Filomat 31:3 (2017), 559–573
DOI 10.2298/FIL1703559Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Solving the General Split Common Fixed-Point Problem of Quasi-Nonexpansive Mappings without Prior Knowledge of Operator Norms

Jing Zhao[a], Songnian He[a]

[a]*College of Science, Civil Aviation University of China, Tianjin, 300300, P.R. China*

Abstract. Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The general multiple-set split common fixed-point problem under consideration in this paper is to

$$\text{find } \ x \in \cap_{i=1}^{p} F(U_i), \ \ y \in \cap_{j=1}^{r} F(T_j) \ \text{ such that } \ Ax = By, \tag{1}$$

where $p, r \geq 1$ are integers, $U_i : H_1 \to H_1$ $(1 \leq i \leq p)$ and $T_j : H_2 \to H_2$ $(1 \leq j \leq r)$ are quasi-nonexpansive mappings with nonempty common fixed-point sets $\cap_{i=1}^{p} F(U_i) = \cap_{i=1}^{p}\{x \in H_1 : U_i x = x\}$ and $\cap_{j=1}^{r} F(T_j) = \cap_{j=1}^{r}\{x \in H_2 : T_j x = x\}$. Note that, the above problem (1) allows asymmetric and partial relations between the variables x and y. If $H_2 = H_3$ and $B = I$, then the general multiple-set split common fixed-point problem (1) reduces to the multiple-set split common fixed-point problem proposed by Censor and Segal [J. Convex Anal. 16(2009), 587-600]. In this paper, we introduce simultaneous parallel and cyclic algorithms for the general split common fixed-point problems (1). We introduce a way of selecting the stepsizes such that the implementation of our algorithms does not need any prior information about the operator norms. We prove the weak convergence of the proposed algorithms and apply the proposed algorithms to the multiple-set split feasibility problems. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let I denote the identity operator on H. Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set.

Recall that the convex feasibility problem (CFP) is formulated as finding a point x^* satisfying the property:

$$x^* \in \bigcap_{i=1}^{p} C_i, \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 47H09, 47H10; Secondary 54H25
Keywords. split common fixed-point problem, multiple-set split feasibility problem, quasi-nonexpansive mapping, weak convergence, iterative algorithm, Hilbert space
Received: 06 December 2014; Accepted: 08 July 2015
Communicated by Naseer Shahzad
Email addresses: zhaojing200103@@163.com (Jing Zhao), hesongnian@@163.com (Songnian He)

where $p \geq 1$ is an integer and each C_i is a nonempty closed convex subset of H. Note that the CFP has received a lot of attention due to its extensive applications in many applied disciplines as diverse as approximation theory, image recovery and signal processing, control theory, biomedical engineering, communications and geophysics (see [2, 12, 21] and the references therein).

The multiple-set split feasibility problem (MSFP) which finds application in intensity modulated radiation therapy was proposed in [7] and is formulated as finding a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} C_i \ \text{ such that } \ Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{3}$$

where $p, r \geq 1$ are integers, $\{C_i\}_{i=1}^{p}$ are nonempty closed convex subsets of real Hilbert space H_1, $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of real Hilbert space H_2 and $A : H_1 \to H_2$ is a bounded linear operator. The MSFP (3) with $p = r = 1$ is known as the split feasibility problem (SFP) originally introduced in Censor and Elfving [8] which is formulated as finding a point x^* with the property:

$$x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{4}$$

where C and Q are nonempty closed convex subset of H_1 and H_2, respectively. The SFP (4) and MSFP (3) model image retrieval [8] and intensity-modulated radiation therapy [6], and have recently been investigated by many researchers([4, 9, 20, 22, 23, 25, 26]).

Let $A^{-1}(Q) = \{x : Ax \in Q\}$, then the MSFP (3) can be viewed as a special case of the CFP (2) since (3) can be rewritten as

$$x^* \in \bigcap_{i=1}^{p+r} C_i, \ \ C_{p+j} = A^{-1}(Q_j), \ \ 1 \leq j \leq r.$$

However, the methodologies for studying the MSFP (3) are actually different from those for the CFP (2) in order to avoid usage of the inverse A^{-1}. In other words, the methods for solving CFP (2) may not apply to solve the MSFP (3) straightforwardly without involving the inverse A^{-1}.

Assuming that the SFP (4) is consistent (i.e., (4) has a solution), it is not hard to see that $x^* \in C$ solves (4) if and only if it solves the fixed point equation

$$x = P_C(I - \gamma A^*(I - P_Q)A)x, \ \ x \in C, \tag{5}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. To solve the SFP (4), Byrne [5] proposed his CQ algorithm that involves only the orthogonal projections onto C and Q and does not need to compute the inverse A^{-1} to solve the SFP (4). The CQ algorithm is defined as follows:

$$x_{k+1} = P_C(I - \gamma A^*(I - P_Q)A)x_k, \ \ k \geq 1,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Since every closed convex subset of a Hilbert space is the fixed point set of its associating projection, the problems (3) and (4) are all special cases of the so-called multiple-set split common fixed-point problem (MSCFP) which is formulated as find a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} F(U_i) \ \text{ such that } \ Ax^* \in \bigcap_{j=1}^{r} F(T_j), \tag{6}$$

where $p, r \geq 1$ are integers, $\{U_i\}_{i=1}^{p} : H_1 \to H_1$, $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ are nonlinear operators and $A : H_1 \to H_2$ is a bounded linear operator. In particular, if $p = r = 1$, then (6) reduces to find a point x^* with the property:

$$x^* \in F(U) \ \text{ such that } \ Ax^* \in F(T), \tag{7}$$

which is usually called the solution set of the two-sets of SCFP.

The concept of SCFP in finite-dimensional Hilbert spaces was first introduced by Censor and Segal [10] who proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm for the SCFP of nonexpansive operators with nonempty fixed-point sets:

$$x_{k+1} = U(x_k + \gamma A^t(T - I)Ax_k), \ \ k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition).

Recently, Moudafi [17] introduced a new split common fixed-point problem (SCFP). Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two firmly quasi-nonexpansive operators. The SCFP in [17] is to find x^*, y^* with the property:

$$x^* \in F(U), \ \ y^* \in F(T) \ \text{ such that } \ Ax^* = By^*, \tag{8}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situation, for instance in decomposition methods for PDEs, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see [1]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [6]). If $H_2 = H_3$ and $B = I$, then the SCFP (8) reduces to the two-sets of the SCFP (7).

For solving the SCFP (8), Moudafi [17] introduced the following alternating algorithm

$$\begin{aligned} x_{k+1} &= U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} &= T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)) \end{aligned} \tag{9}$$

for firmly quasi-nonexpansive operators U and T, where non-decreasing sequence $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, and λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively.

In [3], Byrne and Moudafi consider and study the algorithms to solve the approximate split equality problem (ASEP), which can be regarded as obtaining the consistent case and the inconsistent case of the split equality problem (SEP):

$$x \in C, \ \ y \in Q \ \text{ such that } \ Ax = By, \tag{10}$$

where $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets. There they proposed a simultaneous iterative algorithm:

$$\begin{aligned} x_{k+1} &= P_C(x_k - \gamma_k A^T(Ax_k - By_k)), \\ y_{k+1} &= P_Q(y_k + \gamma_k B^T(Ax_k - By_k)), \end{aligned} \tag{11}$$

where $\varepsilon \leq \gamma_k \leq \frac{2}{\lambda_G} - \varepsilon$, λ_G stand for the spectral radius of G^TG and $G = [A \ \ -B]$.

