbb{C} \setminus \{0\}$  and  $m \in \mathbb{N} \setminus \{1\}$ ,

$$E_{\alpha, \beta}(z) = \frac{1}{\alpha} \sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O(|z|^{-m}), \quad |z| \rightarrow \infty,$$

where  $Z_s$  is defined by  $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$  and the first summation is taken over all those integers  $s$  satisfying  $|\arg z + 2\pi s| < \alpha(\frac{\pi}{2} + \sigma)$ .

## 2. The Main Structural Properties of Degenerate $(a, k)$ -Regularized $C$ -Resolvent Families

We start this section by introducing the following definition (cf. [20, Subsection 2.1.1] and [33] for the case  $B = I$ ):

**Definition 2.1.** Let  $0 < \tau \leq \infty$ . A function  $u \in C([0, \tau) : E)$  is said to be:

- (i) a (mild) solution of (1) iff  $(a * u)(t) \in D(A)$ ,  $t \in [0, \tau)$ ,  $A(a * u)(t) = Bu(t) - f(t)$ ,  $t \in [0, \tau)$  and the mapping  $t \mapsto Bu(t)$ ,  $t \in [0, \tau)$  is continuous,
- (ii) a strong solution of (1) iff the mapping  $t \mapsto Au(t)$ ,  $t \in [0, \tau)$  is continuous,  $(a * Au)(t) = Bu(t) - f(t)$ ,  $t \in [0, \tau)$  and the mapping  $t \mapsto Bu(t)$ ,  $t \in [0, \tau)$  is continuous,
- (iii) a weak solution of (1) iff for every  $(x^*, y^*) \in A^*$  and for every  $t \in [0, \tau)$ , one has  $\langle x^*, Bu(t) \rangle = \langle x^*, f(t) \rangle + \langle y^*, (a * u)(t) \rangle$ ,  $t \in [0, \tau)$ .

It is clear that any strong solution of (1) is also a mild solution of the same problem, the converse statement is not true in general. Since [21, Lemma 2.4] continues to hold in SCLCSs, the concepts mild and weak solution of (1) coincide actually.

We introduce the notion of an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1) as follows (cf. [41, Definition 1.4] for the case  $a(t) = k(t) = 1$ , and [43, Definition 2.3] for the case  $a(t) = g_n(t)$ ,  $k(t) = 1$ ):

**Definition 2.2.** Suppose that the functions  $a(t)$  and  $k(t)$  satisfy (P1), as well as that  $R(t) : D(B) \rightarrow E$  is a linear mapping ( $t \geq 0$ ). Let  $C \in L(E)$  be injective, and let  $CA \subseteq AC$ . Then the operator family  $(R(t))_{t \geq 0}$  is said to be an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1) iff there exists  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$  such that the following holds:

- (i) The mapping  $t \mapsto R(t)x$ ,  $t \geq 0$  is continuous for every fixed element  $x \in D(B)$ .
- (ii) The family  $\{e^{-\omega t}R(t) : t \geq 0\}$  is equicontinuous, i.e., for every  $p \in \mathfrak{O}$ , there exist  $c > 0$  and  $q \in \mathfrak{O}$  such that

$$p(e^{-\omega t}R(t)x) \leq cq(x), \quad x \in D(B), t \geq 0. \tag{3}$$

- (iii) For every  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega$  and  $\tilde{k}(\lambda) \neq 0$ , the operator  $B - \tilde{a}(\lambda)A$  is injective,  $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$  and

$$\tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx = \int_0^\infty e^{-\lambda t}R(t)x dt, \quad x \in D(B). \tag{4}$$

If  $k(t) = g_{r+1}(t)$  for some  $r \geq 0$ , then it is also said that  $(R(t))_{t \geq 0}$  is an exponentially equicontinuous  $r$ -times integrated  $(a, C)$ -regularized resolvent family for (1); an exponentially equicontinuous 0-times integrated  $(a, C)$ -regularized resolvent family for (1) is also said to be an exponentially equicontinuous  $(a, C)$ -regularized resolvent family for (1).

**Remark 2.3.** (i) If  $B = I$ , then the above simply means by [19, Theorem 2.7] that  $A$  is a subgenerator of the exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  in the sense of [19, Definition 2.1]. The case  $B \neq I$  is more difficult to deal with; for example, the validity of some very simple equalities, like  $R(t)Ax = AR(t)x$ ,  $t \geq 0$  or  $R(t)Bx = BR(t)x$ ,  $t \geq 0$ , cannot be proved without making some new assumptions. Furthermore, it is not clear how one can define, by using a method similar to that employed in Definition 2.2, the notion of an exponentially equicontinuous  $(a, k)$ -regularized  $(C_1, C_2)$ -existence and uniqueness family for (1) in a satisfactory way.

- (ii) In contrast to [41]-[43], we do not assume in Definition 2.2 that  $CB \subseteq BC$  or  $R(C) \subseteq R(B - \tilde{a}(\lambda)A)$  ( $\Re \lambda > \omega$ ,  $\tilde{k}(\lambda) \neq 0$ ).

- (iii) The uniqueness theorem for Laplace transform implies that there exists at most one exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1).
- (iv) If  $E$  is complete and  $B$  is densely defined, then [30, Lemma 22.19] combined with (ii) of Definition 2.2 implies that, for every  $t \geq 0$  and for every  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega$  and  $\tilde{k}(\lambda) \neq 0$ , there exist two operators  $\hat{R}(t), G(\lambda) \in L(E)$  such that  $\hat{R}(t)x = R(t)x, x \in D(B)$  and  $G(\lambda)x = (B - \tilde{a}(\lambda)A)^{-1}CBx, x \in D(B)$ . The operator family  $(\hat{R}(t))_{t \geq 0} \subseteq L(E)$  is strongly continuous and, for every  $p \in \mathfrak{O}$ , there exist  $c > 0$  and  $q \in \mathfrak{O}$  such that (3) holds for all  $x \in E$  and  $t \geq 0$ , with  $(R(t))_{t \geq 0}$  replaced by  $(\hat{R}(t))_{t \geq 0}$ . Furthermore,  $\tilde{k}(\lambda)G(\lambda)x = \int_0^\infty e^{-\lambda t} R(t)x dt, x \in E$ .
- (v) The notion of a (local)  $(a, k)$ -regularized  $C$ -resolvent family  $(R(t))_{t \in [0, \tau]}$  for (1) can be defined in many different ways, but it seems that there is no satisfactory option that would provide us a general approach to the Volterra problems of the kind (1). Observe also that we can simply construct a great number of examples of strongly continuous operator families  $(R(t))_{t \in [0, \tau]} \subseteq L(E)$  for which neither  $R(t)A \subseteq R(t)A, t \in [0, \tau]$  nor  $R(t)B \subseteq R(t)B, t \in [0, \tau]$ , or only  $R(t)B \not\subseteq R(t)B, t \in [0, \tau]$ , but the solution of (1) exists and has the form  $u(t) = R(t)x, t \in [0, \tau]$  for some  $x \in E$ ; see e.g. [20, Example 2.8.11] and [21, Example 2.31]. Because of that, hereafter we tend to pay attention primarily to the notion introduced in Definition 2.2.

The proof of following proposition is standard and therefore omitted (cf. [19, Proposition 2.4] for more details).

**Proposition 2.4.** (i) Let  $(R(t))_{t \geq 0}$  be an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1), and let  $b \neq 0$  in  $L^1_{loc}([0, \infty))$ . If there exist  $M \geq 1$  and  $\omega \geq 0$  such that  $\int_0^t |b(s)| ds \leq Me^{\omega t}, t \geq 0$ , then  $((b * R)(t))_{t \geq 0}$  is an exponentially equicontinuous  $(a, k * b)$ -regularized  $C$ -resolvent family for (1).

(ii) Let  $(R_i(t))_{t \geq 0}$  be an exponentially equicontinuous  $(a, k_i)$ -regularized  $C$ -resolvent family for (1),  $i = 1, 2$ . Then  $(k_2 * R_1)(t)x = (k_1 * R_2)(t)x, t \geq 0, x \in D(B)$ .

(iii) Let  $(R(t))_{t \geq 0}$  be an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1). Suppose that  $k(0) \neq 0, b(t)$  satisfies (P1),  $(b * k)(t) + k(t)k(0)^{-1} = 1, t \geq 0$  and the function  $t \mapsto \int_0^t |b(s)| ds, t \geq 0$  is exponentially bounded. Then  $(S(t) \cdot \equiv k(0)^{-1}R(t) \cdot + (b * R(\cdot))(t))_{t \geq 0}$  is an exponentially equicontinuous  $(a, 1)$ -regularized  $C$ -resolvent family for (1).

It should be noted that our analysis covers many important subjects that have not been considered in [41]-[43]. For example, we are in a position to clarify the Hille-Yosida's type theorems for degenerate exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent families.

**Theorem 2.5.** (cf. [20, Theorem 2.1.6] for the case  $B = I$ )

(i) Let  $\omega_0 > \max(0, \text{abs}(a), \text{abs}(k))$ , and let  $a(t)$  and  $k(t)$  satisfy (P1). Assume that, for every  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega_0$  and  $\tilde{k}(\lambda) \neq 0$ , the operator  $B - \tilde{a}(\lambda)A$  is injective and  $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$ . If for each  $x \in D(B)$  there exists a function  $\Upsilon_x : \{\lambda \in \mathbb{C} : \Re \lambda > \omega_0\} \rightarrow E$  which satisfies:

- (a)  $\Upsilon_x(\lambda) = \tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx, \Re \lambda > \omega_0, \tilde{k}(\lambda) \neq 0$ ,
- (b) the mapping  $\lambda \mapsto \Upsilon_x(\lambda), \Re \lambda > \omega_0$  is analytic, and
- (c) there exists  $r \geq -1$  such that for each  $p \in \mathfrak{O}$  there exist  $M_p > 0$  and  $q_p \in \mathfrak{O}$  satisfying

$$p(\Upsilon_x(\lambda)) \leq M_p q_p(x) |\lambda|^r, \quad \Re \lambda > \omega_0, x \in D(B),$$

then, for every  $\alpha > 1$ , there exists an exponentially equicontinuous  $(a, k * g_{\alpha+r})$ -regularized  $C$ -resolvent family  $(R_\alpha(t))_{t \geq 0}$  for (1), and there exists a constant  $c_\alpha > 0$  such that

$$p(R_\alpha(t)x) \leq c_\alpha M_p q_p(x) e^{\omega_0 t}, \quad p \in \mathfrak{O}, x \in D(B), t \geq 0.$$

(ii) Let  $\omega \in \mathbb{R}$ ,  $\omega_0 > \max(0, \omega, \text{abs}(a), \text{abs}(k))$ , and let  $a(t)$  and  $k(t)$  satisfy (P1). Assume that for each  $x \in D(B)$  there exists an infinitely differentiable function  $\Upsilon_x : \{\lambda \in \mathbb{C} : \Re \lambda > \omega_0\} \rightarrow E$  which satisfies the item (i)(a) for real values of parameter  $\lambda$ , as well as that for each  $p \in \mathbb{N}$  there exist  $c_p > 0$  and  $q_p \in \mathbb{N}$  such that

$$p \left( n!^{-1} (\lambda - \omega)^{n+1} \frac{d^n}{d\lambda^n} \Upsilon_x(\lambda) \right) \leq c_p q_p(x), \tag{5}$$

provided  $\tilde{k}(\lambda) \neq 0$ ,  $\lambda > \omega_0$ ,  $x \in D(B)$ ,  $n \in \mathbb{N}_0$ . Then, for every  $r \in (0, 1]$ , there exists an exponentially equicontinuous  $(a, k * g_r)$ -regularized C-resolvent family  $(R_r(t))_{t \geq 0}$  for (1), and

$$p(R_r(t+h)x - R_r(t)x) \leq \frac{2c_p q_p(x)}{r\Gamma(r)} \max(e^{\omega(t+h)}, 1) h^r,$$

provided  $p \in \mathbb{N}$ ,  $t \geq 0$ ,  $h > 0$ ,  $x \in D(B)$ . Furthermore, if  $B$  is densely defined and the mapping  $t \mapsto R_1(t)x$ ,  $t \geq 0$  is continuously differentiable for all  $x \in D(B)$ , then there exists an exponentially equicontinuous  $(a, k)$ -regularized C-resolvent family for (1).

In the subsequent theorem ([3], [20], [27], [33]-[34]), we analyze subordination principles for degenerate  $(a, k)$ -regularized C-resolvent families.

**Theorem 2.6.** (i) Let  $a(t)$ ,  $b(t)$  and  $c(t)$  satisfy (P1) and let  $\int_0^\infty e^{-\beta t} |b(t)| dt < \infty$  for some  $\beta \geq 0$ . Let

$$\alpha = \tilde{c}^{-1}\left(\frac{1}{\beta}\right) \text{ if } \int_0^\infty c(t) dt > \frac{1}{\beta}, \alpha = 0 \text{ otherwise,}$$

and let  $\tilde{a}(\lambda) = \tilde{b}\left(\frac{1}{\tilde{c}(\lambda)}\right)$ ,  $\lambda \geq \alpha$ . Assume that there exists an exponentially equicontinuous  $(b, k)$ -regularized C-resolvent family  $(R_b(t))_{t \geq 0}$  for (1), with  $a(t)$  replaced by  $b(t)$ , satisfying that the family  $\{e^{-\omega_b t} R_b(t) : t \geq 0\}$  is equicontinuous for some  $\omega_b \geq 0$  (with the clear meaning). Assume, further, that  $c(t)$  is completely positive and that there exists a function  $k_1(t)$  satisfying (P1) and

$$\tilde{k}_1(\lambda) = \frac{1}{\lambda \tilde{c}(\lambda)} \tilde{k}\left(\frac{1}{\tilde{c}(\lambda)}\right), \lambda > \omega_0, \tilde{k}\left(\frac{1}{\tilde{c}(\lambda)}\right) \neq 0, \text{ for some } \omega_0 > 0.$$

Let

$$\omega_a = \tilde{c}^{-1}\left(\frac{1}{\omega_b}\right) \text{ if } \int_0^\infty c(t) dt > \frac{1}{\omega_b}, \omega_a = 0 \text{ otherwise.}$$

Then, for every  $r \in (0, 1]$ , there exists an exponentially equicontinuous  $(a, k_1 * g_r)$ -regularized C-resolvent family  $(R_r(t))_{t \geq 0}$  for (1), satisfying that the family  $\{e^{-\omega_a t} R_r(t) : t \geq 0\}$  is equicontinuous, if  $\omega_b = 0$  or  $\omega_b \tilde{c}(0) \neq 1$ , resp., for every  $\varepsilon > 0$ , there exists  $M_\varepsilon \geq 1$  such that the family  $\{e^{-\varepsilon t} R_r(t) : t \geq 0\}$  is equicontinuous, if  $\omega_b > 0$  and  $\omega_b \tilde{c}(0) = 1$ .

