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On Hermite-Hadamard Inequalities for h-Preinvex Functions

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Abstract. The objective of this paper is to obtain some Hermite-Hadamard type inequalities for h-preinvex functions. Firstly, a new kind of generalized h-convex functions, termed h-preinvex functions, is introduced through relaxing the concept of h-convexity introduced by Varosanec. Some Hermite-Hadamard type inequalities for h-preinvex functions are established under certain conditions. Our results can be viewed as generalization of several previously known results. Results proved in this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of preinvex functions introduced by Weir and Mond [33]. It is well known that the preinvex functions and invex sets may not be convex functions and convex sets. For the applications, properties and other aspects of the preinvex functions, see [1, 3, 11, 15, 16, 24, 33].

Various refinements of the Hermite-Hadamard inequalities for the convex functions and their variant forms are being obtained in the literature by many researchers (see [2–4, 6–10, 14, 15, 17, 19, 21, 23, 25, 29–31]). In [32], Varosanec introduced the concept of h-convex functions, which include classical convex functions, s-convex functions [5], Godunova-Levin functions [12] and P-functions [10] as special cases. Sarikaya et al. [31] proved some Hermite-Hadamard inequalities for h-convex functions. Noor [19, 21] has established several Hermite-Hadamard inequalities for preinvex, log-preinvex functions and product of two log-preinvex functions. For some recent developments, see [26–28].

Motivated and inspired by the recent activities in this area, we obtain some new Hermite-Hadamard type inequalities for h-preinvex functions through relaxing the concept of h-convex functions. This is the main motivation of this paper. In particular, our results include the previously known results for h-convex functions as special cases. The interested readers are encouraged to find the novel and innovative applications of these results in other areas.

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2. Preliminaries

Let K be a nonempty closed set in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : K \times K \to \mathbb{R}^n$ be a continuous bifunction. First of all, we recall some known results and concepts.

Definition 2.1 ([33]). *A set K is said to be invex set with respect to $\eta(.,.)$, if*

$$u + t\eta(v,u) \in K, \quad \forall u,v \in K, t \in [0,1]. \tag{1}$$

The invex set K is also called η-connected set.

Remark 2.2 ([1]). *We would like to mention that Definition 2.1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in K$, then $\eta(v,u) = v - u$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(v,u) = v - u$, but the converse is not necessarily true, see [18, 34] and the references therein. For the sake of simplicity, we always assume that $K = [u, u + \eta(v,u)]$, unless otherwise specified.*

Definition 2.3 ([33]). *A function f is said to be preinvex with respect to arbitrary bifunction $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le (1-t)f(u) + tf(v), \quad \forall u,v \in K, t \in [0,1]. \tag{2}$$

The function f is said to be preincave if and only if $-f$ is preinvex. For the basic properties and applications of the preinvex functions in variational inequalities, see [1, 11, 16, 18, 20, 22, 24, 33, 34].
For $\eta(v,u) = v - u$ in (2), the preinvex functions becomes convex functions in the classical sense.

Definition 2.4. *A function $f : K \to \mathbb{R}$ is said to be convex in the classical sense, if*

$$f(u + t(v-u)) \le (1-t)f(u) + tf(v), \quad \forall u,v \in K, t \in [0,1].$$

From Definitions 2.3 and 2.4 it is obvious that every convex function is a preinvex function. However it is known [33] that preinvex functions may not be convex functions.

Definition 2.5 ([32]). *Let I, J be intervals in $\mathbb{R}$, $(0,1) \subseteq J$, and let $h : J \to \mathbb{R}$ be a non-negative function. We say that a non-negative function $f : I \to \mathbb{R}$ is called h-convex function, or that f belongs to the class $SX(h,I)$, if we have*

$$f((1-t)u + tv) \le h(1-t)f(u) + h(t)f(v), \quad \forall u,v \in I, t \in (0,1). \tag{3}$$

If the above inequality is reversed, then f is said to be h-concave function.