Very recently, Moudafi [18] introduced the following simultaneous iterative method to solve SCFP (8):

$$\begin{aligned} x_{k+1} &= U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} &= T(y_k + \gamma_k B^*(Ax_k - By_k)) \end{aligned} \tag{12}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_k \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively.

In this paper, inspired and motivated by the works mentioned above, the MSFP under consideration is nothing but to find x^*, y^* with the property:

$$x^* \in \bigcap_{i=1}^{p} C_i, \ \ y^* \in \bigcap_{j=1}^{r} Q_j, \ \text{ such that } \ Ax^* = By^*, \tag{13}$$

and the general MSCFP is to find x^*, y^* with the property:

$$x^* \in \bigcap_{i=1}^{p} F(U_i), \quad y^* \in \bigcap_{j=1}^{r} F(T_j), \quad \text{such that} \quad Ax^* = By^*. \tag{14}$$

For example, let $H_1 = H_2 = H_3 = l_2$, we define bounded linear operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$ by $Ax = x$ and $B(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$, respectively. Let $U_1, U_2 : H_1 \to H_1$ be defined by $U_1(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots)$ and $U_2(x_1, x_2, x_3, \cdots) = (0, 0, x_1, \cdots)$. Let $T : H_2 \to H_2$ be defined by $Tx = x$. Then the general MSCFP under consideration is to find $x^* = 0$, $y^* \in \{(x_1, 0, 0, \cdots) : x_1 \in \mathbb{R}\}$ such that $x^* \in F(U_1) \cap F(U_2)$, $y^* \in F(T)$ and $Ax^* = By^* = 0$.

Note that in the algorithms (9), (11) and (12) mentioned above, the determination of the stepsize $\{\gamma_k\}$ depends on the operator (matrix) norms $||A||$ and $||B||$ (or the largest eigenvalues of A^*A and B^*B). In order to implement the above algorithms, one needs to know the operator norms of A and B (or, at least, estimate), which is in general not an easy work in practice. To overcome this difficulty, López et al [14] and Zhao and Yang [28] presented a helpful method for estimating the stepsizes which don't need prior knowledge of the operator norms for solving the SFP and MSFP, respectively. Inspired by them, in this paper, we introduce a new choice of the stepsize sequence $\{\gamma_k\}$ for the simultaneous parallel and cyclic algorithms to solve the general MSCFP (14) governed by quasi-nonexpansive operators as follows

$$\gamma_k \in \Big(0, \frac{2||Ax_k - By_k||^2}{||A^*(Ax_k - By_k)||^2 + ||B^*(Ax_k - By_k)||^2}\Big). \tag{15}$$

The advantage of our choice (15) of the stepsizes lies in the fact that no prior information about the operator norms of A and B is required, and still convergence is guaranteed. At last, we apply the proposed parallel and cyclic algorithms to solve the MSFP (13) and variational problems by resolvent mappings.

2. Preliminaries

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$ and use Γ stand for the solution set of the general SCFP (14).

- A mapping $T : H \to H$ belongs to the set Φ_N of nonexpansive mappings if

$$||Tx - Ty|| \le ||x - y||, \quad \forall (x, y) \in H \times H.$$

- A mapping $T : H \to H$ belongs in the general class Φ_Q of (possibly discontinuous) quasi-nonexpansive mappings if $F(T) \neq \emptyset$ and

$$||Tx - q|| \le ||x - q||, \quad \forall (x, q) \in H \times F(T).$$

- A mapping $T : H \to H$ belongs to the set Φ_{FN} of firmly nonexpansive mappings if

$$||Tx - Ty||^2 \le ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \quad \forall (x, y) \in H \times H.$$

- A mapping $T : H \to H$ belongs to the set Φ_{FQ} of firmly quasi-nonexpansive mappings if $F(T) \neq \emptyset$ and

$$||Tx - q||^2 \le ||x - q||^2 - ||x - Tx||^2, \quad \forall (x, q) \in H \times F(T).$$

It is easily observed that $\Phi_{FN} \subset \Phi_N \subset \Phi_Q$ and that $\Phi_{FN} \subset \Phi_{FQ} \subset \Phi_Q$. Furthermore, Φ_{FN} is well known to include resolvents and projection operators, while Φ_{FQ} contains subgradient projection operators (see, for instance, [15] and the reference therein).

A mapping $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

We remark here that a quasi-nonexpansive operator T may be not nonexpansive. See the following examples.

Example 2.1. *([13]) Let $H = \mathbb{R}$, and define a mapping by $T : H \to H$ by*

$$Tx := \begin{cases} \frac{x}{2} sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $F(T) = \{0\}$ and T is quasi-nonexpansive but not nonexpansive. It is easily to see that $T - I$ is demiclosed at origin. Similarly, let $H = l_2$, and define $T : H \to H$ by

$$Tx := \begin{cases} (\frac{x_1}{2} sin\frac{1}{x_1}, 0, 0, \cdots), & x_1 \neq 0, \\ (0, 0, \cdots), & x_1 = 0, \end{cases}$$

where $x = (x_1, x_2, x_3, \cdots)$. Then T is quasi-nonexpansive but not nonexpansive.

Example 2.2. *([11]) Let $K := \{x \in l_\infty : \|x\|_\infty \leq 1\}$. Define $T : K \to K$ by $Tx := (0, x_1^2, x_2^2, x_3^2, \cdots)$ for $x = (x_1, x_2, x_3, \cdots)$ in K. Then it is clear that T is continuous and maps K into K. Moreover, $Tx^* = x^*$ if and only if $x^* = 0$. Furthermore,*

$$\begin{aligned} \|Tx - x^*\|_\infty &= \|Tx\|_\infty = \|(0, x_1^2, x_2^2, x_3^2, \cdots)\|_\infty \\ &\leq \|(0, x_1, x_2, x_3, \cdots)\|_\infty = \|x\|_\infty = \|x - x^*\|_\infty \end{aligned}$$

for all $x \in K$. Therefore, T is quasi-nonexpansive. However, T is not nonexpansive, for if $x = (\frac{3}{4}, \frac{3}{4}, \cdots)$ and $y = (\frac{1}{2}, \frac{1}{2}, \cdots)$, it is clear that x and y belong to K. Furthermore, $\|x - y\|_\infty = \|(\frac{1}{4}, \frac{1}{4}, \cdots)\|_\infty = \frac{1}{4}$, and $\|Tx - Ty\|_\infty = \|(0, \frac{5}{16}, \frac{5}{16}, \cdots)\|_\infty = \frac{5}{16} > \|x - y\|_\infty$.