(ii) Suppose  $\alpha \geq 0$  and there exists an exponentially equicontinuous  $(1, g_\alpha)$ -regularized C-resolvent family for (1). Assume, further, that  $a(t)$  and  $k(t)$  satisfies (P1), as well as that  $\tilde{k}(\lambda) = \tilde{a}(\lambda)^\alpha$  for  $\lambda$  sufficiently large, and  $a(t)$  is completely positive. Then, for every  $r \in (0, 1]$ , there exists an exponentially equicontinuous  $(a, k * g_r)$ -regularized C-resolvent family for (1)  $((a, a^{*n} * g_r)$ -regularized C-resolvent family for (1) if  $\alpha = n \in \mathbb{N}$ , resp.  $(a, g_{r+1})$ -regularized C-resolvent family if  $\alpha = 0$ ).

(iii) Suppose  $\alpha \geq 0$  and there exists an exponentially equicontinuous  $(g_2, g_\alpha)$ -regularized C-resolvent function for (1). Let  $L_{loc}^1([0, \infty)) \ni c$  be completely positive and let  $a(t) = (c * c)(t)$ ,  $t \geq 0$ . ( Recall that for any function  $a \in L_{loc}^1([0, \infty))$  given in advance, such a function  $c(t)$  always exists provided  $a(t)$  is completely positive or  $a(t) \neq 0$  is a creep function and  $a_1(t)$  is log-convex. ) Assume  $k(t)$  satisfies (P1) and  $\tilde{k}(\lambda) = \tilde{c}(\lambda)^\alpha / \lambda$ ,  $\lambda$  sufficiently large. Then, for every  $r \in (0, 1]$ , there exists an exponentially equicontinuous  $(a, k * g_r)$ -regularized C-resolvent family for (1)  $((a, c^{*n} * g_r)$ -regularized C-resolvent family for (1) if  $\alpha = n \in \mathbb{N}$ , resp.  $(a, g_{r+1})$ -regularized C-resolvent family for (1) if  $\alpha = 0$ ).

**Remark 2.7.** In the case that  $B = I$  and  $A$  is densely defined, the mapping  $t \mapsto R_1(t)x$ ,  $t \geq 0$ , which appears in the formulation of Theorem 2.5(ii), is continuously differentiable for all  $x \in E$  and, in the situation of Theorem 2.6(i), there exists an exponentially equicontinuous  $(a, k_1)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1) satisfying that the family  $\{e^{-\omega_n t} R_r(t) : t \geq 0\}$  is equicontinuous, resp., for every  $\varepsilon > 0$ , the family  $\{e^{-\varepsilon t} R_r(t) : t \geq 0\}$  is equicontinuous (similar statements hold in the case of Theorem 2.6(ii)-(iii), cf. [20, Theorem 2.1.8] for further information). It is not clear whether the above results can be reformulated for degenerate abstract Volterra equations.

The following theorem provides an extension of [41, Theorem 1.6] and [43, Theorem 3.1].

**Theorem 2.8.** Let  $\tau = \infty$ , let the functions  $a(t)$  and  $k(t)$  satisfy (P1), and let  $(R(t))_{t \geq 0}$  be an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1), satisfying (ii) of Definition 2.2 with some number  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$ .

(i) Suppose that  $v_0 \in D(B)$  and the following condition holds:

(i.1) for every  $x \in D(B)$ , there exist a number  $\omega_0 > \omega$  and a function  $h(\lambda; x) \in LT - E$  such that  $h(\lambda; x) = \tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}CBx$  provided  $\Re \lambda > \omega_0$  and  $\tilde{k}(\lambda) \neq 0$ .

Then the function  $u(t) = R(t)v_0$ ,  $t \geq 0$  is a mild solution of (1) with  $f(t) = k(t)CBv_0$ ,  $t \geq 0$ . The uniqueness of mild solutions holds if we suppose additionally that  $CB \subseteq BC$ , as well as:

(i.2) the function  $k(t)$  satisfies (P2), and

(i.3) there exists  $\sigma > 0$  such that, for every  $t \geq 0$ ,

$$\lim_{\lambda \rightarrow +\infty} \int_0^t \int_s^\infty e^{\lambda(s-r-\sigma)} |a(r)| dr ds = 0. \tag{6}$$

(ii) Suppose that  $v_0 \in D(A) \cap D(B)$ ,  $CB \subseteq BC$ , and the following condition holds:

(ii.1) for every  $x \in E$ , there exist a number  $\omega_1 > \omega$  and a function  $h(\lambda; x) \in LT - E$  such that  $h(\lambda; x) = \tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}Cx$  provided  $\Re \lambda > \omega_1$  and  $\tilde{k}(\lambda) \neq 0$ .

Then the function  $u(t) = R(t)v_0$ ,  $t \geq 0$  is a strong solution of (1) with  $f(t) = k(t)CBv_0$ ,  $t \geq 0$ . The uniqueness of strong solutions holds if we suppose additionally the validity of (i.2)-(i.3).

*Proof.* Let  $v_0 \in D(B)$ . Then, for every  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega$  and  $\tilde{k}(\lambda) \neq 0$ , we have

$$\tilde{a}(\lambda)\tilde{k}(\lambda)A(B - \tilde{a}(\lambda)A)^{-1}CBv_0 = \tilde{k}(\lambda)[-CBv_0 + B(B - \tilde{a}(\lambda)A)^{-1}CBv_0]. \tag{7}$$

Taking into account (i.1), the equation (4), as well as the closedness of  $B$  and Lemma 1.1, it can be simply proved that  $R(t)v_0 \in D(B)$  for all  $t \geq 0$ , as well as that the mapping  $t \mapsto BR(t)v_0$ ,  $t \geq 0$  is continuous and

$$\tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}CBv_0 = \int_0^\infty e^{-\lambda t} BR(t)v_0 dt, \quad \Re \lambda > \omega_0, \tilde{k}(\lambda) \neq 0.$$

The previous equality in combination with (7) and Lemma 1.1 implies that

$$A(a * R)(t)v_0 = BR(t)v_0 - k(t)CBv_0, \quad t \geq 0, \tag{8}$$

so that the function  $t \mapsto u(t) \equiv R(t)v_0$ ,  $t \geq 0$  is a mild solution of (1) with  $f(t) = k(t)CBv_0$ ,  $t \geq 0$ . In order to prove (ii), fix an element  $v_0 \in D(A) \cap D(B)$ . Since we have assumed that  $CB \subseteq BC$ , it readily follows that

$$\tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}CBv_0 - \tilde{k}(\lambda)BCv_0 = \tilde{k}(\lambda)\tilde{a}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}CAv_0, \tag{9}$$

for any  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega$  and  $\tilde{k}(\lambda) \neq 0$ . By (ii.1), we know that there exists a continuous function  $t \mapsto G(t)$ ,  $t \geq 0$  such that

$$\int_0^\infty e^{-\lambda t} G(t) dt = \tilde{k}(\lambda) B(B - \tilde{a}(\lambda)A)^{-1} CA v_0, \tag{10}$$

for any  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega_1$  and  $\tilde{k}(\lambda) \neq 0$ . Then it is not difficult to prove with the help of (8)-(10) that:

$$\int_0^\infty e^{-\lambda t} A \int_0^t a(t-s)R(s)v_0 ds dt = \int_0^\infty e^{-\lambda t} [BR(t)v_0 - k(t)CBv_0] dt = \int_0^\infty e^{-\lambda t} \int_0^t a(t-s)G(s) ds dt,$$

for any  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega_1$ . By the uniqueness theorem for the Laplace transform, we get that  $A \int_0^t a(t-s)R(s)v_0 ds = \int_0^t a(t-s)G(s) ds$ ,  $t \geq 0$ . Let  $(x^*, y^*) \in A^*$ . Then  $\langle x^*, (a * G)(t) \rangle = \langle y^*, (a * R(\cdot)v_0)(t) \rangle$ ,  $t \geq 0$ , i.e.,  $\int_0^t a(t-s)\langle x^*, G(s) \rangle ds = \int_0^t a(t-s)\langle y^*, R(s)v_0 \rangle ds$ ,  $t \geq 0$ . Since the function  $a(t)$  is a kernel on  $[0, \infty)$ , we obtain that  $\langle x^*, G(t) \rangle = \langle y^*, R(t)v_0 \rangle$ ,  $t \geq 0$ . Because of that, the equality  $AR(t)v_0 = G(t)$  holds for any  $t \geq 0$ , and the function  $u(t) = R(t)v_0$ ,  $t \geq 0$  is a strong solution of (1) with  $f(t) = k(t)CBv_0$ ,  $t \geq 0$ . It remains to be proved the uniqueness of mild solutions under the additional assumptions  $CB \subseteq BC$  and (i.2)-(i.3). Towards this end, suppose that the function  $t \mapsto u(t)$ ,  $t \geq 0$  is a mild solution of (1) with  $f(t) \equiv 0$ . Put  $v(t) := Cu(t)$ ,  $t \geq 0$ . Since  $CB \subseteq BC$ , the function  $v(t)$  is also a mild solution of (1) with  $f(t) \equiv 0$ . Then we obtain

$$\begin{aligned} (B - \tilde{a}(\lambda)A)(e^{\lambda \cdot} * a * v)(t) &= B(e^{\lambda \cdot} * a * v)(t) - \tilde{a}(\lambda)(e^{\lambda \cdot} * Bv)(t) \\ &= -\left( Bv(\cdot) * \int_0^\infty e^{\lambda(\cdot-s)} a(s) ds \right)(t), \quad \Re \lambda > \omega, t \geq 0. \end{aligned}$$

Combined with (4), the above implies:

$$(e^{\lambda \cdot} * a * v)(t) = -\left( \int_0^\infty e^{-\lambda r} R(r)u(\cdot) dr * \int_0^\infty e^{\lambda(\cdot-s)} a(s) ds \right)(t), \tag{11}$$

for  $\Re \lambda > \omega$  and  $t \geq 0$ . By (i.3), (11), and (ii) of Definition 2.2, we get that for each  $p \in \mathfrak{D}$  there exist  $c > 0$  and  $q \in \mathfrak{D}$  such that

$$e^{-\lambda \sigma} p((e^{\lambda \cdot} * a * v)(t)) \leq \frac{ce^{-\lambda \sigma}}{|\tilde{k}(\lambda)|(\lambda - \omega)} \int_0^t \int_s^\infty e^{\lambda(s-r)} |a(r)| q(u(t-s)) dr ds, \tag{12}$$

for  $\lambda > \omega$ ,  $t \geq 0$  and  $\sigma \geq 0$ . On the other hand, we can always find constants  $\sigma_0 > 0$  and  $M \geq 1$  such that

$$\frac{e^{-\lambda \sigma_0}}{|\tilde{k}(\lambda)|} \leq M, \quad \lambda > \omega + 1. \tag{13}$$

If not so, then there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  in  $(\omega + 1, \infty)$  such that  $|\tilde{k}(\lambda_n)| \leq e^{-n\lambda_n}$ ,  $n \in \mathbb{N}$ . Making use of the condition (P2) and the Bolzano-Weierstrass theorem, it can be easily seen that the sequence  $(\lambda_n)_{n \in \mathbb{N}}$  must be unbounded; hence,  $\limsup_{\lambda \rightarrow +\infty} (\ln |\tilde{k}(\lambda)| / \lambda) = -\infty$  and [1, Proposition 2.4.3] implies that  $k(t) = 0$ ,  $t \geq 0$ , which is a contradiction. Applying now (6) and (12)-(13), we obtain that  $\lim_{\lambda \rightarrow +\infty} e^{-\lambda \sigma} p((e^{\lambda \cdot} * a * v)(t)) = 0$ ,  $t \geq 0$ ,  $p \in \mathfrak{D}$ . By the argumentation given in the final part of the proof of [41, Theorem 1.6], it readily follows that  $(a * v)(t) = 0$ ,  $t \geq 0$ . Since the function  $a(t)$  is a kernel on  $[0, \infty)$  and  $C$  is injective, we get that  $0 = v(t) = Cu(t) = u(t)$ ,  $t \geq 0$ .  $\square$

**Remark 2.9.** (i) Suppose  $v_0 \in D(B)$  and, instead of (i.1), a slightly stronger condition

(i.1)' for every  $x \in D(B)$ , there exist a number  $\omega_0 > \omega$  and two functions  $h_1(\lambda; x), h_2(\lambda) \in LT - E$  such that  $h_1(\lambda; x) = \tilde{k}(\lambda)\tilde{a}(\lambda)^{-1}B(B - \tilde{a}(\lambda)A)^{-1}CBx$  and  $h_2(\lambda) = \tilde{k}(\lambda)\tilde{a}(\lambda)^{-1}$ , provided  $x \in D(B)$ ,  $\Re \lambda > \omega_0$  and  $\tilde{k}(\lambda) \neq 0$ .