Remark 2.6. *It is worth pointing out that not all convex functions belong to the class of h-convex ones and this inconvenience can be avoided omitting the assumption that f is nonnegative, see [4]. So, in the further text we assume that h and f are real functions without assumption of non-negativity. Obviously, if $h(t) = t$, then all convex (concave) functions belong to the class of h-convex (concave) functions; if $h(t) = t^s, s \in (0,1)$, then all s-convex (concave) functions in the second sense belong to the class of h-convex (concave) functions. Additional, this large class of h-convex functions contains Godunova-Levin functions and P-functions, if $h(t) = t^{-1}$, $h(t) = 1$, respectively.*

Definition 2.7. *Let $h : J \to \mathbb{R}$ where $(0,1) \subseteq J$ be an interval in $\mathbb{R}$, and let K be an invex set with respect to $\eta(.,.)$. A function $f : K \to \mathbb{R}$ is called h-preinvex with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le h(1-t)f(u) + h(t)f(v), \quad u,v \in K, t \in (0,1).$$

If above inequality is reversed, then f is said to be h-preconcave with respect to bifunction $\eta(.,.)$.
Now, we discuss some special cases of h-preinvex functions.
I. If $h(t) = t$, then Definition 2.7 reduces to Definition 2.3.
II. If $h(t) = t^s$ with $s \in (0,1)$, then we have the definition of s-preinvex functions.

Definition 2.8. *A function $f : K \to \mathbb{R}$ is said to be s-preinvex with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le (1-t)^s f(u) + (t)^s f(v), \quad u, v \in K, t \in [0,1], s \in (0,1).$$

III. If $h(t) = t^{-1}$, then we have the definition of Q-preinvex functions.

Definition 2.9. *A function $f : K \to \mathbb{R}$ is said to be Q-preinvex with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le \frac{1}{1-t} f(u) + \frac{1}{t} f(v), \quad u, v \in K, t \in (0,1).$$

IV. If $h(t) = 1$, then we have the definition of P-preinvex functions.

Definition 2.10. *A function $f : K \to \mathbb{R}$ is said to be P-preinvex with respect to $\eta(.,.)$, if*

$$f(u + t\eta(v,u)) \le f(u) + f(v), \quad u, v \in K, t \in [0,1].$$

Now we give an example of h-preinvex function. Which also illustrates the fact that an h-preinvex functions may not be preinvex functions.

Example 2.11. *Let us consider the function $f(u) = -|u|$ with*

$$\eta(v,u) = \begin{cases} v - u, \ if & uv < 0, \\ u - v, \ if & uv > 0. \end{cases}$$

If function $h(t) = t^s$, $t \in (0,1)$, where $s \ge 1$, then by Definition 2.7, f is h-preinvex function with respect to $\eta(.,.)$ on $\mathbb{R} \setminus \{0\}$. However, if $h(t) = t^s, t \in (0,1)$ where $s < 1$, by letting $x = 1, y = 1, t = 1/2, s = 1/2$, we have

$$f(u + t\eta(v,u)) = f(1) = -1 > -\sqrt{2} = t^s f(u) + (1-t)^s f(v),$$

which shows that f is not h-preinvex with respect to the same $\eta(.,.)$.

To prove some results in the paper, we need the well-known Condition C introduced by Mohan and Neogy in [18].

Condition C. Let $K \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(.,.)$. Then for any $x, y \in K$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

Note that for every $x, y \in K$, $t_1, t_2 \in [0,1]$ and from Condition C, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y).$$

It is worth mentioning that Condition C has played a crucial and significant role in the development of the variational-like inequalities and optimization problems, see [11, 18, 22] and the references therein.
From now onwards $I = [a, a + \eta(b,a)]$ will be the interval unless otherwise specified.

3. Main Results

In this section, we prove our main results.
Using the technique of Varosanec [32], we prove that the product of two h-preinvex functions is again a h-preinvex function.

Theorem 3.1. *Let f and w be two h-preinvex functions. Then their product fw is also h-preinvex function provided if f and w are similarly ordered functions and $h(t) + h(1-t) \le 1$.*

Proof. Since f and w are h-preinvex functions, then

$$\begin{aligned}
&f(x + t\eta(y, x))w(x + t\eta(y, x))\\
&\leq [h(1-t)f(x) + h(t)f(y)][h(1-t)w(x) + h(t)w(y)]\\
&= [h(1-t)]^2 f(x)w(x) + h(t)h(1-t)[f(x)w(y) + f(y)w(x)] + [h(t)]^2 f(y)w(y)\\
&\leq [h(1-t)]^2 f(x)w(x) + h(t)h(1-t)[f(x)w(x) + f(y)w(y)] + [h(t)]^2 f(y)w(y)\\
&= [h(1-t)f(x)w(x) + h(t)f(y)w(y)][h(t) + h(1-t)]\\
&\leq h(1-t)f(x)w(x) + h(t)f(y)w(y),
\end{aligned}$$

where we have used the fact that $h(t) + h(1-t) \leq 1$. This shows that the product of two h-preinvex functions is also h-preinvex. □

We prove the following result which will be helpful in proving our next result of Fejer type inequality for h-preinvex function.