Recall that, given a nonempty closed convex subset C of a Hilbert space H, the projection $P_C : H \to C$ assigns each $x \in H$ to its closest point from C defined by

$$P_C x = argmin_{z \in C} \|x - z\|.$$

It is well known that P_C is firmly nonexpansive and $P_C x$ is characterized by the inequality:

$$P_C x \in C, \ \ \langle x - P_C x, z - P_C x \rangle \leq 0, \ \ z \in C.$$

In real Hilbert space, we easily get the following equality:

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \ \ \forall x, y \in H. \tag{16}$$

In what follows, we give some key properties of the α-relaxed operator $T_\alpha = \alpha I + (1 - \alpha)T$ which will be needed in the convergence analysis of our algorithms.

Lemma 2.3. *([19]) Let H be a real Hilbert space and $T : H \to H$ a quasi-nonexpansive mapping. Set $T_\alpha = \alpha I + (1 - \alpha)T$ for $\alpha \in [0, 1)$. Then, the following properties are reached for all $(x, q) \in H \times F(T)$:*
(i) $\langle x - Tx, x - q \rangle \geq \frac{1}{2}\|x - Tx\|^2$ and $\langle x - Tx, q - Tx \rangle \leq \frac{1}{2}\|x - Tx\|^2$,
(ii) $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2$;
(iii) $\langle x - T_\alpha x, x - q \rangle \geq \frac{1-\alpha}{2}\|x - Tx\|^2$.

Remark 2.4. *Let $T_\alpha = \alpha I + (1 - \alpha)T$, where $T : H \to H$ is a quasi-nonexpansive mapping and $\alpha \in [0, 1)$. We have $F(T_\alpha) = F(T)$ and $\|T_\alpha x - x\|^2 = (1 - \alpha)^2\|Tx - x\|^2$. It follows from (ii) of Lemma 2.3 that $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \frac{\alpha}{1-\alpha}\|T_\alpha x - x\|^2$, which implies that T_α is firmly quasi-nonexpansive when $\alpha = \frac{1}{2}$. On the other hand, if $\hat{T}$ is a firmly quasi-nonexpansive mapping, we can obtain $\hat{T} = \frac{1}{2}I + \frac{1}{2}T$, where T is quasi-nonexpansive. This is proved by the following inequalities.*

For all $x \in H$ and $q \in F(\hat{T}) = F(T)$,

$$\begin{aligned}\|Tx - q\|^2 &= \|(2\hat{T} - I)x - q\|^2 = \|(\hat{T}x - q) + (\hat{T}x - x)\|^2 \\ &= \|\hat{T}x - q\|^2 + \|\hat{T}x - x\|^2 + 2\langle \hat{T}x - q, \hat{T}x - x\rangle \\ &= \|\hat{T}x - q\|^2 + \|\hat{T}x - x\|^2 + \|\hat{T}x - q\|^2 + \|\hat{T}x - x\|^2 - \|x - q\|^2 \\ &= 2\|\hat{T}x - q\|^2 + 2\|\hat{T}x - x\|^2 - \|x - q\|^2 \\ &\le 2\|x - q\|^2 - 2\|\hat{T}x - x\|^2 + 2\|\hat{T}x - x\|^2 - \|x - q\|^2 \\ &= \|x - q\|^2,\end{aligned}$$

where $\hat{T}$ is firmly quasi-nonexpansive mapping.

Lemma 2.5. *([24]) Let $R > 0$. If E is uniformly convex then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, such that for all $x, y \in B_R(0) := \{x \in E : \|x\| \le R\}$ and for any $\alpha \in [0, 1]$, we have*

$$\|\alpha x + (1 - \alpha)y\|^2 \le \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|).$$

Lemma 2.6. *([27], Lemma 2.10) Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_r x_r\|^2 \le \sum_{i=0}^{r} \alpha_i\|x_i\|^2 - \alpha_s\alpha_t g(\|x_s - x_t\|),$$

for any $s, t \in \{0, 1, 2, \cdots, r\}$ and for $x_i \in B_R(0) := \{x \in E : \|x_k\| \le R\}$, $i = 0, 1, 2, \cdots, r$ with $\alpha_0 + \alpha_1 + \cdots + \alpha_r = 1$ and $0 \le \alpha_i \le 1$.

Lemma 2.7. *([16]) Let H be a real Hilbert space. Then for all $t \in [0, 1]$ and $x, y \in H$,*

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

Similar to technology from Lemma 2.5 to Lemma 2.6, we can get the following result from Lemma 2.7.

Lemma 2.8. *Let H be a real Hilbert space. Then*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_r x_r\|^2 \le \sum_{i=0}^{r} \alpha_i\|x_i\|^2 - \alpha_s\alpha_t\|x_s - x_t\|^2,$$

for any $s, t \in \{0, 1, 2, \cdots, r\}$ and for $x_i \in H$, $i = 0, 1, 2, \cdots, r$ with $\alpha_0 + \alpha_1 + \cdots + \alpha_r = 1$ and $0 \le \alpha_i \le 1$.

3. Algorithms without Prior Knowledge of Operator Norms

Firstly, we propose simultaneous parallel and cyclic algorithms for solving the general MSCFP (14) of quasi-nonexpansive mappings where the stepsizes don't depend on the operator norms $\|A\|$ and $\|B\|$ and prove the weak convergence of the proposed algorithms. Let $p, r \ge 1$ be integers and H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$, $B : H_2 \to H_3$, let $U_i : H_1 \to H_1$ $(1 \le i \le p)$ and $T_j : H_2 \to H_2$ $(1 \le j \le r)$ be quasi-nonexpansive mappings.

3.1. Parallel Algorithms

Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary. Let the sequences $\{\alpha_k^i\}_{k=1}^{\infty}$, $\{\beta_k^j\}_{k=1}^{\infty}$, $\{s_k^l\}_{k=1}^{\infty} \subset [0,1]$, $(0 \le i \le p,\ 0 \le j \le r,\ 1 \le l \le r)$ such that $\Sigma_{i=0}^{p}\alpha_k^i = 1$, $\Sigma_{j=0}^{r}\beta_k^j = 1$, $\alpha_k^0 + \Sigma_{l=1}^{r} s_k^l = 1$ for every $k \ge 0$. Assume that the kth iterate $x_k \in H_1$, $y_k \in H_2$ has been constructed and $Ax_k - By_k \neq 0$; then we calculate the (k + 1)th iterate (x_{k+1}, y_{k+1}) via the formula:

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k^0 u_k + \alpha_k^1 U_1(u_k) + \cdots + \alpha_k^p U_p(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k^0 v_k + \beta_k^1 T_1(v_k) + \cdots + \beta_k^r T_r(v_k) \end{cases} \qquad (Parallel\ \ Algorithm\ \ 1)$$

or

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k^0 x_k + \alpha_k^1 U_1(u_k) + \cdots + \alpha_k^p U_p(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k^0 y_k + s_k^1 T_1(v_k) + \cdots + s_k^r T_r(v_k). \end{cases} \qquad (Parallel\ \ Algorithm\ \ 2)$$