Then we can simply prove with the help of closedness of  $A$  and the resolvent equation that there exists a function  $h_3(\lambda; x) \in LT - E$  such that  $h_3(\lambda; x) = \tilde{k}(\lambda)A(B - \tilde{a}(\lambda)A)^{-1}CBx$ , provided  $x \in D(B)$ ,  $\Re \lambda > \omega_0$  and  $\tilde{k}(\lambda) \neq 0$ . Keeping in mind Lemma 1.1, it readily follows that the mapping  $t \mapsto AR(t)v_0, t \geq 0$  is well defined and continuous. In conclusion, the function  $u(t) = R(t)v_0, t \geq 0$  is a strong solution of (1) with  $f(t) = k(t)CBv_0, t \geq 0$ .

(ii) If the function  $a(t)$  satisfies (P1) and there exist constants  $M \geq 1$  and  $\omega_2 \geq 0$  such that  $\int_0^t |a(r)| dr \leq Me^{\omega_2 t}, t \geq 0$ , then (6) holds for any number  $\sigma > 0$ . Speaking-matter-of-factly, for every  $\sigma > 0$ , we have that

$$\begin{aligned} & \int_0^t \int_s^\infty e^{\lambda(s-r-\sigma)} |a(r)| dr ds = e^{-\lambda\sigma} \int_0^t \int_0^\infty e^{-\lambda\eta} |a(s+\eta)| d\eta ds \\ & = e^{-\lambda\sigma} \int_0^\infty \int_0^t e^{-\lambda\eta} |a(s+\eta)| ds d\eta = e^{-\lambda\sigma} \int_0^\infty e^{-\lambda\eta} \int_\eta^{t+\eta} |a(r)| dr d\eta \\ & \leq Me^{\omega_2 t - \lambda\sigma} \int_0^\infty e^{-(\lambda - \omega_2)\eta} d\eta = Me^{\omega_2 t - \lambda\sigma} / (\lambda - \omega_2), \quad t \geq 0, \lambda \text{ suff. large.} \end{aligned}$$

(iii) It is not clear how one can prove the uniqueness of mild solutions of (1) in the case that the equation (6) does not hold or that the function  $k(t)$  does not satisfy (P2). Concerning the last condition, the following comment should be made: suppose that the function  $a(t)$  satisfies the assumptions stated in the part (ii) of this remark, and that the assertion of [32, Lemma 4.1.1, p. 100] continues to hold with the sequence  $\lambda_n = n$  replaced by any strictly increasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of positive real numbers tending to infinity (yet unproven hypothesis, unknown in the existing literature to the best knowledge of the author). Then the uniqueness of solutions clarified in Theorem 2.8 holds even in the case that the function  $k(t)$  does not satisfy (P2), and this can be proved by using the estimate (12) and the fact that there exist a positive real number  $\sigma' > 0$  and a strictly increasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of positive real numbers such that  $\lim_{n \rightarrow \infty} \lambda_n = \infty$  and  $|\tilde{k}(\lambda_n)| \geq e^{-\sigma' \lambda_n}, n \in \mathbb{N}$  (cf. [1, Proposition 2.4.3]).

Finally, we state the following rescaling result for degenerate  $K$ -convoluted  $C$ -semigroups in locally convex spaces; observe, however, that it is very difficult to say something more about perturbation properties of abstract degenerate Volterra equations (cf. [19, Theorem 4.2(ii)] and [20, Section 2.6] for further information concerning non-degenerate case).

**Proposition 2.10.** (i) Suppose  $z \in \mathbb{C}, K(t)$  satisfy (P1),  $F(t)$  is exponentially bounded,  $k(t) = \int_0^t K(s) ds, t \geq 0$ , there exists  $\omega_0 > 0$  such that

$$\frac{\tilde{K}(\lambda) - \tilde{K}(\lambda + z)}{\tilde{K}(\lambda + z)} = \int_0^\infty e^{-\lambda t} F(t) dt, \quad \Re \lambda > \omega_0, \tilde{K}(\lambda + z) \neq 0, \tag{14}$$

and there exists an exponentially equicontinuous  $K$ -convoluted  $C$ -semigroup  $(S_K(t))_{t \in [0, \tau]}$  for (1), i.e.,  $(a, k)$ -regularized  $C$ -resolvent family for (1) with  $a(t) = 1$ . Then there exists an exponentially equicontinuous  $K$ -convoluted  $C$ -semigroup  $(S_{K,z}(t))_{t \in [0, \tau]}$  for (1), with  $A$  replaced by  $A - zB$ . Furthermore,

$$S_{K,z}(t)x = e^{-tz}S_K(t)x + \int_0^t F(t-s)e^{-zs}S_K(s)x ds, \quad t \geq 0, x \in D(B).$$

(ii) Suppose  $z \in \mathbb{C}, \alpha > 0$  and there exists an exponentially equicontinuous  $\alpha$ -times integrated  $C$ -semigroup  $(S_\alpha(t))_{t \in [0, \tau]}$  for (1), i.e.  $(a, k)$ -regularized  $C$ -resolvent family for (1) with  $a(t) = 1$  and  $k(t) = g_{\alpha+1}(t)$ . Then there

exists an exponentially equicontinuous  $\alpha$ -times integrated  $C$ -semigroup  $(S_{\alpha,z}(t))_{t \in [0,\tau]}$  for (1), with  $A$  replaced by  $A - zB$ . Furthermore,

$$S_{\alpha,z}(t)x = e^{-zt}S_\alpha(t)x + \int_0^t \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{z^n t^{n-1}}{(n-1)!} e^{-zs} S_\alpha(s)x ds, \quad t \geq 0, x \in D(B).$$

### 3. Differential and Analytical Properties of Degenerate $(a, k)$ -Regularized $C$ -Resolvent Families

In this section, we shall clarify the most important differential and analytical properties of degenerate  $(a, k)$ -regularized  $C$ -resolvent families. Set  $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\}$  ( $\alpha \in (0, \pi]$ ).

**Definition 3.1.** (cf. [19, Definition 3.1] for the case  $B = I$ ) Suppose that the functions  $a(t)$  and  $k(t)$  satisfy (P1), as well as that  $C \in L(E)$  is an injective mapping satisfying  $CA \subseteq AC$ . Let  $(R(t))_{t \geq 0}$  be an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family for (1), and let  $0 < \alpha \leq \pi$ . Then it is said that  $(R(t))_{t \geq 0}$  is an exponentially equicontinuous (equicontinuous), analytic  $(a, k)$ -regularized  $C$ -resolvent family for (1), of angle  $\alpha$ , iff there exists  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$  ( $\omega = 0$ ) such that the following holds:

- (i) For every  $x \in D(B)$ , the mapping  $t \mapsto R(t)x, t > 0$  can be analytically extended to the sector  $\Sigma_\alpha$ ; since no confusion seems likely, we shall denote the extension by the same symbol.
- (ii) For every  $x \in D(B)$  and  $\beta \in (0, \alpha)$ , one has  $\lim_{z \rightarrow 0, z \in \Sigma_\beta} R(z)x = R(0)x$ .
- (iii) The family  $\{e^{-\omega z}R(z) : z \in \Sigma_\beta\}$  is equicontinuous for all  $\beta \in (0, \alpha)$ , i.e., for every  $p \in \otimes$ , there exist  $c > 0$  and  $q \in \otimes$  such that

$$p(e^{-\omega z}R(z)x) \leq cq(x), \quad x \in D(B), z \in \Sigma_\beta. \tag{15}$$

Before going any further, we would like to observe that the assertion of [19, Theorem 3.6] cannot be transferred to degenerate  $(a, k)$ -regularized  $C$ -resolvent families without imposing some restrictive assumptions, including the injectivity of the operator  $B$ . This is not the case with the assertion of [19, Theorem 3.7], as the following theorem shows.

**Theorem 3.2.** Assume that  $a(t)$  and  $k(t)$  satisfy (P1),  $A$  and  $B$  are closed linear operators,  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$ ,  $\alpha \in (0, \pi/2]$ ,  $C \in L(E)$  is injective and satisfies  $CA \subseteq AC$ . Assume, further, that for every  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \omega$  and  $\tilde{k}(\lambda) \neq 0$ , we have that the operator  $B - \tilde{a}(\lambda)A$  is injective and  $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$ . Let for each  $x \in D(B)$  there is an analytic function  $q_x : \omega + \Sigma_{\frac{\pi}{2} + \alpha} \rightarrow E$  such that

$$q_x(\lambda) = \tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx, \quad \Re \lambda > \omega, \tilde{k}(\lambda) \neq 0.$$

Suppose that, for every  $\beta \in (0, \alpha)$  and  $p \in \otimes$ , there exist  $c_{p,\beta} > 0$  and  $r_{p,\beta} \in \otimes$  such that  $p((\lambda - \omega)q_x(\lambda)) \leq c_{p,\beta}r_{p,\beta}(x)$ ,  $x \in D(B)$ ,  $\lambda \in \omega + \Sigma_{\beta + (\pi/2)}$  and that, for every  $x \in D(B)$ , there exists the limit  $\lim_{\lambda \rightarrow +\infty} \lambda q_x(\lambda)$  in  $E$ . Then there exists an exponentially equicontinuous, analytic  $(a, k)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1), of angle  $\alpha$ , and for each  $\beta \in (0, \alpha)$  the family  $\{e^{-\omega z}R(z) : z \in \Sigma_\beta\}$  is equicontinuous.

The assertion of [20, Theorem 2.4.2] can be also reformulated for degenerate  $(a, k)$ -regularized  $C$ -resolvent families:

**Theorem 3.3.** Assume that  $k_\beta(t)$  satisfies (P1),  $0 < \alpha < \beta$ ,  $\gamma = \alpha/\beta$  and there exists an exponentially equicontinuous  $(g_\beta, k_\beta)$ -regularized  $C$ -resolvent family  $(R_\beta(t))_{t \geq 0}$  for (1), with  $a(t) = g_\beta(t)$  and  $k(t) = k_\beta(t)$ , satisfying that the family  $\{e^{-\omega t}R_\beta(t) : t \geq 0\}$  is equicontinuous for some  $\omega \geq \max(0, \text{abs}(k_\beta))$ . Assume that there exist a function  $k_\alpha(t)$  satisfying (P1) and a number  $\eta > 0$  such that  $k_\alpha(0) = k_\beta(0)$  and  $\tilde{k}_\alpha(\lambda) = \lambda^{\gamma-1}\tilde{k}_\beta(\lambda^\gamma)$ ,  $\lambda > \eta$ . Then there exists an exponentially

equicontinuous  $(g_\alpha, k_\alpha)$ -regularized  $C$ -resolvent family  $(R_\alpha(t))_{t \geq 0}$  for (1), with  $a(t) = g_\alpha(t)$  and  $k(t) = k_\alpha(t)$ , satisfying that the family  $\{e^{-\omega^{1/\gamma}t} S_\alpha(t) : t \geq 0\}$  is equicontinuous and

$$R_\alpha(t)x = \int_0^\infty t^{-\gamma} \Phi_\gamma(st^{-\gamma}) R_\beta(s)x \, ds, \quad x \in D(B), \quad t > 0.$$

Furthermore, for every  $\zeta \geq 0$ , the equicontinuity of the family  $\{e^{-\omega t}(1+t^\zeta)^{-1} R_\beta(t) : t \geq 0\}$ , resp.  $\{e^{-\omega t} t^{-\zeta} R_\beta(t) : t \geq 0\}$ , implies the equicontinuity of the family  $\{e^{-\omega^{1/\gamma}t}(1+t^{\gamma\zeta})^{-1}(1+\omega t^{\zeta(1-\gamma)})^{-1} R_\alpha(t) : t \geq 0\}$ , resp.  $\{e^{-\omega^{1/\gamma}t} t^{-\gamma\zeta}(1+\omega t^{\zeta(1-\gamma)})^{-1} R_\alpha(t) : t \geq 0\}$ , and the following holds:

- (i) The mapping  $t \mapsto R_\alpha(t)x, t > 0$  admits an analytic extension to the sector  $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi)}$  for all  $x \in D(B)$ .
- (ii) If  $\omega = 0$  and  $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi))$ , then the family  $\{R_\alpha(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi)-\varepsilon}\}$  is equicontinuous and  $\lim_{z \rightarrow 0, z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi)-\varepsilon}} R_\alpha(z)x = R_\alpha(0)x$  for all  $x \in D(B)$ .
- (iii) If  $\omega > 0$  and  $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2}))$ , then there exists  $\delta_{\gamma, \varepsilon} > 0$  such that the family  $\{e^{-\delta_{\gamma, \varepsilon} \Re z} R_\alpha(z) : z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2})-\varepsilon}\}$  is equicontinuous. Moreover,  $\lim_{z \rightarrow 0, z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2})-\varepsilon}} R_\alpha(z)x = R_\alpha(0)x$  for all  $x \in D(B)$ .

Concerning differential properties of degenerate  $(a, k)$ -regularized  $C$ -resolvent families, the following statements can be verified to be true based on the information provided in the final part of the proof of [17, Theorem 3.2.15] (cf. also [19, Theorem 3.18], [19, Theorem 3.20] with  $m = 2$ , the proof of [17, Theorem 2.4.8]), and Theorem 2.5(i); recall only that for each sequence  $(M_n)$  of positive real numbers satisfying  $M_0 = 1$ , as well as

(M.1)  $M_n^2 \leq M_{n+1} M_{n-1}, n \in \mathbb{N}$ ,

(M.2)  $M_n \leq AH^n \min_{n_1, n_2 \in \mathbb{N}, n_1+n_2=n} M_{n_1} M_{n_2}, n \in \mathbb{N}$ , for some  $A > 1$  and  $H > 1$ , and

(M.3)'  $\sum_{n=1}^\infty \frac{M_{n-1}}{M_n} < \infty$ ,

we define the function  $\omega_L(\cdot)$  by  $\omega_L(t) := \sum_{n=0}^\infty \frac{t^n}{M_n}, t \geq 0$ .