The stepsize γ_k is chosen in such a way that

$$\gamma_k \in \Big(\epsilon, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} - \epsilon\Big), \ \ k \in \Omega \tag{17}$$

for small enough $\epsilon > 0$, otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{k : Ax_k - By_k \neq 0\}$. If $Ax_k - By_k = 0$, then we take $u_k = x_k$, $v_k = y_k$ and

$$\begin{cases} x_{k+1} = \alpha_k^0 x_k + \alpha_k^1 U_1(x_k) + \cdots + \alpha_k^p U_p(x_k), \\ y_{k+1} = \beta_k^0 y_k + \beta_k^1 T_1(y_k) + \cdots + \beta_k^r T_r(y_k). \end{cases}$$

3.2. Cyclic Algorithms

Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary. Let the sequences $\{\alpha_k\}$, $\{\beta_k\} \subset [0,1]$, $i(k) = k(\text{mod } p) + 1$ and $j(k) = k(\text{mod } r) + 1$. Assume that the kth iterate $x_k \in H_1$, $y_k \in H_2$ has been constructed and $Ax_k - By_k \neq 0$; then we calculate the $(k+1)$th iterate (x_{k+1}, y_{k+1}) via the formula:

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1 - \alpha_k) U_{i(k)}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k v_k + (1 - \beta_k) T_{j(k)}(v_k) \end{cases} \qquad (Cyclic\ \ Algorithm\ \ 1)$$

or

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1 - \alpha_k) U_{i(k)}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1 - \alpha_k) T_{j(k)}(v_k). \end{cases} \qquad (Cyclic\ \ Algorithm\ \ 2)$$

The stepsize γ_k is chosen in such a way that

$$\gamma_k \in \Big(\epsilon, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} - \epsilon\Big) \ \ k \in \Omega,$$

for small enough $\epsilon > 0$, otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{k : Ax_k - By_k \neq 0\}$. If $Ax_k - By_k = 0$, then we take $u_k = x_k$, $v_k = y_k$ and

$$\begin{cases} x_{k+1} = \alpha_k x_k + (1 - \alpha_k) U_{i(k)}(x_k), \\ y_{k+1} = \beta_k y_k + (1 - \beta_k) T_{j(k)}(y_k). \end{cases}$$

Lemma 3.1. *Assume the solution set* Γ *of (14) is nonempty. Then* γ_k *defined by (17) is well-defined.*

Proof. Taking $(x, y) \in \Gamma$, i.e., $x \in \cap_{i=1}^{p} F(U_i)$, $y \in \cap_{j=1}^{r} F(T_j)$ and $Ax = By$, we have

$$\langle A^*(Ax_k - By_k), x_k - x\rangle = \langle Ax_k - By_k, Ax_k - Ax\rangle$$

and

$$\langle B^*(Ax_k - By_k), y - y_k\rangle = \langle Ax_k - By_k, By - By_k\rangle.$$

By adding the two above equalities and by taking into account the fact that $Ax = By$, we obtain

$$\begin{aligned}\|Ax_k - By_k\|^2 &= \langle A^*(Ax_k - By_k), x_k - x\rangle + \langle B^*(Ax_k - By_k), y - y_k\rangle\\ &\le \|A^*(Ax_k - By_k)\| \cdot \|x_k - x\| + \|B^*(Ax_k - By_k)\| \cdot \|y - y_k\|.\end{aligned}$$

Consequently, for $k \in \Omega$, that is, $\|Ax_k - By_k\| > 0$, we have $\|A^*(Ax_k - By_k)\| \neq 0$ or $\|B^*(Ax_k - By_k)\| \neq 0$. This leads that γ_k is well-defined. □

Remark 3.2. *Note that in (17) the choice of the stepsize* γ_k *is independent of the norms* $\|A\|$ *and* $\|B\|$*. The value of* γ *does not influence the considered algorithm, but it was introduced just for the sake of clarity.*

Theorem 3.3. *Assume that* $U_i - I$ $(1 \le i \le p)$, $T_j - I$ $(1 \le j \le r)$ *are demiclosed at origin and the solution set* Γ *of (14) is nonempty. Then, the sequence* $\{(x_k, y_k)\}$ *generated by Parallel Algorithm 1 weakly converges to a solution* (x^*, y^*) *of (14), provided that* $\liminf_{k\to\infty} \alpha_k^0 \alpha_k^i > 0$, $\forall 1 \le i \le p$ *and* $\liminf_{k\to\infty} \beta_k^0 \beta_k^j > 0$, $\forall 1 \le j \le r$.

Moreover $\|Ax_k - By_k\| \to 0$, $\|x_{k+1} - x_k\| \to 0$ *and* $\|y_{k+1} - y_k\| \to 0$ *as* $k \to \infty$.

Proof. From the condition on γ_k, we have

$$\inf_{k\in\Omega}\Big\{\frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} - \gamma_k\Big\} > 0.$$

It follows that $\sup_{k\in\Omega} \gamma_k < +\infty$ and $\{\gamma_k\}_{k\ge1}$ is bounded.

Taking $(x, y) \in \Gamma$, i.e., $x \in \cap_{i=1}^{p} F(U_i)$; $y \in \cap_{j=1}^{r} F(T_j)$ and $Ax = By$. We have

$$\begin{aligned}\|u_k - x\|^2 &= \|x_k - \gamma_k A^*(Ax_k - By_k) - x\|^2\\ &= \|x_k - x\|^2 - 2\gamma_k\langle x_k - x, A^*(Ax_k - By_k)\rangle + \gamma_k^2\|A^*(Ax_k - By_k)\|^2.\end{aligned} \tag{18}$$

Using the equality (16), we have

$$\begin{aligned}-2\langle x_k - x, A^*(Ax_k - By_k)\rangle &= -2\langle Ax_k - Ax, Ax_k - By_k\rangle\\ &= -\|Ax_k - Ax\|^2 - \|Ax_k - By_k\|^2 + \|By_k - Ax\|^2.\end{aligned} \tag{19}$$

By (18) and (19) we obtain

$$\begin{aligned}\|u_k - x\|^2 = \|x_k - x\|^2 &- \gamma_k\|Ax_k - Ax\|^2 - \gamma_k\|Ax_k - By_k\|^2\\ &+ \gamma_k\|By_k - Ax\|^2 + \gamma_k^2\|A^*(Ax_k - By_k)\|^2.\end{aligned} \tag{20}$$

Similarly, we have

$$\begin{aligned}\|v_k - y\|^2 = \|y_k - y\|^2 &- \gamma_k\|By_k - By\|^2 - \gamma_k\|Ax_k - By_k\|^2\\ &+ \gamma_k\|Ax_k - By\|^2 + \gamma_k^2\|B^*(Ax_k - By_k)\|^2.\end{aligned} \tag{21}$$

By adding the two last equalities and the fact that $Ax = By$, we obtain

$$\|u_k - x\|^2 + \|v_k - y\|^2 = \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]. \tag{22}$$

Using the fact that U_i is quasi-nonexpansive mapping and $x \in F(U_i)$ for every $1 \le i \le p$, it follows from Lemma 3.1 that

$$\begin{aligned}\|x_{k+1} - x\|^2 &\le \alpha_k^0\|u_k - x\|^2 + \alpha_k^1\|U_1(u_k) - x\|^2 + \cdots + \alpha_k^p\|U_p(u_k) - x\|^2 - \alpha_k^0\alpha_k^1\|U_1(u_k) - u_k\|^2 \\ &\le \alpha_k^0\|u_k - x\|^2 + \alpha_k^1\|u_k - x\|^2 + \cdots + \alpha_k^p\|u_k - x\|^2 - \alpha_k^0\alpha_k^1\|U_1(u_k) - u_k\|^2 \\ &= \|u_k - x\|^2 - \alpha_k^0\alpha_k^1\|U_1(u_k) - u_k\|^2.\end{aligned} \tag{23}$$