**Theorem 3.4.** Suppose  $A$  and  $B$  are closed linear operators,  $a(t)$  and  $k(t)$  satisfy (P1),  $r \geq -1$  and there exists  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$  such that, for every  $z \in \{\lambda \in \mathbb{C} : \Re \lambda > \omega, \tilde{k}(\lambda) \neq 0\}$ , we have that the operator  $B - \tilde{a}(z)A$  is injective and  $C(R(B)) \subseteq R(B - \tilde{a}(z)A)$ . Suppose, additionally, that for every  $\sigma > 0$  and  $x \in D(B)$ , there exist a number  $c_{\sigma, x} > 0$ , an open neighborhood  $\Omega_{\sigma, x, \omega}$  of the region

$$\Lambda_{\sigma, x, \omega} := \{\lambda \in \mathbb{C} : \Re \lambda \leq \omega, \Re \lambda \geq -\sigma \ln |\Im \lambda| + c_{\sigma, x}\} \cup \{\lambda \in \mathbb{C} : \Re \lambda \geq \omega\},$$

and an analytic function  $h_{\sigma, x} : \Omega_{\sigma, x, \omega} \rightarrow L(E)$  such that  $h_{\sigma, x}(\lambda) = \tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx, \Re \lambda > \omega, \tilde{k}(\lambda) \neq 0$  and that the set  $\{|\lambda|^{-r} h_{\sigma, x}(\lambda) : \lambda \in \Lambda_{\sigma, x, \omega}, \Re \lambda \leq \omega\}$  is bounded. If, for every  $\sigma > 0$  and  $p \in \mathfrak{O}$ , there exist  $c_p > 0$  and  $q_p \in \mathfrak{O}$  such that  $p(h_{\sigma, x}(\lambda)) \leq c_p |\lambda|^r q_p(x), \Re \lambda > \omega, x \in D(B)$ , then, for every  $\zeta > 1$ , there exists an exponentially equicontinuous  $(a, k * g_{\zeta+r})$ -regularized  $C$ -resolvent family  $(R_\zeta(t))_{t \geq 0}$  for (1), satisfying that the mapping  $t \mapsto R_\zeta(t)x, t > 0$  is infinitely differentiable for all  $x \in D(B)$ .

**Theorem 3.5.** (i) Suppose that  $a(t)$  and  $k(t)$  satisfy (P1), there exists an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1), satisfying (ii) of Definition 2.2 with some number  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$ . Let  $\omega_0 > \omega$ . Denote, for every  $x \in D(B), \varepsilon \in (0, 1)$ , and a corresponding  $K_{\varepsilon, x} > 0$ ,

$$F_{\varepsilon, \omega_0, x} := \{\lambda \in \mathbb{C} : \Re \lambda \geq -\ln \omega_L(K_{\varepsilon, x} |\Im \lambda|) + \omega_0\}.$$

Assume that, for every  $x \in D(B)$  and  $\varepsilon \in (0, 1)$ , there exist  $K_{\varepsilon, x} > 0$ , an open neighborhood  $O_{\varepsilon, \omega_0, x}$  of the region  $G_{\varepsilon, \omega_0, x} := \{\lambda \in \mathbb{C} : \Re \lambda \geq \omega_0, \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon, \omega_0} : \Re \lambda \leq \omega_0\}$  and the analytic mappings  $h_{\varepsilon, x} : O_{\varepsilon, \omega_0, x} \rightarrow E, f_{\varepsilon, x} : O_{\varepsilon, \omega_0, x} \rightarrow \mathbb{C}, g_{\varepsilon, x} : O_{\varepsilon, \omega_0, x} \rightarrow \mathbb{C}$  such that:

- (a)  $f_{\varepsilon,x}(\lambda) = \tilde{k}(\lambda)$ ,  $\Re\lambda > \omega_0$ ;  $g_{\varepsilon,x}(\lambda) = \tilde{a}(\lambda)$ ,  $\Re\lambda > \omega_0$ ,
- (b) for every  $\lambda \in F_{\varepsilon,\omega_0,x}$ , the operator  $B - g_{\varepsilon,x}(\lambda)A$  is injective and  $C(R(B)) \subseteq R(B - g_{\varepsilon,x}(\lambda)A)$ ,
- (c) for every  $x \in D(B)$ ,  $h_{\varepsilon,x}(\lambda) = f_{\varepsilon,x}(\lambda)(B - g_{\varepsilon,x}(\lambda)A)^{-1}CBx$ ,  $\lambda \in G_{\varepsilon,\omega_0,x}$ ,
- (d) the set  $\{(1 + |\lambda|)^{-m}e^{-\varepsilon|\Re\lambda|}h_{\varepsilon,x}(\lambda) : \lambda \in F_{\varepsilon,\omega_0}, \Re\lambda \leq \omega_0\}$  is bounded.

Then, for every  $x \in D(B)$ , the mapping  $t \mapsto R(t)x$ ,  $t > 0$  is infinitely differentiable and, for every compact set  $K \subseteq (0, \infty)$ , there exists  $h_K > 0$  such that the set  $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)x}{M_n} : t \in K, n \in \mathbb{N}_0\}$  is bounded; furthermore, if  $K_{\varepsilon,x}$  is independent of  $x \in D(B)$  and if for each  $p \in \otimes$  there exist  $c_p > 0$  and  $q_p \in \otimes$  such that  $p((1 + |\lambda|)^{-m}e^{-\varepsilon|\Re\lambda|}h_{\varepsilon,x}(\lambda)) \leq c_p q_p(x)$ ,  $x \in D(B)$ ,  $\lambda \in F_{\varepsilon,\omega_0}$ ,  $\Re\lambda \leq \omega_0$ , then the family  $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)}{M_n} : t \in K, n \in \mathbb{N}_0\}$  is equicontinuous.

- (ii) Suppose that  $a(t)$  and  $k(t)$  satisfy (P1), there exists an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1), satisfying (ii) of Definition 2.2 with some number  $\omega \geq \max(0, \text{abs}(a), \text{abs}(k))$ . Let  $\omega_0 > \omega$ . Denote, for every  $x \in D(B)$ ,  $\varepsilon \in (0, 1)$ ,  $\rho \in [1, \infty)$  and a corresponding  $K_{\varepsilon,x} > 0$ ,

$$F_{\varepsilon,\omega_0,\rho,x} := \{\lambda \in \mathbb{C} : \Re\lambda \geq -K_{\varepsilon,x}|\Im\lambda|^{1/\rho} + \omega_0\}.$$

Assume that, for every  $x \in D(B)$  and  $\varepsilon \in (0, 1)$ , there exist  $K_{\varepsilon,x} > 0$ , an open neighborhood  $O_{\varepsilon,\omega_0,x}$  of the region  $G_{\varepsilon,\omega_0,\rho,x} := \{\lambda \in \mathbb{C} : \Re\lambda \geq \omega_0, \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega_0,\rho,x} : \Re\lambda \leq \omega_0\}$ , and analytic mappings  $h_{\varepsilon,x} : O_{\varepsilon,\omega_0,x} \rightarrow E$ ,  $f_{\varepsilon,x} : O_{\varepsilon,\omega_0,x} \rightarrow \mathbb{C}$  and  $g_{\varepsilon,x} : O_{\varepsilon,\omega_0,x} \rightarrow \mathbb{C}$  such that the conditions (i)(a)-(d) of this theorem hold with  $F_{\varepsilon,\omega_0,x}$ , resp.  $G_{\varepsilon,\omega_0,x}$ , replaced by  $F_{\varepsilon,\omega_0,\rho,x}$ , resp.  $G_{\varepsilon,\omega_0,\rho,x}$ . Then, for every  $x \in D(B)$ , the mapping  $t \mapsto R(t)$ ,  $t > 0$  is infinitely differentiable and, for every compact set  $K \subseteq (0, \infty)$ , there exists  $h_K > 0$  such that the set  $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)x}{n!^p} : t \in K, n \in \mathbb{N}_0\}$  is equicontinuous; furthermore, if  $K_{\varepsilon,x}$  is independent of  $x \in D(B)$  and if for each  $p \in \otimes$  there exist  $c_p > 0$  and  $q_p \in \otimes$  such that  $p((1 + |\lambda|)^{-m}e^{-\varepsilon|\Re\lambda|}h_{\varepsilon,x}(\lambda)) \leq c_p q_p(x)$ ,  $x \in D(B)$ ,  $\lambda \in F_{\varepsilon,\omega_0,\rho,x}$ ,  $\Re\lambda \leq \omega_0$ , then the family  $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)}{n!^p} : t \in K, n \in \mathbb{N}_0\}$  is equicontinuous.

**Theorem 3.6.** (The abstract Weierstrass formula)

- (i) Assume that  $a(t)$  and  $k(t)$  satisfy (P1), and there exist  $M > 0$  and  $\omega > 0$  such that  $|k(t)| \leq Me^{\omega t}$ ,  $t \geq 0$ . Assume, further, that there exist a number  $\omega' \geq \omega$  and a function  $a_1(t)$  satisfying (P1) and  $\tilde{a}_1(\lambda) = \tilde{a}(\sqrt{\lambda})$ ,  $\Re\lambda > \omega'$ . (Let us recall that the above holds if  $a(t)$  is exponentially bounded; in this case,  $a_1(t) = \int_0^\infty s \frac{e^{-s^2/4t}}{2\sqrt{\pi t^{3/2}}} a(s) ds$ ,  $t > 0$ .) Let there exist an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family  $(C(t))_{t \geq 0}$  for (1). Then there exists an exponentially equicontinuous, analytic  $(a_1, k_1)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1), with  $a(t)$  replaced by  $a_1(t)$ , of angle  $\frac{\pi}{2}$ , where:

$$k_1(t) := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} k(s) ds, \quad t > 0, \quad k_1(0) := k(0), \quad \text{and} \tag{16}$$

$$R(t)x := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} C(s)x ds, \quad t > 0, \quad x \in D(B), \quad R(0)x := C(0)x, \quad x \in D(B). \tag{17}$$

- (ii) Assume that  $k(t)$  satisfy (P1),  $\beta > 0$  and there exist  $M > 0$  and  $\omega > 0$  such that  $|k(t)| \leq Me^{\omega t}$ ,  $t \geq 0$ . Let there exist an exponentially equicontinuous  $(g_{2\beta}, k)$ -regularized  $C$ -resolvent family  $(C(t))_{t \geq 0}$  for (1). Then there exists an exponentially equicontinuous, analytic  $(g_\beta, k_1)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1), with  $a(t)$  replaced by  $a_1(t)$ , of angle  $\frac{\pi}{2}$ , where  $k_1(t)$  and  $R(t)$  are defined through (16)-(17).

**Remark 3.7.** As Theorem 2.8 shows, the existence of an exponentially equicontinuous  $(a, k)$ -regularized  $C$ -resolvent family  $(R(t))_{t \geq 0}$  for (1) does not automatically imply the existence of mild (strong) solutions of this problem, we need to impose the conditions like (i.1), (i.1)' or (ii.1). In this remark, we will reconsider our results proved so far by including these conditions in a further analysis:

1. Proposition 2.4(i): If the function  $b(t)$  satisfies the prescribed assumptions and  $(R(t))_{t \geq 0}$  satisfies any of the conditions (i.1), (i.1)' or (ii.1), then  $((b * R)(t))_{t \geq 0}$  satisfies the same condition as well, with the function  $k(t)$  replaced by  $(b * k)(t)$ ; Proposition 2.4(iii): If the functions  $k(t)$  and  $b(t)$  satisfy the prescribed assumptions and  $(R(t))_{t \geq 0}$  satisfies any of the conditions (i.1), (i.1)' or (ii.1), then  $(S(t))_{t \geq 0}$  satisfies the same condition as well, with the function  $k(t)$  replaced by 1.
2. In the formulations of Theorem 2.5(i)-(ii), as well as Theorem 3.2 and Theorems 3.4-3.5, we must add some very natural conditions ensuring the validity of (i.1), (i.1)' or (ii.1) for the corresponding resolvent families.
3. The conditions (i.1) and (ii.1) are invariant under the action of subordination principles stated in Theorem 2.6, while some additional assumptions must be imposed for the invariance of the condition (i.1)'.
4. The rescaling of degenerate  $K$ -convoluted  $C$ -semigroups (cf. Proposition 2.10) preserves the conditions (i.1), (i.1)' and (ii.1).
5. Using the arguments given in the proof of [3, Theorem 3.1], it is not difficult to verify that the conditions (i.1), (i.1)' and (ii.1) are invariant under the action of subordination principles stated in Theorem 3.3 and Theorem 3.6.

We round off this section with the following example.