Similarly, we have

$$\|y_{k+1} - y\|^2 \le \|v_k - y\|^2 - \beta_k^0\beta_k^1\|T_1(v_k) - v_k\|^2. \tag{24}$$

So, by (22) we have

$$\begin{aligned}\|x_{k+1} - x\|^2 + \|y_{k+1} - y\|^2 \le{}& \|x_k - x\|^2 + \|y_k - y\|^2 - \alpha_k^0\alpha_k^1\|U_1(u_k) - u_k\|^2 - \beta_k^0\beta_k^1\|T_1(v_k) - v_k\|^2 \\ &- \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)].\end{aligned} \tag{25}$$

Now, by setting $\rho_k(x, y) := \|x_k - x\|^2 + \|y_k - y\|^2$, we obtain the following inequality

$$\begin{aligned}&\rho_{k+1}(x, y) \le \rho_k(x, y) - \alpha_k^0\alpha_k^1\|U_1(u_k) - u_k\|^2 - \beta_k^0\beta_k^1\|T_1(v_k) - v_k\|^2 \\ &phantom\rho_{k+1}(x, y) \le - \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)].\end{aligned} \tag{26}$$

By (17) we see the sequence $\{\rho_k(x, y)\}$ being decreasing and lower bounded by 0, consequently it converges to some finite limit, says $\rho(x, y)$. So the sequences $\{x_k\}$ and $\{y_k\}$ are bounded.

Again from (26) we have

$$\rho_{k+1}(x, y) \le \rho_k(x, y) - \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]$$

and hence

$$\lim_{k\to\infty} \|Ax_k - By_k\| = 0$$

by the assumption on $\{\gamma_k\}$. Similarly, by the conditions on $\{\alpha_k^i\}$ $(0 \le i \le p)$ and $\{\beta_k^j\}$ $(0 \le j \le r)$ we obtain

$$\lim_{k\to\infty} \|U_1(u_k) - u_k\| = \lim_{k\to\infty} \|T_1(v_k) - v_k\| = 0.$$

Since

$$\|u_k - x_k\| = \gamma_k\|A^*(Ax_k - By_k)\|$$

and the fact that $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty} \|u_k - x_k\| = 0$. Similarly, $\lim_{k\to\infty} \|v_k - y_k\| = 0$. Repeating the above proof, for $2 \le i \le p$ and $2 \le j \le r$ we can obtain that

$$\lim_{k\to\infty} \|U_i(u_k) - u_k\| = \lim_{k\to\infty} \|T_j(v_k) - v_k\| = 0.$$

Taking $(x^*, y^*) \in \omega_w(x_k, y_k)$, from $\lim_{k\to\infty} \|u_k - x_k\| = 0$ and $\lim_{k\to\infty} \|v_k - y_k\| = 0$ we have $(x^*, y^*) \in \omega_w(u_k, v_k)$. For any $1 \le i \le p$ and $1 \le j \le r$, combined with the demiclosednesses of $U_i - I$ and $T_j - I$ at 0,

$$\lim_{k\to\infty} \|U_i(u_k) - u_k\| = \lim_{k\to\infty} \|T_j(v_k) - v_k\| = 0$$

yields $U_i(x^*) = x^*$ and $T_j(y^*) = y^*$. So $x^* \in \cap_{i=1}^p F(U_i)$ and $y^* \in \cap_{j=1}^r F(T_j)$. On the other hand, $Ax^* - By^* \in \omega_w(Ax_k - By_k)$ and weakly lower semicontinuity of the norm imply

$$\|Ax^* - By^*\| \le \liminf_{k\to\infty} \|Ax_k - By_k\| = 0,$$

hence $(x^*, y^*) \in \Gamma$. Moreover, it follows from $\|U_i(u_k) - u_k\| \to 0$ and $\|u_k - x_k\| \to 0$ as $k \to \infty$ that $\|U_i(u_k) - x_k\| \to 0$ for all $1 \le i \le p$. So

$$\|x_{k+1} - x_k\| \le \alpha_k^0\|u_k - x_k\| + \alpha_k^1\|U_1(u_k) - x_k\| + \cdots + \alpha_k^p\|U_p(u_k) - x_k\| \to 0$$

as $k \to \infty$, which infer that $\{x_k\}$ is asymptotically regular, namely $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$. Similarly, $\{y_k\}$ is asymptotically regular too.

Next, we will show the uniqueness of the weak cluster points of $\{(x_k, y_k)\}$. Indeed, let $(\bar{x}, \bar{y})$ be other weak cluster points of $\{(x_k, y_k)\}$, then $(\bar{x}, \bar{y}) \in \Gamma$. From the definition of $\rho_k(x, y)$ we have

$$\begin{aligned}\rho_k(x^*, y^*) &= \|x_k - \bar{x}\|^2 + \|\bar{x} - x^*\|^2 + 2\langle x_k - \bar{x}, \bar{x} - x^*\rangle + \|y_k - \bar{y}\|^2 + \|\bar{y} - y^*\|^2 + 2\langle y_k - \bar{y}, \bar{y} - y^*\rangle \\ &= \rho_k(\bar{x}, \bar{y}) + \|\bar{x} - x^*\|^2 + \|\bar{y} - y^*\|^2 + 2\langle x_k - \bar{x}, \bar{x} - x^*\rangle + 2\langle y_k - \bar{y}, \bar{y} - y^*\rangle.\end{aligned} \tag{27}$$

Without of generality, we may assume that $x_k \rightharpoonup \bar{x}$ and $y_k \rightharpoonup \bar{y}$. By passing to the limit in the relation (3.11), we obtain

$$\rho(x^*, y^*) = \rho(\bar{x}, \bar{y}) + \|\bar{x} - x^*\|^2 + \|\bar{y} - y^*\|^2.$$

Reversing the role of (x^*, y^*) and $(\bar{x}, \bar{y})$, we also have

$$\rho(\bar{x}, \bar{y}) = \rho(x^*, y^*) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2.$$

By adding the two last equalities, we obtain $x^* = \bar{x}$ and $y^* = \bar{y}$, which implies that the whole sequence $\{(x_k, y_k)\}$ weakly converges to a solutions of problem (14). This completes the proof. □

Theorem 3.4. *Assume that $U_i - I$ $(1 \le i \le p)$, $T_j - I$ $(1 \le j \le r)$ are demiclosed at origin and the solution set Γ of (1.13) is nonempty. Then, the sequence $\{(x_k, y_k)\}$ generated by Parallel Algorithm 2 weakly converges to a solution (x^*, y^*) of (14), provided that* $\liminf_{k\to\infty} \alpha_k^0 \alpha_k^i > 0$, $\forall 1 \le i \le p$ *and* $\liminf_{k\to\infty} \alpha_k^0 s_k^l > 0$, $\forall 1 \le l \le r$. *Moreover $\|Ax_k - By_k\| \to 0$, $\|x_{k+1} - x_k\| \to 0$ and $\|y_{k+1} - y_k\| \to 0$ as $k \to \infty$.*

Proof. Taking $(x, y) \in \Gamma$, i.e., $x \in \cap_{i=1}^p F(U_i)$; $y \in \cap_{j=1}^r F(T_j)$ and $Ax = By$. By repeating the proof of Theorem 3.3, we have (22) is true.