**Example 3.8.** (cf. [17, Example 2.8.3(iii)], [20, Example 2.6.10]) Let  $s > 1$ ,

$$E := \left\{ f \in C^\infty[0, 1] ; \|f\| := \sup_{p \geq 0} \frac{\|f^{(p)}\|_\infty}{p!^s} < \infty \right\}$$

and

$$A := -d/ds, D(A) := \{f \in E ; f' \in E, f(0) = 0\}.$$

If  $f \in E, t \in [0, 1]$  and  $\lambda \in \mathbb{C}$ , set  $f_\lambda^1(t) := \int_0^t e^{-\lambda(t-s)} f(s) ds$  and  $f_\lambda^2(t) := \int_0^t e^{\lambda(t-s)} f(s) ds$ . Then  $f_\lambda^1(\cdot), f_\lambda^2(\cdot) \in E, \lambda \in \mathbb{C}$  and there exist  $b' > 0$  and  $M \geq 1$ , independent of  $f(\cdot)$ , such that

$$\|f_\lambda^1(\cdot)\| \leq M \|f\| e^{b'|\lambda|^{1/s}}, \Re \lambda \geq 0, f \in E. \tag{18}$$

Furthermore, for each  $\eta > 1$  there exists  $M_\eta \geq 1$ , independent of  $f(\cdot)$ , such that

$$\|f_\lambda^2(\cdot)\| \leq M_\eta \|f\| e^{\eta|\lambda|}, \Re \lambda \geq 0, f \in E. \tag{19}$$

Let  $P_1(z) = \sum_{j=0}^{N_1} a_{j,1} z^j, z \in \mathbb{C}, a_{N_1,1} \neq 0$  be a complex non-zero polynomial, and let  $P_2(z) = \sum_{j=0}^{N_2} a_{j,2} z^j, z \in \mathbb{C}, a_{N_2,2} \neq 0$  be a complex non-zero polynomial so that  $N_1 = dg(P_1) > 1 + dg(P_2) = 1 + N_2$ . For any complex non-zero polynomial  $P(z)$ , we define the operator  $P(A)$  in the obvious way; then we know that

$$\rho(P(A)) = \mathbb{C} \text{ and } R(\lambda : A)f = f_\lambda^1, f \in E, \lambda \in \mathbb{C}. \tag{20}$$

Set  $P_\lambda(z) := \lambda P_2(z) - P_1(z), z \in \mathbb{C} (\lambda \in \mathbb{C})$ . Let  $\{z_1, \dots, z_s\}$  be the set which consists of joint multiple roots of polynomials  $P_1(z)$  and  $P_2(z)$ . Then there exist uniquely determined integers  $k_1, \dots, k_s \in \mathbb{N}$  such that

$$P_\lambda(z) = (z - z_1)^{k_1} \cdots (z - z_s)^{k_s} (\lambda Q_2(z) - Q_1(z)), \quad z \in \mathbb{C}, \lambda \in \mathbb{C},$$

with  $Q_1(z)$  and  $Q_2(z)$  being two non-zero complex polynomials without joint multiple roots, satisfying additionally that  $dg(Q_1) > dg(Q_2) + 1$ . This implies that there exists  $d > 0$  such that, for every  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq d$ , the polynomial  $Q_\lambda(z) := \lambda Q_2(z) - Q_1(z)$  is square-free. Denote by  $z_{1,\lambda}, \dots, z_{N_1,\lambda}$  the roots of  $P_\lambda(z) (\lambda \in \mathbb{C})$ . Using [12, Corollary 5.6] (this is an elementary result on root localization of complex polynomials), we get the existence of positive real number  $\vartheta \geq 1$  such that

$$|z_{i,\lambda}| \leq \vartheta (1 + |\lambda|)^{\frac{1}{N_1 - N_2}}, \quad 1 \leq i \leq N_1 (\lambda \in \mathbb{C}). \tag{21}$$

It is quite easy to prove that the operator  $\lambda P_2(A) - P_1(A)$  has the bounded inverse for all  $\lambda \in \mathbb{C}$ , as well as that

$$(\lambda P_2(A) - P_1(A))^{-1} = (-1)^{N_1+1} a_{N_1,1}^{-1} R(z_{1,\lambda} : A) \cdots R(z_{N_1,\lambda} : A), \lambda \in \mathbb{C}. \tag{22}$$

Observe further that, for every  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq d$ , the discriminant of polynomial  $Q_\lambda(z)$ , for which it is well known that can be represented by a homogenous polynomial of degree  $2(dg(Q_1) - 1)$  in the coefficients of  $Q_\lambda(z)$ , is a complex non-zero polynomial in  $\lambda$ . Hence, there exist numbers  $d_1 \geq d$  and  $\eta > 0$  such that  $|D(Q_\lambda(z))| \geq \eta$ ,  $|\lambda| \geq d_1$ . Making use of this fact and [35, Theorem 1], we obtain the existence of a sufficiently small number  $\zeta > 0$  such that, for every  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq d_1$ , and for every two distinct roots  $z_{i,\lambda}, z_{j,\lambda}$  of polynomial  $Q_\lambda(z)$ , we have  $|z_{i,\lambda} - z_{j,\lambda}| \geq \zeta$ . Then the computation contained in the analysis made in [20, Example 2.6.10], combined with the equality (22) and the above fact, shows that the norm of operator  $(\lambda P_2(A) - P_1(A))^{-1}$  does not exceed  $M \sum \|R(z_{j,\lambda} : A)\|$ , where the summation is taken over all roots  $z_{j,\lambda}$  of polynomial  $Q_\lambda(z)$ . Taken together with (18)-(21) and the generalized resolvent equation, the above implies that there exist numbers  $b > 0, c > 0$  and  $\zeta > 0$  such that

$$\|(\lambda P_2(A) - P_1(A))^{-1}\| = O(e^{b|\lambda|^{1/(N_1-N_2)s+c|\lambda|^{1/(N_1-N_2)}}}), \lambda \in \mathbb{C}, \tag{23}$$

and

$$\|(\lambda P_2(A) - P_1(A))^{-1} P_2(A) f\| \leq \zeta \|f\| e^{b|\lambda|^{1/(N_1-N_2)s+c|\lambda|^{1/(N_1-N_2)}}}, \tag{24}$$

for all  $\lambda \in \mathbb{C}$  and  $f \in D(P_2(A))$ . Before proceeding further, it should be noted that the above estimates can be used in proving the existence of convoluted solutions of fractional analogs of equation  $(\lambda - \Delta)u_t = \alpha\Delta - \beta\Delta^2$  ( $\alpha, \beta > 0, \lambda \in \mathbb{R}$ ), in contrast with the assertions of Theorem 4.2-Theorem 4.3 below, which can be applied only in the case that  $\lambda > 0$  (cf. [22] for more details); as observed by G. A. Sviridyuk, this equation is important in evolution modeling of some problems appearing in the theory of liquid filtration, see e.g. [11, p. 6]. Suppose  $N_1 - N_2 > \alpha \geq 1, \delta \in (0, \pi/2], (\pi/2 + \delta)\alpha/(N_1 - N_2) < \pi/2, \rho > c/\cos((\pi/2 + \delta)\alpha/(N_1 - N_2))$  and  $k(t) = \mathcal{L}^{-1}(e^{-\rho\lambda^{\alpha/n}})(t), t \geq 0$ . By Theorem 3.2 and (24), there exists an exponentially bounded, analytic  $(g_\alpha, k)$ -regularized resolvent family  $(R_\alpha(t))_{t \geq 0}$  for the corresponding problem (1), of angle  $\delta$ ; it is clear that the conditions (i.2)-(i.3), (i.1)' and (ii.1) stated in Theorem 2.8 and Remark 2.9(i) holds for  $(R_\alpha(t))_{t \geq 0}$ . Observe finally that the case  $N_1 = N_2 + 1$  is critical and that we always have the existence of integrated solution families in the case  $N_2 \geq N_1$  (this follows from the above analysis and the fact that, for every  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq d$ , the roots  $z_{1,\lambda}, \dots, z_{N_2,\lambda}$  of polynomial  $P_\lambda(z)$  belong to a compact set  $K \subseteq \mathbb{C}$  which does not depend on  $\lambda$ ; see e.g. [12, Theorem 5.4]).

#### 4. Degenerate Time-Fractional Equations Associated with Abstract Differential Operators

With the exceptions of Remark 4.5 and Remark 4.8(ii), we shall always assume in this section that  $n \in \mathbb{N}$  and  $iA_j, 1 \leq j \leq n$  are commuting generators of bounded  $C_0$ -groups on a Banach space  $E$ . Denote by  $\mathcal{S}(\mathbb{R}^n)$  the Schwartz space of rapidly decreasing functions on  $\mathbb{R}^n$ . Put  $k := 1 + \lfloor n/2 \rfloor, A := (A_1, \dots, A_n)$  and  $A^\eta := A_1^{\eta_1} \cdots A_n^{\eta_n}$  for any  $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{N}_0^n$ . For every  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  and  $u \in \mathcal{FL}^1(\mathbb{R}^n) = \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$ , we set  $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}, (\xi, A) := \sum_{j=1}^n \xi_j A_j$  and

$$u(A)x := \int_{\mathbb{R}^n} \mathcal{F}^{-1}u(\xi)e^{-i(\xi,A)}x d\xi, x \in E.$$

Then  $u(A) \in L(E), u \in \mathcal{FL}^1(\mathbb{R}^n)$  and there exists a finite constant  $M \geq 1$  such that  $\|u(A)\| \leq M\|\mathcal{F}^{-1}u\|_{L^1(\mathbb{R}^n)}, u \in \mathcal{FL}^1(\mathbb{R}^n)$ . Let  $N \in \mathbb{N}$ , and let  $P(x) = \sum_{|\eta| \leq N} a_\eta x^\eta, x \in \mathbb{R}^n$  be a complex polynomial. Then we define  $P(A) := \sum_{|\eta| \leq N} a_\eta A^\eta$  and  $E_0 := \{\phi(A)x : \phi \in \mathcal{S}(\mathbb{R}^n), x \in E\}$ . We know that the operator  $P(A)$  is closable and that the following holds (cf. [46], [25], [20] and [18] for further information):

$$(\text{p}) \quad \overline{E_0} = E, E_0 \subseteq \bigcap_{\eta \in \mathbb{N}_0^n} D(A^\eta), \overline{P(A)|_{E_0}} = \overline{P(A)} \text{ and } \phi(A)P(A) \subseteq P(A)\phi(A) = (\phi P)(A), \phi \in \mathcal{S}(\mathbb{R}^n).$$

Assuming that  $E$  is a function space on which translations are uniformly bounded and strongly continuous, the obvious choice for  $A_j$  is  $-i\partial/\partial x_j$  (notice also that  $E$  can be consisted of functions defined on some bounded domain [5], [25], [46], [47]). If  $P(x) = \sum_{|\eta| \leq N} a_\eta x^\eta$ ,  $x \in \mathbb{R}^n$  and  $E$  is such a space (for example,  $L^p(\mathbb{R}^n)$  with  $p \in [1, \infty)$ ,  $C_0(\mathbb{R}^n)$  or  $BUC(\mathbb{R}^n)$ ), then it is not difficult to prove that  $\overline{P(A)}$  is nothing else but the operator  $\sum_{|\eta| \leq N} a_\eta (-i)^{|\eta|} \partial^{|\eta|} / \partial x_1^{\eta_1} \cdots \partial x_n^{\eta_n} \equiv \sum_{|\eta| \leq N} a_\eta D^\eta$ , acting with its maximal distributional domain. Recall that  $P(x)$  is called  $r$ -coercive ( $0 < r \leq N$ ) if there exist  $M, L > 0$  such that  $|P(x)| \geq M|x|^r$ ,  $|x| \geq L$ ; by a corollary of the Seidenberg-Tarski theorem, the equality  $\lim_{|x| \rightarrow \infty} |P(x)| = \infty$  implies in particular that  $P(x)$  is  $r$ -coercive for some  $r \in (0, N]$  (cf. [1, Remark 8.2.7]). Henceforward,  $M > 0$  denotes a generic constant whose value may change from line to line.

Let  $p \in [1, \infty]$ . Following L. Hörmander [14], it will be said that a function  $u \in L^\infty(\mathbb{R}^n)$  is a Fourier multiplier on  $L^p(\mathbb{R}^n)$  if  $\mathcal{F}^{-1}(u\mathcal{F}\phi) \in L^p(\mathbb{R}^n)$  for all  $\phi \in \mathcal{S}(\mathbb{R}^n)$  and if

$$\|u\|_{\mathcal{M}_p} := \sup \left\{ \left\| \mathcal{F}^{-1}(u\mathcal{F}\phi) \right\|_{L^p(\mathbb{R}^n)} : \phi \in \mathcal{S}(\mathbb{R}^n), \|\phi\|_{L^p(\mathbb{R}^n)} \leq 1 \right\} < \infty.$$

We use the abbreviation  $\mathcal{M}_p$  for the space of all Fourier multipliers on  $L^p(\mathbb{R}^n)$ ; cf. [13] for more details. Then  $\mathcal{M}_p$  is a Banach algebra under pointwise multiplication and  $\mathcal{FL}^1(\mathbb{R}^n)$  is continuously embedded in  $\mathcal{M}_p$ . We need the following lemma (see e.g. [42, Lemma 5.2, Lemma 5.4, pp. 20-22]).

**Lemma 4.1.** (i) Let  $1 \leq p \leq \infty$ ,  $j, n \in \mathbb{N}$ ,  $j > n/2$  and  $\{f_t\}_{t \geq 0}$  be a family of  $C^j(\mathbb{R}^n)$ -functions. Assume that for each  $x \in \mathbb{R}^n$ ,  $\eta \in \mathbb{N}_0^n$  with  $|\eta| \leq j$ ,  $t \mapsto D^\eta f_t(x)$ ,  $t \geq 0$  is continuous and that there exist  $a > 0$ ,  $r > n|\frac{1}{p} - \frac{1}{2}|$  and  $M_t > 0$  ( $M_t$  is bounded on compacts of  $t \geq 0$ ) such that

$$|D^\eta f_t(x)| \leq M_t^{|\eta|} (1 + |x|)^{(a-1)|\eta| - ar}, \quad |\eta| \leq j, \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

Then, for any  $t \geq 0$ ,  $p = 1, \infty$  (resp.  $1 < p < \infty$ ), we have  $f_t \in \mathcal{FL}^1(\mathbb{R}^n)$  (resp.  $f_t \in \mathcal{M}_p$ ),  $t \mapsto f_t$ ,  $t \geq 0$  is continuous with respect to  $\|\cdot\|_{\mathcal{FL}^1(\mathbb{R}^n)}$  (resp.  $\|\cdot\|_{\mathcal{M}_p}$ ) and there exists a constant  $M > 0$  independent of  $t \geq 0$  such that

$$\|f_t\|_{\mathcal{FL}^1(\mathbb{R}^n)} \quad (\text{resp. } \|f_t\|_{\mathcal{M}_p}) \leq MM_t^{n|\frac{1}{p} - \frac{1}{2}|}, \quad t \geq 0.$$

(ii) Let  $1 < p < \infty$ ,  $j, n \in \mathbb{N}$ ,  $j > n/2$  and  $f \in C^j(\mathbb{R}^n)$ . Assume that there exist  $a \geq 0$ ,  $r \geq n|\frac{1}{p} - \frac{1}{2}|$ ,  $M_f \geq 1$  and  $L_f > 0$  such that

$$|D^\eta f(x)| \leq L_f M_f^{|\eta|} (1 + |x|)^{(a-1)|\eta| - ar}, \quad |\eta| \leq j, \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

Then  $f \in \mathcal{M}_p$  and there exists a constant  $M > 0$  independent of  $f(\cdot)$  such that

$$\|f\|_{\mathcal{M}_p} \leq ML_f M_f^{n|\frac{1}{p} - \frac{1}{2}|}.$$

Suppose now that  $P_1(x)$  and  $P_2(x)$  are non-zero complex polynomials in  $n$  variables and  $0 < \alpha \leq 2$ ; put  $N_1 := dg(P_1(x))$ ,  $N_2 := dg(P_2(x))$  and  $m := \lceil \alpha \rceil$ . We investigate the generation of some very specific classes of  $(g_\alpha, C)$ -regularized resolvent families associated with the following degenerate abstract Cauchy problem

$$(DFP) : \begin{cases} \mathbf{D}_t^\alpha \overline{P_2(A)} u(t) = \overline{P_1(A)} u(t), & t \geq 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0, & 1 \leq j \leq \lceil \alpha \rceil - 1; \end{cases}$$

the basic information about fractional calculus and non-degenerate fractional differential equations can be obtained by consulting the monographs [15], [17]-[20], [28], [33], [36] and the doctoral dissertation of E. Bazhlekova [3]. Convoluting the both sides of (DFP) with  $g_\alpha(t)$ , and using the equality [3, (1.21)], it follows that every solution of (DFP) is, in fact, a strong solution of problem (1) with  $B = \overline{P_2(A)}$ ,  $\tau = \infty$ ,  $a(t) = g_\alpha(t)$  and  $f(t) \equiv \overline{P_2(A)}Cx$ . It can be easily checked that any such a solution of problem (1) is also a solution of

problem (DFP). A continuous  $E$ -valued function  $t \mapsto u(t)$ ,  $t \geq 0$  is said to be a mild solution of (DFP) iff  $\overline{P_1(A)}(g_\alpha * u)(t) = \overline{P_2(A)}u(t) - \overline{P_2(A)}Cx$ ,  $t \geq 0$ .

We start by stating the following extension of [18, Theorem 2.1]; observe only that we do not assume here the coercivity of  $P_1(x)$  or  $P_2(x)$ , and that in the formulation of the afore-mentioned theorem we have that  $P_1(x) = Q(x) = P(x)$  and  $P_2(x) = 1$  ( $x \in \mathbb{R}^n$ ).

**Theorem 4.2.** *Suppose  $0 < \alpha < 2$ ,  $\omega \geq 0$ ,  $P_1(x)$  and  $P_2(x)$  are non-zero complex polynomials,  $N_1 = dg(P_1(x))$ ,  $N_2 = dg(P_2(x))$ ,  $N \in \mathbb{N}$  and  $r \in (0, N]$ . Let  $Q(x)$  be an  $r$ -coercive complex polynomial of degree  $N$ ,  $a \in \mathbb{C} \setminus Q(\mathbb{R}^n)$ ,  $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$  (resp.  $\gamma = \frac{n}{r}|\frac{1}{p} - \frac{1}{2}| \max(N, \frac{N_1+N_2}{\min(1,\alpha)})$ ), if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ ,  $P_2(x) \neq 0$ ,  $x \in \mathbb{R}^n$  and*

$$\sup_{x \in \mathbb{R}^n} \Re \left( \left( \frac{P_1(x)}{P_2(x)} \right)^{1/\alpha} \right) \leq \omega. \tag{25}$$

Set

$$R_\alpha(t) := \left( E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) (a - Q(x))^{-\gamma} \right) (A), \quad t \geq 0.$$

Then  $(R_\alpha(t))_{t \geq 0} \subseteq L(E)$  is a global exponentially bounded  $(g_\alpha, R_\alpha(0))$ -regularized resolvent family for (DFP),  $(R_\alpha(t))_{t \geq 0}$  is norm continuous provided  $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$ , and the following holds:

$$\|R_\alpha(t)\| \leq M \left( 1 + t^{\max(1,\alpha)n/2} \right) e^{\omega t}, \quad t \geq 0, \text{ resp. } \|R_\alpha(t)\| \leq M \left( 1 + t^{\max(1,\alpha)n \left( \frac{1}{p} - \frac{1}{2} \right)} \right) e^{\omega t}, \quad t \geq 0. \tag{26}$$

*Proof.* Put  $C := R_\alpha(0)$ . Then  $C$  is injective and it can be easily proved with the help of  $(\triangleright)$  that  $\overline{CP(A)} \subseteq \overline{P(A)C}$  for any complex polynomial  $P(x)$ ; see e.g. [18]. Furthermore,  $\sup_{x \in \mathbb{R}^n} |P_2(x)|^{-1} < \infty$  and, for every multi-index  $\eta \in \mathbb{N}_0^n$  with  $|\eta| > 0$ , there exists  $c_\eta > 0$  such that

$$\left| D^\eta \left( \frac{P_1(x)}{P_2(x)} \right) \right| \leq c_\eta (1 + |x|)^{|\eta|(N_1+N_2-1)}, \quad x \in \mathbb{R}^n. \tag{27}$$

By induction, one can prove that, for every multi-index  $\eta \in \mathbb{N}_0^n$  with  $|\eta| > 0$ , the following holds:

$$D^\eta E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) = \sum_{j=1}^{|\eta|} t^{\alpha j} E_\alpha^{(j)} \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) R_{\eta,j}(x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{28}$$

where  $R_{\eta,j}(x)$  can be represented as a finite sum of terms like  $\prod_{w=1}^{s_j} D^{\eta_{j,w}} \left( \frac{P_1(x)}{P_2(x)} \right)$  with  $|\eta_{j,w}| > 0$  ( $1 \leq w \leq s_j$ ) and  $|\eta_{j,1}| + \dots + |\eta_{j,s_j}| \leq |\eta|$ . Consider now the assertion of Lemma 1.2. Taking the number  $\sigma > 0$  sufficiently small, and keeping in mind that  $0 < \alpha < 2$ , we obtain that, for every  $m \in \mathbb{N} \setminus \{1\}$  and for every  $t \geq 0$ ,  $x \in \mathbb{R}^n$  with  $|t^\alpha P_1(x)/P_2(x)| \geq 1$ , the term

$$\left| E_{\alpha,\alpha j-(j-l)} \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) - \frac{1}{\alpha} \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right)^{(1-(\alpha j-(j-l)))/\alpha} e^{(t^\alpha P_1(x)/P_2(x))^{1/\alpha}} - \sum_{j=1}^{m-1} \frac{\left( t^\alpha P_1(x)/P_2(x) \right)^{-j}}{\Gamma(\alpha j - (j-l) - \alpha j)} \right|$$

can be majorized by  $M|t^\alpha P_1(x)/P_2(x)|^{-m}$ . Clearly, the function  $E_{\alpha,\alpha j-(j-l)}(\cdot)$  is bounded on compacts of  $\mathbb{C}$ , and (25) implies that  $\Re \left( (t^\alpha P_1(x)/P_2(x))^{1/\alpha} \right) \leq \omega t$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^n$ . As in the proof of [19, Theorem 2.1], the above implies that, for every  $t \geq 0$ ,  $x \in \mathbb{R}^n$  and  $1 \leq l \leq j \leq k$ ,

$$\left| E_{\alpha,\alpha j-(j-l)} \left( t^\alpha P_1(x)/P_2(x) \right) \right| \leq M \left[ 1 + t^{1-(\alpha j-(j-l))} |P_1(x)/P_2(x)|^{\frac{1-(\alpha j-(j-l))}{\alpha}} e^{\omega t} \right]. \tag{29}$$

By (2), (27)-(28) and the boundedness of derivatives of the Mittag-Leffler function  $E_\alpha(\cdot)$  on compacts of  $\mathbb{C}$ , we obtain that, for every  $t \geq 0$  and  $x \in \mathbb{R}^n$  with  $|t^\alpha P_1(x)/P_2(x)| \leq 1$  :

$$\left| D^\eta E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) \right| \leq M (t^\alpha + t^{\alpha|\eta|}) (1 + |x|)^{\eta(N_1+N_2-1)}, \quad 0 < |\eta| \leq k. \tag{30}$$

If  $0 < |\eta| \leq k, t \geq 0, x \in \mathbb{R}^n$  and  $|t^\alpha P_1(x)/P_2(x)| \geq 1$ , then the following holds (cf. (29) and [18, (2.6)-(2.7)]):

$$\begin{aligned} & \left| D^\eta E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) \right| \\ & \leq M \sum_{j=1}^{|\eta|} t^{\alpha j} \sum_{l=1}^j \left[ 1 + \left| t^\alpha P_1(x)/P_2(x) \right|^{\frac{1-(\alpha j-(j-l))}{\alpha}} e^{\omega t} \right] (1 + |x|)^{\eta(N_1+N_2-1)} \\ & \leq M \sum_{j=1}^{|\eta|} t^{\alpha j} \sum_{l=1}^j \left[ 1 + e^{\omega t} (1 + t^{1-(\alpha j-(j-l))}) \right] (1 + |x|)^{\eta \left( \frac{N_1+N_2}{\min(1,\alpha)} - 1 \right)} \\ & \leq M (1 + t^{\max(1,\alpha)|\eta|}) e^{\omega t} (1 + |x|)^{\eta \left( \frac{N_1+N_2}{\min(1,\alpha)} - 1 \right)}. \end{aligned}$$

Taking into account (30), we obtain from the previous estimate that, for every  $t \geq 0$  and  $x \in \mathbb{R}^n$ ,

$$\left| D^\eta E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) \right| \leq M (1 + t^{\max(1,\alpha)|\eta|}) e^{\omega t} (1 + |x|)^{\eta \left( \frac{N_1+N_2}{\min(1,\alpha)} - 1 \right)}, \quad 0 < |\eta| \leq k; \tag{31}$$

observe that the inequality  $\Re((t^\alpha(P_1(x)/P_2(x)))^{1/\alpha}) \leq \omega t, t \geq 0, x \in \mathbb{R}^n$  and Lemma 1.2 together imply that the previous estimate also holds in the case that  $|\eta| = 0$ . Set  $f_t(x) := E_\alpha(t^\alpha P_1(x)/P_2(x))(a - Q(x))^{-\gamma}, t \geq 0, x \in \mathbb{R}^n$ . It is clear that there exists  $L > 0$  such that  $|Q(x)| \geq M|x|^r, |x| \geq L$  and  $|a - Q(x)| \geq M|x|^r, |x| \geq L$ . Using [26, (3.19)], (31) and the product rule, it readily follows that, for every  $t \geq 0, x \in \mathbb{R}^n$ , and for every  $\eta \in \mathbb{N}_0^n$  with  $|\eta| \leq k$ ,

$$\begin{aligned} & \left| D^\eta \left( E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) (a - Q(x))^{-\gamma} \right) \right| \\ & \leq M (1 + t^{\max(1,\alpha)|\eta|}) e^{\omega t} (1 + |x|)^{\eta \left( \max \left( N, \frac{N_1+N_2}{\min(1,\alpha)} \right) - 1 \right) - r\gamma}. \end{aligned} \tag{32}$$

We obtain similarly that, for every  $t \geq 0, x \in \mathbb{R}^n$ , and for every  $\eta \in \mathbb{N}_0^n$  with  $|\eta| \leq k$ ,

$$\begin{aligned} & \left| D^\eta \left( P_2(x)^{-1} E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) (a - Q(x))^{-\gamma} \right) \right| \\ & \leq M (1 + t^{\max(1,\alpha)|\eta|}) e^{\omega t} (1 + |x|)^{\eta \left( \max \left( N, \frac{N_1+N_2}{\min(1,\alpha)} \right) - 1 \right) - r\gamma}. \end{aligned} \tag{33}$$

Suppose first that  $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$ . By performing the Laplace transform and using [3, (1.26)] we get that

$$\begin{aligned} (a - Q(x))^{-\gamma} P_2(x) &= P_2(x) E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) (a - Q(x))^{-\gamma} \\ &\quad - \int_0^t g_\alpha(t-s) P_1(x) E_\alpha \left( s^\alpha \frac{P_1(x)}{P_2(x)} \right) (a - Q(x))^{-\gamma} ds, \quad t \geq 0, x \in \mathbb{R}^n. \end{aligned}$$

Keeping in mind this equality, Lemma 4.1 and the facts that  $R_\alpha(t)\overline{P(A)} \subseteq \overline{P(A)}R_\alpha(t)$ ,  $t \geq 0$  for any complex polynomial  $P(x)$ , we can repeat literally the arguments used in the proof of [18, Theorem 2.1] so as to conclude that

$$\overline{P_1(A)} \int_0^t g_\alpha(t-s)R_\alpha(s)x \, ds = R_\alpha(t)\overline{P_2(A)}x - C\overline{P_2(A)}x, \quad t \geq 0, x \in D(\overline{P_2(A)}),$$

which clearly implies by Theorem 2.8 that, for every  $x \in D(\overline{P_2(A)})$  ( $x \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$ ), the function  $t \mapsto u(t) \equiv R_\alpha(t)x$ ,  $t \geq 0$  is a mild (strong) solution of (DFP). The proof of theorem in the case of general space  $E$  can be completed by performing the Laplace transform once more. The proof is quite similar if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ , and the only non-trivial thing here is to show the strong continuity of the operator family  $(R_\alpha(t))_{t \geq 0}$ . The arguments used in the proof of [18, Theorem 2.1] show that the mapping  $t \mapsto R_\alpha(t)f$ ,  $t \geq 0$  is continuous for every  $f \in R(\overline{P_2(A)})$ . But,  $\overline{P_2(A)}|_{\mathcal{S}(\mathbb{R}^n)}$  is a linear topological homeomorphism of the space  $\mathcal{S}(\mathbb{R}^n)$ , which along with the exponential boundedness of  $(R_\alpha(t))_{t \geq 0}$  implies the continuity of mapping  $t \mapsto R_\alpha(t)f$ ,  $t \geq 0$  for any  $f \in L^p(\mathbb{R}^n)$ .  $\square$

We can prove in a similar way the following extension of [18, Theorem 2.2], let us only note that the choice of regularizing operator  $C$  is slightly different now.