Using the fact that U_i $(1 \le i \le p)$ and T_j $(1 \le j \le r)$ are quasi-nonexpansive mappings, it follows from Lemma 3.1 that

$$\begin{aligned}\|x_{k+1} - x\|^2 &\le \alpha_k^0 \|x_k - x\|^2 + \alpha_k^1 \|U_1(u_k) - x\|^2 + \cdots + \alpha_k^p \|U_p(u_k) - x\|^2 - \alpha_k^0 \alpha_k^1 \|U_1(u_k) - x_k\|^2 \\ &\le \alpha_k^0 \|x_k - x\|^2 + \alpha_k^1 \|u_k - x\|^2 + \cdots + \alpha_k^p \|u_k - x\|^2 - \alpha_k^0 \alpha_k^1 \|U_1(u_k) - x_k\|^2 \\ &= \alpha_k^0 \|x_k - x\|^2 + (1 - \alpha_k^0)\|u_k - x\|^2 - \alpha_k^0 \alpha_k^1 \|U_1(u_k) - x_k\|^2\end{aligned} \tag{28}$$

and

$$\|y_{k+1} - y\|^2 \le \alpha_k^0 \|y_k - y\|^2 + (1 - \alpha_k^0)\|v_k - y\|^2 - \alpha_k^0 s_k^1 \|T_1(v_k) - y_k\|^2.$$

So, by (22) we have

$$\begin{aligned}\|x_{k+1} - x\|^2 + \|y_{k+1} - y\|^2 \le{}& \|x_k - x\|^2 + \|y_k - y\|^2 - \alpha_k^0 \alpha_k^1 \|U_1(u_k) - x_k\|^2 - \alpha_k^0 s_k^1 \|T_1(v_k) - y_k\|^2 \\ &- (1 - \alpha_k^0)\gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)].\end{aligned} \tag{29}$$

Now, by setting $\rho_k(x, y) := \|x_k - x\|^2 + \|y_k - y\|^2$, we obtain the following inequality

$$\begin{aligned}\rho_{k+1}(x, y) \le{}& \rho_k(x, y) - \alpha_k^0 \alpha_k^1 \|U_1(u_k) - x_k\|^2 - \alpha_k^0 s_k^1 \|T_1(v_k) - y_k\|^2 \\ &- (1 - \alpha_k^0)\gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)].\end{aligned} \tag{30}$$

Following the lines of the proof of Theorem 3.3, by the conditions on $\{\gamma_k\}$, $\{\alpha_k^i\}$ $(0 \le i \le p)$ and $\{s_k^l\}$ $(1 \le l \le r)$ we have that the sequence $\{\rho_k(x, y)\}$ converges to some finite limit, say $\rho(x, y)$. Furthermore, we obtain

$$\lim_{k\to\infty} \|Ax_k - By_k\| = \lim_{k\to\infty} \|U_1(u_k) - x_k\| = \lim_{k\to\infty} \|T_1(v_k) - y_k\| = 0.$$

Similarly, for $2 \le i \le p$ and $2 \le l \le r$, we have

$$\lim_{k\to\infty} \|U_i(u_k) - x_k\| = \lim_{k\to\infty} \|T_l(v_k) - y_k\| = 0.$$

It follows that

$$\|x_{k+1} - x_k\| \le \alpha_k^1 \|U_1(u_k) - x_k\| + \alpha_k^2 \|U_2(u_k) - x_k\| + \cdots + \alpha_k^p \|U_p(u_k) - x_k\| \to 0$$

as $k \to \infty$. So, $\{x_k\}$ is asymptotically regular. Since

$$\|u_k - x_k\| = \gamma_k \|A^*(Ax_k - By_k)\|$$

and the fact that $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty} \|u_k - x_k\| = 0$. Hence

$$\lim_{k\to\infty} \|U_i(u_k) - u_k\| = 0, \ \ \forall 1 \le i \le p.$$

Similarly, $\lim_{k\to\infty} \|v_k - y_k\| = 0$, $\lim_{k\to\infty} \|T_l(v_k) - v_k\| = 0$ $(\forall 1 \le l \le r)$ and $\{y_k\}$ is asymptotically regular too.

The rest of the proof is analogous to that of Theorem 3.3 □

Theorem 3.5. *Assume that $U_i - I$ $(1 \le i \le p)$, $T_j - I$ $(1 \le j \le r)$ are demiclosed at origin and the solution set Γ of (14) is nonempty. Then, the sequence $\{(x_k, y_k)\}$ generated by Cyclic Algorithm 1 weakly converges to a solution (x^*, y^*) of (14), provided that $\{\alpha_k\} \subset (\delta, 1-\delta)$ and $\{\beta_k\} \subset (\sigma, 1-\sigma)$ for small enough $\delta, \sigma > 0$. Moreover $\|Ax_k - By_k\| \to 0$, $\|x_{k+1} - x_k\| \to 0$ and $\|y_{k+1} - y_k\| \to 0$ as $k \to \infty$.*

Proof. Taking $(x, y) \in \Gamma$, i.e., $x \in \cap_{i=1}^p F(U_i)$; $y \in \cap_{j=1}^r F(T_j)$ and $Ax = By$. By repeating the proof of Theorem 3.3, we have (3.6) is true.

Using the fact that U_i $(1 \le i \le p)$ and T_j $(1 \le j \le r)$ are quasi-nonexpansive mappings, it follows from the property (ii) of Lemma 2.3 that

$$\|x_{k+1} - x\|^2 \le \|u_k - x\|^2 - \alpha_k(1-\alpha_k)\|U_{i(k)}(u_k) - u_k\|^2$$

and

$$\|y_{k+1} - y\|^2 \le \|v_k - y\|^2 - \beta_k(1-\beta_k)\|T_{j(k)}(v_k) - v_k\|^2.$$

So, by (22) we have

$$\begin{aligned}\|x_{k+1} - x\|^2 + \|y_{k+1} - y\|^2 \le{}& \|x_k - x\|^2 + \|y_k - y\|^2 - \alpha_k(1-\alpha_k)\|U_{i(k)}(u_k) - u_k\|^2 - \beta_k(1-\beta_k)\|T_{j(k)}(v_k) - v_k\|^2 \\ &- \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]. \end{aligned} \tag{31}$$

Now, by setting $\rho_k(x, y) := \|x_k - x\|^2 + \|y_k - y\|^2$, we obtain the following inequality

$$\begin{aligned}\rho_{k+1}(x, y) \le{}& \rho_k(x, y) - \alpha_k(1-\alpha_k)\|U_{i(k)}(u_k) - u_k\|^2 - \beta_k(1-\beta_k)\|T_{j(k)}(v_k) - v_k\|^2 \\ &- \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]. \end{aligned} \tag{32}$$

We see the sequence $\{\rho_k(x, y)\}$ being decreasing and lower bounded by 0, consequently it converges to some finite limit, says $\rho(x, y)$. So the sequences $\{x_k\}$ and $\{y_k\}$ are bounded. Again from (32) we have

$$\rho_{k+1}(x, y) \le \rho_k(x, y) - \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]$$

and hence

$$\lim_{k\to\infty} \|Ax_k - By_k\| = 0$$

by the assumption on $\{\gamma_k\}$. Similarly, by the conditions on $\{\alpha_k\}$ and $\{\beta_k\}$ we obtain

$$\lim_{k\to\infty} \|U_{i(k)}(u_k) - u_k\| = \lim_{k\to\infty} \|T_{j(k)}v_k - v_k\| = 0.$$

Since

$$\|u_k - x_k\| = \gamma_k \|A^*(Ax_k - By_k)\|$$

and the fact that $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty}\|u_k - x_k\| = 0$. It follows from $\lim_{k\to\infty}\|U_{i(k)}(u_k) - u_k\| = 0$ that $\lim_{k\to\infty}\|U_{i(k)}(u_k) - x_k\| = 0$. So

$$\|x_{k+1} - x_k\| \le \alpha_k\|u_k - x_k\| + (1-\alpha_k)\|U_{i(k)}(u_k) - x_k\| \to 0$$

as $n \to \infty$, which infers that $\{x_k\}$ is asymptotically regular, namely $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$. It follows that

$$\|u_{k+1} - u_k\| \le \|u_{k+1} - x_{k+1}\| + \|x_{k+1} - x_k\| + \|x_k - u_k\| \to 0$$

as $k \to \infty$, which implies that

$$\lim_{k\to\infty}\|u_{k+i} - u_k\| = 0, \ \ \forall 1 \le i \le p. \tag{33}$$

Similarly, $\lim_{k\to\infty}\|v_k - y_k\| = 0$, $\{y_k\}$ is asymptotically regular too and

$$\lim_{k\to\infty}\|v_{k+j} - v_k\| = 0, \ \ \forall 1 \le j \le r. \tag{34}$$

Taking $(x^*, y^*) \in \omega_w(x_k, y_k)$, from $\lim_{k\to\infty}\|u_k - x_k\| = 0$ and $\lim_{k\to\infty}\|v_k - y_k\| = 0$ we have $(x^*, y^*) \in \omega_w(u_k, v_k)$. Let an index $i \in \{1, 2, \cdots, p\}$ be fixed. Noticing that the pool of indexes is finite, from (33) we can find a subsequence $\{u_{k_m}\}$ of $\{u_k\}$ such that $u_{k_m} \rightharpoonup x^*$ as $m \to \infty$ and $i(k_m) = i$ for all m. It turns out that

$$\lim_{m\to\infty}\|U_i(u_{k_m}) - u_{k_m}\| = \lim_{m\to\infty}\|U_{i(k_m)}(u_{k_m}) - u_{k_m}\| = 0.$$

Combined with the demiclosednesses of $U_i - I$ at 0, we get $U_i(x^*) = x^*$. So, $x^* \in F(U_i)$ and hence $x^* \in \cap_{i=1}^{p} F(U_i)$. Similarly, we have $y^* \in \cap_{j=1}^{r} F(T_j)$.

The rest of the proof is analogous to that of Theorem 3.3. □

By Lemma 3.1, similar to technology from Theorem 3.3 to Theorem 3.4, we can get the following result from Theorem 3.5.

Theorem 3.6. *Assume that $U_i - I$ $(1 \le i \le p)$, $T_j - I$ $(1 \le j \le r)$ are demiclosed at origin and the solution set Γ of (1.13) is nonempty. Then, the sequence $\{(x_k, y_k)\}$ generated by Cyclic Algorithm 2 weakly converges to a solution (x^*, y^*) of (1.13), provided that $\{\alpha_k\} \subset (\delta, 1-\delta)$ for small enough $\delta > 0$. Moreover $\|Ax_k - By_k\| \to 0$, $\|x_{k+1} - x_k\| \to 0$ and $\|y_{k+1} - y_k\| \to 0$ as $k \to \infty$.*

Remark 3.7. *(Relationship to Moudafi's work) In [19], A. Moudafi considered the multiple-set split common fixed-point problem (6) on bounded linear operator A. In this paper, we consider the general multiple-set split common fixed-point problem (14) on bounded linear operator A and B. When $H_2 = H_3$ and $B = I$, (14) becomes (6). In his algorithms, the determination of the stepsize γ depends on the operator (matrix) norm $\|A\|$(or the largest eigenvalues of A^*A). In order to implement the above algorithms, one needs to know the operator norm of A(or, at least, estimate), which is in general not an easy work in practice. In this paper, we introduce a new choice of the stepsize which does not need any prior information about the operator norm of A and B, and still convergence is guaranteed.*

Remark 3.8. *In general, to get strong convergence we use Halpern-type iterative process or projection-type iterative process. But Halpern-type iterative process converges slowly to solution and projection-type iterative process is not easily be realized. On the other hand, there are better properties for operators after parallel iteration, see Lemma 2.3. It would be future works to propose fast iterative algorithms for the general MSCFP (14) to get strong convergence result.*

We now turn our attention to apply the proposed algorithms to the general MSCFP (14) governed by firmly quasi-nonexpansive mappings. Since $\Phi_{FQ} \subset \Phi_Q$, we can straightly get Parallel Algorithm 1, 2 and Cyclic Algorithm 1, 2 for solving the general MSCFP (14) of firmly quasi-nonexpansive mappings. Noticing

Remarks 3.2 and 3.8, we know that any firmly quasi-nonexpansive mapping can be expressed be the $\frac{1}{2}$-relaxed operator of quasi-nonexpansive mapping. For any positive numbers $0 < \lambda_i < 1$ $(1 \le i \le p)$ and $0 < \mu_j < 1$ $(1 \le j \le r)$, setting

$$\alpha_k^0 = \frac{\lambda_1 + \cdots + \lambda_p}{2}, \ \ \alpha_k^1 = \frac{\lambda_1}{2}, \cdots, \alpha_k^p = \frac{\lambda_p}{2},$$

$$\beta_k^0 = \frac{\mu_1 + \cdots + \mu_r}{2}, \ \ \beta_k^1 = \frac{\mu_1}{2}, \cdots, \beta_k^r = \frac{\mu_r}{2},$$

Parallel Algorithm 1 take the following equivalent form for solving the general MSCFP (14) of firmly quasi-nonexpansive mappings $\{U_i\}$ $(1 \le i \le p)$ and $\{T_j\}$ $(1 \le j \le r)$:

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \lambda_1 U_1(u_k) + \cdots + \lambda_p U_p(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \mu_1 T_1(v_k) + \cdots + \mu_r T_r(v_k), \end{cases} \tag{35}$$

where the stepsize γ_k is chosen by (3.1). Setting $\alpha_k \equiv \beta_k \equiv \frac{1}{2}$ for all $k \ge 0$, Cyclic Algorithm 1 take the following equivalent form for solving the general MSCFP (1.13) of firmly quasi-nonexpansive mappings $\{U_i\}$ $(1 \le i \le p)$ and $\{T_j\}$ $(1 \le j \le r)$:

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = U_{i(k)}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = T_{j(k)}(v_k), \end{cases} \tag{36}$$

where the stepsize γ_k is chosen by (18). Finally, we apply our algorithms to the MSFP (13). Taking $U_i = P_{C_i}$ $(1 \le i \le p)$ and $T_j = P_{Q_j}$ $(1 \le j \le r)$, we have the following simultaneous parallel and cyclic iterative algorithms:

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \lambda_1 P_{C_1}(u_k) + \cdots + \lambda_p P_{C_p}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \mu_1 P_{Q_1}(v_k) + \cdots + \mu_r P_{Q_r}(v_k) \end{cases} \tag{37}$$

and

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = P_{C_{i(k)}}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = P_{Q_{j(k)}}(v_k) \end{cases} \tag{38}$$

where the stepsize γ_k is chosen by (18).