**Theorem 4.3.** *Suppose  $0 < \alpha < 2$ ,  $\omega \geq 0$ ,  $P_1(x)$  and  $P_2(x)$  are non-zero complex polynomials,  $N_1 = dg(P_1(x))$ ,  $N_2 = dg(P_2(x))$ ,  $\beta > \frac{n(N_1+N_2)}{2 \min(1,\alpha)}$  (resp.  $\beta \geq n|\frac{1}{p} - \frac{1}{2}| \frac{(N_1+N_2)}{\min(1,\alpha)}$ ), if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ ,  $P_2(x) \neq 0$ ,  $x \in \mathbb{R}^n$  and (25) holds. Set*

$$R_\alpha(t) := \left( E_\alpha \left( t^\alpha \frac{P_1(x)}{P_2(x)} \right) (1 + |x|^2)^{-\beta/2} \right) (A), \quad t \geq 0.$$

Then  $(R_\alpha(t))_{t \geq 0} \subseteq L(E)$  is a global exponentially bounded  $(g_\alpha, R_\alpha(0))$ -regularized resolvent family for (DFP),  $(R_\alpha(t))_{t \geq 0}$  is norm continuous provided  $\beta > \frac{n(N_1+N_2)}{2 \min(1,\alpha)}$ , and (26) holds.

**Remark 4.4.** (i) *The assumption  $P_2(x) \neq 0$ ,  $x \in \mathbb{R}^n$  implies that the operator  $\overline{P_2(A)}$  is injective. Speaking-matter-of-factly, the assumption  $\overline{P_2(A)}f = 0$ , in combination with  $(\triangleright)$  and the fact that  $P_2(\cdot)^{-1}\phi(\cdot) \in \mathcal{S}(\mathbb{R}^n)$ ,  $\phi \in \mathcal{S}(\mathbb{R}^n)$  implies that  $\phi(A)f = 0$ ,  $\phi \in \mathcal{S}(\mathbb{R}^n)$ ; hence,  $f = 0$ . Consider now the situation of Theorem 4.2, with  $E$  being a general space and  $\gamma > \frac{n \max(N_1, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$ . Set  $G_\alpha(t) := (P_2(\cdot)^{-1}f_i(\cdot))(A)$ ,  $t \geq 0$ . Applying again  $(\triangleright)$ , we get that  $\overline{P_2(A)}G_\alpha(t)x = R_\alpha(t)x$ ,  $t \geq 0$ ,  $x \in E_0$ . By the closedness of  $\overline{P_2(A)}$ , the above equality holds for any  $x \in E$  so that  $G_\alpha(t)x = \overline{P_2(A)}^{-1}R_\alpha(t)x$ ,  $t \geq 0$ ; furthermore,  $(G_\alpha(t))_{t \geq 0} \subseteq L(E)$  is a strongly continuous operator family. Then the Laplace transform and the identity  $A(g_\alpha * G_\alpha)(t)x = R_\alpha(t)x - Cx$ ,  $t \geq 0$ ,  $x \in E$  can be used to prove that  $\lambda^{\alpha-1}(\lambda^\alpha B - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}G_\alpha(t)x \, dt$  for any  $x \in E$  and  $\lambda > 0$  sufficiently large; therefore, the conditions (i.2)–(i.3) and (ii.1) stated in Theorem 2.8 holds, with  $a(t) = g_\alpha(t)$  and  $k(t) = 1$ , which continues to hold in any case set out in Theorem 4.2–Theorem 4.3 and Remark 4.5 below. It should be also observed that  $(G_\alpha(t))_{t \geq 0}$  is an exponentially equicontinuous  $(g_\alpha, C)$ -regularized resolvent family generated by  $\overline{P_1(A)}$ ,  $\overline{P_2(A)}$  (cf. [22] for the notion and more details), and that for each  $f \in D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$ , the function  $u(t) := R_\alpha(t)x$ ,  $t \geq 0$  is a unique solution of the following Cauchy problem:*

$$(P)_L : \begin{cases} u \in C([0, \infty) : [D(\overline{P_1(A)})]) \cap C([0, \infty) : [D(\overline{P_2(A)})]), \\ \overline{P_2(A)}\mathbf{D}_t^\alpha u(t) = \overline{P_1(A)}u(t), \quad t \geq 0, \\ u(0) = Cx; \quad u^{(j)}(0) = 0, \quad 1 \leq j \leq \lceil \alpha \rceil - 1. \end{cases}$$

A similar result holds for the second order degenerate equations considered in the next subsection, and for the first order degenerate equations, in the case that the requirements of Theorem 4.3 (the part (ii) of this remark, or Remark 4.5 below) hold; cf. [22].

(ii) It is worth noting that Theorem 4.2 and Theorem 4.3 can be strengthened in the following way. Suppose that the estimate (27) holds with the number  $N_1 + N_2$  replaced by some other number  $\sigma \geq 0$ , and that  $|P_1(x)/P_2(x)| \leq M(1 + |x|)^{\sigma_1}$ ,  $x \in \mathbb{R}^n$  for some  $\sigma_1 \in [0, N_1]$ . Put  $W := \sigma + \chi_{(0,1)}(\alpha)\sigma_1(\alpha^{-1} - 1)$ . Then the argumentation used in proving the estimate (31), along with the inequality [26, (3.19)] and the first estimate appearing in the proof of [26, Theorem 4.3], implies that the following holds:

Theorem 4.2: The assertion of this theorem continues to hold for any  $\gamma > n \max(N, W)/2r$  (resp.  $\gamma = \frac{n}{r}|_p^1 - \frac{1}{2}|\max(N, W)$ , if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ ).

Theorem 4.3: The assertion of this theorem continues to hold for any  $\beta > \frac{nW}{2}$  (resp.  $\beta = n|_p^1 - \frac{1}{2}|W$ , if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ ).

If  $V_2 \geq 0$  and if for each  $\eta \in \mathbb{N}_0^n$  there exists  $M_\eta > 0$  such that  $|D^\eta(P_2(x)^{-1})| \leq M_\eta(1 + |x|)^{|\eta|(V_2-1)}$ ,  $x \in \mathbb{R}^n$  (this holds with  $V_2 = N_2$ ), then similarly as in the first part of this remark we can prove that the conditions (i.2)-(i.3) and (ii.1) stated in Theorem 2.8 holds, with  $a(t) = g_\alpha(t)$  and  $k(t) = 1$ , if:  $\gamma > \frac{n \max(N, V_2, W)}{2r}$ , resp.  $\gamma \geq \frac{n}{r}|_p^1 - \frac{1}{2}|\max(N, V_2, W)$  (Theorem 4.2);  $\beta > \frac{n \max(V_2, W)}{2}$ , resp.  $\beta \geq n|_p^1 - \frac{1}{2}|\max(V_2, W)$  (Theorem 4.3). Observe, finally, that there is a great number of concrete examples where we can further refine the obtained theoretical results by using direct computations ([17]).

(iii) The estimate (25) is very restrictive in the case that  $\alpha \in (0, 1)$ . If  $1 < \alpha < 2$  and  $\omega \geq 0$ , then by the proof of [26, Theorem 4.2], cf. also [18, Remark 2.1(i)], the condition  $\frac{P_1}{P_2}(\mathbb{R}^n) \subseteq \mathbb{C} \setminus (\omega + \Sigma_{\alpha\pi/2})$  implies the validity of (25).

(iv) Let  $t \mapsto u(t)$ ,  $t \geq 0$  be a mild solution of the problem (1) with the operators  $A$  and  $B$  replaced respectively by  $\overline{P_1(A)}$  and  $\overline{P_2(A)}$ . Then it can be simply proved that the variation of parameters formula  $(R_\alpha * f)(t) = (\overline{CP_2(A)} * u)(t)$  holds for any  $t \geq 0$ . This implies that we can consider the  $C$ -wellposedness of the inhomogeneous degenerate Cauchy problem  $(DFP)_f$ , obtained by adding the term  $f(t)$  on the right hand side of  $(DFP)$ . All this has been seen many times and we shall skip details for the sake of brevity; the interested reader may consult [20, Subsection 2.1.1] for further information concerning abstract fractional Cauchy problems.

(v) Let  $0 < \alpha_0 < \alpha < 2$ , and let the assumptions of Theorem 4.2 (Theorem 4.3) hold. Using Theorem 3.3(ii), it readily follows that there exists an exponentially bounded, analytic  $(g_{\alpha_0}, R_{\alpha_0}(0))$ -regularized resolvent family  $(R_{\alpha_0}(t))_{t \geq 0} \subseteq L(E)$  for  $(DFP)$ , of angle  $\min(((\alpha/\alpha_0) - 1)\pi/2, \pi/2)$ ; furthermore, if  $E = L^2(\mathbb{R}^n)$  and  $\omega = 0$ , then the angle of analyticity equals  $\min(((\alpha/\alpha_0) - 1)\pi/2, \pi)$  and can be strictly greater than  $\pi/2$ .

**Remark 4.5.** In this remark, we would like to explain how one can reformulate the assertions of Theorem 4.2 and Theorem 4.3 in  $E_l$ -type spaces. Let  $E$  be one of the spaces  $L^p(\mathbb{R}^n)$  ( $1 \leq p \leq \infty$ ),  $C_0(\mathbb{R}^n)$ ,  $C_b(\mathbb{R}^n)$ ,  $BUC(\mathbb{R}^n)$  and let  $0 \leq l \leq n$ . Let  $\mathbf{T}_1\langle \cdot \rangle$  possess the same meaning as in [41], let  $a_\eta \in \mathbb{C}$ ,  $0 \leq |\eta| \leq N_1$ , and let  $b_\eta \in \mathbb{C}$ ,  $0 \leq |\eta| \leq N_2$ . Assume that the operators  $P_1(D)f \equiv \sum_{|\eta| \leq N_1} a_\eta D^\eta f$  and  $P_2(D)f \equiv \sum_{|\eta| \leq N_2} b_\eta D^\eta f$  act with their maximal distributional domains. Then  $P_1(D)$  and  $P_2(D)$  are closed linear operators on  $E_l$ ; in the sequel, we will assume that  $P_1(D) \neq 0$  and  $P_2(D) \neq 0$ . Let  $\omega \geq 0$  be such that (25) holds. Then it can be easily seen that  $P(D)$  generates an exponentially equicontinuous  $(g_\alpha, I)$ -regularized resolvent family  $(S_\alpha(t))_{t \geq 0}$  in the space  $E_n$  and that the conditions (i.2)-(i.3) and (ii.1) stated in Theorem 2.8 hold, with  $a(t) = g_\alpha(t)$  and  $k(t) = 1$ . Let  $\gamma$  ( $\beta$ ) have the same value as in the formulation of Theorem 4.2 (Theorem 4.3). Then the estimates (32)-(33) continue to hold, and slight modifications of the proofs of [41, Theorem 2.2, Theorem 2.4] show that the following holds (in our opinion, the proofs of Theorem 4.2 and Theorem 4.3 are much simpler than those of [41, Theorem 2.2, Theorem 2.4] in the case that  $l = 0$  and  $E \neq L^\infty(\mathbb{R}^n)$ ,  $E \neq C_b(\mathbb{R}^n)$ ):

(i) Theorem 4.2: Set  $R_\alpha(t) =: \mathbf{T}_1\langle E_\alpha(t^\alpha P_1(x)/P_2(x))(a - Q(x))^{-\gamma} \rangle$ ,  $t \geq 0$ . Then  $(R_\alpha(t))_{t \geq 0}$  is an exponentially equicontinuous  $(g_\alpha, R_\alpha(0))$ -regularized resolvent family for  $(DFP)$ ,  $(R_\alpha(t))_{t \geq 0}$  is 'norm continuous' provided  $\gamma > \frac{n \max(N, \frac{N_1+N_2}{\min(1,\alpha)})}{2r}$ , in the sense that, for every bounded subset  $B$  of  $E_l$  and for every  $\eta \in \mathbb{N}_0^l$ , the mapping  $t \mapsto \sup_{f \in B} q_\eta(R_\alpha(t)f)$ ,  $t \geq 0$  is continuous. The estimate (26) becomes

$$q_\eta(R_\alpha(t)f) \leq M(1 + t^{\max(1,\alpha)n/2})e^{\omega t} q_\eta(f), \quad t \geq 0, \quad f \in E_l, \quad \eta \in \mathbb{N}_0^l, \text{ resp.},$$

$$q_\eta(R_\alpha(t)f) \leq M(1 + t^{\max(1,\alpha)n|_p^1 - \frac{1}{2}|})e^{\omega t} q_\eta(f), \quad t \geq 0, \quad f \in E_l, \quad \eta \in \mathbb{N}_0^l, \tag{34}$$

with  $M$  being independent of  $f \in E_1$  and  $\eta \in \mathbb{N}_0^l$ .

(ii) *Theorem 4.3:* Set  $R_\alpha(t) =: \mathbf{T}_1 \langle E_\alpha(t^\alpha P_1(x)/P_2(x))(1 + |x|^2)^{-\beta/2} \rangle$ ,  $t \geq 0$ . Then  $(R_\alpha(t))_{t \geq 0}$  is an exponentially equicontinuous  $(g_\alpha, R_\alpha(0))$ -regularized resolvent family for (DFP),  $(R_\alpha(t))_{t \geq 0}$  is ‘norm continuous’ provided  $\beta > \frac{n}{2} \frac{(N_1+N_2)}{\min(1,\alpha)}$  and (34) hold.

Notice also that  $(g_\alpha, R_\alpha(0))$ -regularized resolvent families for (DFP), constructed in this remark (Theorem 4.2-Theorem 4.3), satisfy that  $R_\alpha(t)R_\alpha(s) = R_\alpha(s)R_\alpha(t)$ ,  $t, s \geq 0$ , as well as that  $R_\alpha(t)P(D) \subseteq P(D)R_\alpha(t)$ ,  $t \geq 0$  ( $R_\alpha(t)\overline{P(A)} \subseteq \overline{P(A)}R_\alpha(t)$ ,  $t \geq 0$ ) for any complex polynomial  $P(x)$ . The final conclusions of Remark 4.4(ii) remain true for  $(g_\alpha, R_\alpha(0))$ -regularized resolvent families for (DFP), when act on  $E_l$ -type spaces.

#### 4.1. Degenerate Second Order Equations Associated with Abstract Differential Operators

The main objective in this subsection is to prove some results on the C-wellposedness of the following abstract degenerate Cauchy problem of second order

$$(DFP)_2 : \begin{cases} \frac{d^2}{dt^2} \overline{P_2(A)}u(t) = \overline{P_1(A)}u(t), & t \geq 0, \\ u(0) = Cx, u'(0) = 0. \end{cases}$$

Keeping in mind the results clarified by now, as well as the analyses contained in the papers [45] and [41]-[43], the consideration of degenerate second order equations is similar to that of degenerate fractional equations of order  $\alpha \in (0, 2)$ ; because of that, we shall only outline the main details and omit the proofs. As before, we assume that  $P_1(x)$  and  $P_2(x)$  are non-zero complex polynomials in  $n$  variables, as well as that  $N_1 = dg(P_1(x))$  and  $N_2 = dg(P_2(x))$ . Set  $F_t(z) := E_2(t^2z)$ ,  $t \geq 0, z \in \mathbb{C}$ , and  $\Omega(\omega) := \{\lambda^2 : \Re \lambda > \omega\}$ , if  $\omega > 0$  and  $\Omega(\omega) := \mathbb{C} \setminus (-\infty, -\omega^2]$ , if  $\omega \leq 0$ . Given  $l \geq 0$  and  $t \geq 0$  in advance, set

$$Q_l(t) := \begin{cases} (1 + t^l)e^{\omega t}, & \text{if } \omega > 0, \\ 1 + t^{2l}, & \text{if } \omega = 0, \\ 1 + t^l, & \text{if } \omega < 0. \end{cases}$$

Suppose now that  $P_2(x) \neq 0, x \in \mathbb{R}^n$  and  $P_1(x)/P_2(x) \notin \Omega(\omega), x \in \mathbb{R}^n$ . Then, for every  $\eta \in \mathbb{N}_0^n$  with  $|\eta| > 0$ , the equation (28) reads as follows:

$$D^\eta E_2 \left( t^2 \frac{P_1(x)}{P_2(x)} \right) = \sum_{j=1}^{|\eta|} F_t^{(j)} \left( \frac{P_1(x)}{P_2(x)} \right) R_{\eta,j}(x), \quad t \geq 0, x \in \mathbb{R}^n, \tag{35}$$

where  $R_{\eta,j}(x)$  is a finite sum of terms like  $\prod_{w=1}^{s_j} D^{\eta_{jw}} \left( \frac{P_1(x)}{P_2(x)} \right)$  with  $|\eta_{jw}| > 0$  ( $1 \leq w \leq s_j$ ) and  $|\eta_{j1}| + \dots + |\eta_{j,s_j}| \leq |\eta|$ . Due to the computation established in [45, Lemma 2.1], we have that, for every  $l \in \mathbb{N}_0$ ,  $|F_t^{(l)}(P_1(x)/P_2(x))| \leq MQ_l(t)$ ,  $t \geq 0, x \in \mathbb{R}^n$ . Combining this estimate with (35), and repeating verbatim the arguments given in the proof of Theorem 4.2 (cf. also Remark 4.4), it can be easily seen that the following two theorems hold good.

**Theorem 4.6.** *Suppose that  $P_1(x)$  and  $P_2(x)$  are non-zero complex polynomials,  $N_1 = dg(P_1(x)), N_2 = dg(P_2(x)), P_2(x) \neq 0, x \in \mathbb{R}^n, \omega \in \mathbb{R}, P_1(x)/P_2(x) \notin \Omega(\omega), x \in \mathbb{R}^n, N \in \mathbb{N}$  and  $r \in (0, N]$ . Let  $Q(x)$  be an  $r$ -coercive complex polynomial of degree  $N, a \in \mathbb{C} \setminus Q(\mathbb{R}^n)$ , let for each  $\eta \in \mathbb{R}^n$  with  $|\eta| > 0$  the estimate (27) holds with the number  $N_1 + N_2$  replaced by  $\sigma \geq 0$  (the choice  $\sigma = N_1 + N_2$  is always possible), and let  $\gamma > \frac{n \max(N,\sigma)}{2r}$  (resp.  $\gamma = \frac{n}{r} \frac{1}{p} - \frac{1}{2} \max(N, \sigma)$ , if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ ). Set*

$$R_2(t) := \left( E_2 \left( t^2 \frac{P_1(x)}{P_2(x)} \right) (a - Q(x))^{-\gamma} \right) (A), \quad t \geq 0.$$

Then  $(R_2(t))_{t \geq 0} \subseteq L(E)$  is a global exponentially bounded  $(g_2, R_2(0))$ -regularized resolvent family for  $(DFP)_2$ ,  $(R_2(t))_{t \geq 0}$  is norm continuous provided  $\gamma > \frac{n \max(N,\sigma)}{2r}$ , and the following holds:

$$\|R_2(t)\| \leq MQ_{n/2}(t), \quad t \geq 0, \text{ resp., } \|R_2(t)\| \leq MQ_{n \frac{1}{p} - \frac{1}{2}}(t), \quad t \geq 0. \tag{36}$$

**Theorem 4.7.** Suppose that  $P_1(x)$  and  $P_2(x)$  are non-zero complex polynomials,  $N_1 = dg(P_1(x))$ ,  $N_2 = dg(P_2(x))$ ,  $P_2(x) \neq 0$ ,  $x \in \mathbb{R}^n$ ,  $\omega \in \mathbb{R}$  and  $P_1(x)/P_2(x) \notin \Omega(\omega)$ ,  $x \in \mathbb{R}^n$ . Let for each  $\eta \in \mathbb{R}^n$  with  $|\eta| > 0$  the estimate (27) holds with the number  $N_1 + N_2$  replaced by  $\sigma \geq 0$  (the choice  $\sigma = N_1 + N_2$  is always possible), and let  $\beta > \frac{n\sigma}{2}$  (resp.  $\beta = n|\frac{1}{p} - \frac{1}{2}|\sigma$ , if  $E = L^p(\mathbb{R}^n)$  for some  $1 < p < \infty$ ). Set

$$R_2(t) := \left( E_2 \left( t^2 \frac{P_1(x)}{P_2(x)} \right) (1 + |x|^2)^{-\beta/2} \right) (A), \quad t \geq 0.$$

Then  $(R_2(t))_{t \geq 0} \subseteq L(E)$  is a global exponentially bounded  $(g_2, R_2(0))$ -regularized resolvent family for  $(DFP)_2$ ,  $(R_2(t))_{t \geq 0}$  is norm continuous provided  $\beta > \frac{n\sigma}{2}$ , and (36) holds.

**Remark 4.8.** (i) Suppose that

$$\Re \left( P_1(x)/P_2(x) \right) \leq -\zeta|x|^r + \zeta_1, \quad x \in \mathbb{R}^n, \tag{37}$$

for some positive real numbers  $r, \zeta, \zeta_1 > 0$ . Then the Lagrange mean value theorem for vector-valued functions implies that  $\sigma \geq r$ , and by the proof of [45, Lemma 2.1], we have that there exist numbers  $L \geq 1$  and  $M \geq 1$  such that, for every  $j \in \mathbb{N}_0$  with  $j < n/2$ , the following holds  $|F_t^{(j)}(P_1(x)/P_2(x))| \leq M Q_j(t)(1 + |x|)^{-j/2}$ ,  $t \geq 0$ ,  $|x| \geq L$ . Unfortunately, the above does not guarantee that we can refine the results clarified in Theorem 4.6 and Theorem 4.7 by replacing the number  $\sigma$  with  $\sigma - (r/2)$ , unless  $P_2(x) \equiv 1$ . Observe, however, that the refinement of this type is possible if  $\omega \leq 0$  (in this case we can estimate the derivatives of function  $E_2(t^2 P_1(x)/P_2(x))$  by using the formula appearing in the second line of the proof of [43, Theorem 4.1]).

- (ii) Let  $V_2 \geq 0$ . If for each  $\eta \in \mathbb{N}_0^n$  there exists  $M_\eta > 0$  such that  $|D^\eta(P_2(x)^{-1})| \leq M_\eta(1 + |x|)^{\eta(V_2-1)}$ ,  $x \in \mathbb{R}^n$  (recall that the choice  $V_2 = N_2$  is always possible), and if we replace in the formulations of Theorem 4.6 and Theorem 4.7 the number  $\sigma$  with  $\max(\sigma, V_2)$ , then it is not hard to verify that the conditions (i.2)-(i.3) and (ii.1) stated in Theorem 2.8 holds, with  $a(t) = t$  and  $k(t) = 1$ .
- (iii) The assertions of Theorem 4.6 and Theorem 4.7, as well as the conclusions stated in the first and second part of this remark, continue to hold with suitable modifications in the setting of  $E_I$ -type spaces; cf. also [43, Theorem 4.1-Theorem 4.2].

We close the paper with the following illustrative example.

**Example 4.9.** (i) Let  $1 < p < \infty$ ,  $0 < \alpha < 2$ ,  $l \in \mathbb{N}$ ,  $E = L^p(\mathbb{R}^2)$ , and let the fractional Sobolev space  $S^{\alpha,p}(\mathbb{R}^2)$  be defined in the sense of [29, Definition 12.3.1, p. 297]. Consider the following degenerate fractional Cauchy problem:

$$(P) : \begin{cases} \mathbf{D}_t^\alpha [u_{xx} + u_{xy} + u_{yy} - u] = e^{-i\alpha \frac{x}{2}} [(-1)^{l+1} \frac{\partial^{2l}}{\partial x^{2\alpha}} u + u_{yy}], & t \geq 0, \\ u(0, x) = \phi(x); u_t(0, x) = 0 \text{ if } \alpha \geq 1, \end{cases}$$

cf. Theorem 4.3 with  $P_1(x, y) = e^{-i\alpha \frac{x}{2}}(x^{2l} + y^2)$ ,  $P_2(x, y) = x^2 + xy + y^2 + 1$  and  $\omega = 0$ . Then it can be easily seen that the conditions stated in Remark 4.4(ii) hold with  $\sigma = \sigma_1 = 2l - 2$ , so that for each  $\beta \geq n|\frac{1}{p} - \frac{1}{2}|(2l - 2)(1 + \chi_{(0,1)}(\alpha)(\alpha^{-1} - 1))$  there exists a global exponentially bounded  $(g_\alpha, R_\alpha(0))$ -regularized resolvent family for the corresponding problem (DFP), obeying the properties (i.2)-(i.3) and (ii.1) of Theorem 2.8 with  $a(t) = g_\alpha(t)$  and  $k(t) = 1$ . Hence, there exists a unique strong solution of problem (P) provided that  $\phi \in S^{2l+\beta,p}(\mathbb{R}^2)$ .

- (ii) Let  $1 < p < \infty$ ,  $E = L^p(\mathbb{R}^n)$  and  $Q \in \mathbb{N} \setminus \{1\}$ . Consider the following degenerate second order Cauchy problem:

$$(P_2) : \begin{cases} \frac{\partial^2}{\partial t^2} \Delta_x u(t, x) = \sum_{|\eta| \leq Q} a_\eta D^\eta u(t, x), & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = \phi(x), u_t(0, x) = 0, \end{cases}$$

then  $P_1(x) = \sum_{|\eta| \leq Q} a_\eta x^\eta$  and  $P_2(x) = -|x|^2$  ( $x \in \mathbb{R}^n$ ). Assuming that the polynomial  $P_1(x)$  is positive, as well as that the estimate (27) holds with some number  $\sigma \geq 0$  and that the condition stated in Remark 4.8(ii) holds with some number  $V_2 \geq 0$ , then there exists a unique strong solution of problem  $(P_2)$  provided that  $\phi \in S^{Q+n\frac{1}{p}-\frac{1}{2}|\max(\sigma, V_2), p}(\mathbb{R}^n)$ .

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