Remark 3.9. *For the particular case $p = r = 1$, our algorithm (36) and (38) solve the two-sets of SCFP (7) governed by firmly quasi-nonexpansive mappings and the split equality problems (10) without prior knowledge of operator norms, respectively.*

Next we apply our results to variational problems by resolvent mappings. Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well-known that its associated resolvent mapping, $J_\mu^M(x) := (I + \mu M)^{-1}$, is quasi-nonexpansive and $0 \in M(x) \Leftrightarrow x = J_\mu^M(x)$. In other words zeroes of M are exactly fixed-points of its resolvent mapping. For $1 \le i \le p$ and $1 \le j \le r$, by taking $U_i = J_{\mu_i}^{M_i}$, $T_j = J_{\nu_j}^{S_j}$, where $M_i : H_1 \to 2^{H_1}$ and $N_j : H_2 \to 2^{H_2}$ are another maximal monotone operator, the problem under consideration is nothing but to

$$\text{find } x^* \in \bigcap_{i=1}^{p} M_i^{-1}(0), y^* \in \bigcap_{j=1}^{r} N_j^{-1}(0) \text{ such that} Ax^* = By^*, \tag{35}$$

and the algorithms take the following equivalent form.

Parallel Algorithms

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k^0 u_k + \alpha_k^1 J_{\mu_1}^{M_1}(u_k) + \cdots + \alpha_k^p J_{\mu_p}^{M_p}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k^0 v_k + \beta_k^1 J_{\nu_1}^{S_1}(v_k) + \cdots + \beta_k^r J_{\nu_r}^{S_r}(v_k) \end{cases}$$

or

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k^0 x_k + \alpha_k^1 J_{\mu_1}^{M_1}(u_k) + \cdots + \alpha_k^p J_{\mu_p}^{M_p}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k^0 y_k + s_k^1 J_{\nu_1}^{S_1}(v_k) + \cdots + s_k^r J_{\nu_r}^{S_r}(v_k). \end{cases}$$

Cyclic Algorithms

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1 - \alpha_k) J_{\mu_{i(k)}}^{M_{i(k)}}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k v_k + (1 - \beta_k) J_{\nu_{j(k)}}^{S_{j(k)}}(v_k) \end{cases}$$

or

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1 - \alpha_k) J_{\mu_{i(k)}}^{M_{i(k)}}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1 - \alpha_k) J_{\nu_{j(k)}}^{S_{j(k)}}(v_k). \end{cases}$$

Acknowledgements. This work was supported by National Natural Science Foundation of China (No. 61503385) and Fundamental Research Funds for the Central Universities of China (No. 3122016L002).

References

[1] H. Attouch, J. Bolte, P. Redont and A. Soubeyran, Alternating proximal algorithms for weakly coupled minimization problems. Applications to dynamical games and PDEs, J. of Convex Analysis 15(2008) 485-506.
[2] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Review 38(3)(1996) 367-426.
[3] C. Byrne and A. Moudafi, Extensions of the CQ Algorithms for the split feasibility and split equality Problems, J. Nonlinear and Convex A. accepted for publication.
[4] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20(2004) 103-120.
[5] C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem, Inverse Probl. 18(2002) 441-453.
[6] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51(2006) 2353-2365.
[7] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, Inverse Problems 21(2005) 2071-2084.
[8] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8(1994) 221-239.
[9] Y. Censor, A. Gibali and S. Reich, Algorithms for the Split Variational Inequality Problem, Numerical Algorithms 59(2012) 301-323.
[10] Y. Censor and A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16(2009) 587-600.
[11] C.E. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, in: Springer Verlag Series: Lecture Notes in Mathematics, vol. 1965, 2009, p. 326p. XVII.

[12] P.L. Combettes, The convex feasibility problem in image recovery, in: P. Hawkes (Ed.), Advances in Imaging and Electron Physics, Academic Press, New York, vol. 95, 1996, 155-270.
[13] W.G. Dotson Jr., Fixed points of quasi-nonexpansive mappings, J. Aust. Math. Soc. 13(1972) 167-170.
[14] G. López, V. Martín-Márquez, F. Wang, H.K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, Inverse Probl. 27(2012) 085004.
[15] S. Maruster and C. Popirlan, On the Mann-type iteration and convex feasibility problem, J. Comput. Appl. Math. 212(2008) 390-396.
[16] C. Matinez-Yanes and H.K. Xu, Strong convergence of the CQ method for fixed point processes, Nonlinear Anal., 64(2006) 2400-2411.
[17] A. Moudafi, Alternating CQ-algorithm for convex feasibility and split fixed-point problems, J. Nonlinear and Convex A. 15(2014) 809-818.
[18] A. Moudafi, E. Al-Shemas, Simultaneous iterative methods for split equality problems and application, Transactions on Mathematical Programming and Applications 1(2013) 1-11.
[19] A. Moudafi, A note on the split common fixed-point problem for quasi-nonexpansive operators, Nonlinear Anal. 74(2011) 4083-4087.
[20] B. Qu and N. Xiu, A note on the CQ algorithm for the split feasibility problem, Inverse Probl. 21(2005) 1655-1665.
[21] H. Stark (Ed.), Image Recovery Theory and Applications, Academic Press, Orlando, 1987.
[22] H.K. Xu, A variable Krasnosel'skiǐ-Mann algorithm and the multiple-set split feasibility problem, Inverse Probl. 22(2006) 2021-2034.
[23] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Probl. 26(2010) 105018, 17 pages.
[24] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16(1991) 1127-1138.
[25] Q. Yang, The relaxed CQ algorithm solving the split feasibility problem, Inverse Probl. 20(2004) 1261-1266.
[26] Y. Yao, J. Wu and Y.C. Liou, Regularized methods for the split feasibility problem, Abstract and Applied Analysis, 2012(2012) Article ID 140679, 13 pages.
[27] H. Zegeye, A hybrid iteration scheme for equilibrium problems, variational inequality problems and common fixed point problems in Banach spaces, Nonlinear Anal. 72(3)(2010) 2136-2146.
[28] J. Zhao and Q. Yang, A simple projection method for solving the multiple-sets split feasibility problem, Inverse Probl. Sci. Eng. 21(3)(2013) 537-546.