Lemma 3.2. *Let f be h-preinvex function, then we have*

$$f(2a + \eta(b, a) - x) \leq [h(t) + h(1-t)]\,[f(a) + f(b)] - f(x). \tag{4}$$

Proof. Given $x = a + t\eta(b, a) \in I$, then we have

$$\begin{aligned}
f(2a + \eta(b, a) - x) &= f(a + (1-t)\eta(b, a))\\
&\leq h(t)f(a) + h(1-t)f(b)\\
&= [h(t) + h(1-t)]\,[f(a) + f(b)] - [h(1-t)f(a) + h(t)f(b)]\\
&\leq [h(t) + h(1-t)]\,[f(a) + f(b)] - f(a + t\eta(b, a))\\
&= [h(t) + h(1-t)]\,[f(a) + f(b)] - f(x).
\end{aligned}$$

This proves the required result. □

Theorem 3.3. *Let $f : I \to (0, \infty)$ be a h-preinvex function with $a < a + \eta(b, a)$, $h(\frac{1}{2}) \neq 0$ and $w : [a, a + \eta(b, a)] \to \mathbb{R}$ is a non-negative, integrable function and symmetric about $a + \frac{1}{2}\eta(b, a)$, then, using Condition C, we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2a + \eta(b, a)}{2}\right) \int_a^{a+\eta(b,a)} w(x)dx \leq \int_a^{a+\eta(b,a)} f(x)w(x)dx \leq \frac{f(a) + f(b)}{2}(h(t) + h(1-t)) \int_a^{a+\eta(b,a)} w(x)dx.$$

Proof. Since f is h-preinvex function, we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2a + \eta(b, a)}{2}\right) \int_a^{a+\eta(b,a)} w(x)dx\\
&= \frac{1}{2h\left(\frac{1}{2}\right)} \int_a^{a+\eta(b,a)} f\left(\frac{2a + \eta(b, a)}{2}\right) w(x)dx\\
&= \frac{1}{2h\left(\frac{1}{2}\right)} \int_a^{a+\eta(b,a)} f\left(\frac{2a + \eta(b, a) - x + x}{2}\right) w(x)dx\\
&\leq \frac{1}{2h\left(\frac{1}{2}\right)} \int_a^{a+\eta(b,a)} \left[h\left(\frac{1}{2}\right)\{f(2a + \eta(b, a) - x) + f(x)\}\right] w(x)dx
\end{aligned}$$

$$= \frac{1}{2}\int_a^{a+\eta(b,a)} f(2a+\eta(b,a)-x)w(2a+\eta(b,a)-x)dx + \frac{1}{2}\int_a^{a+\eta(b,a)} f(x)w(x)dx$$

$$= \int_a^{a+\eta(b,a)} f(x)w(x)dx$$

$$= \frac{1}{2}\int_a^{a+\eta(b,a)} f(2a+\eta(b,a)-x)w(2a+\eta(b,a)-x)dx + \frac{1}{2}\int_a^{a+\eta(b,a)} f(x)w(x)dx$$

$$= \frac{1}{2}\int_a^{a+\eta(b,a)} f(2a+\eta(b,a)-x)w(x)dx + \frac{1}{2}\int_a^{a+\eta(b,a)} f(x)w(x)dx$$

$$\leq \frac{1}{2}\int_a^{a+\eta(b,a)} [(h(t)+h(1-t))[f(a)+f(b)]-f(x)]w(x)dx + \frac{1}{2}\int_a^{a+\eta(b,a)} f(x)w(x)dx$$

$$\leq \frac{f(a)+f(b)}{2}(h(t)+h(1-t))\int_a^{a+\eta(b,a)} w(x)dx.$$

This completes the proof. □

Theorem 3.4. *Let $f : I \to (0,\infty)$ and $w : I \to (0,\infty)$ be h_1-preinvex and h_2-preinvex functions respectively with $a < a+\eta(b,a)$, $h_1(\frac{1}{2}) \neq 0, h_2(\frac{1}{2}) \neq 0$. If the bifunction $\eta(.,.)$ satisfies Condition C, then*

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{2a+\eta(b,a)}{2}\right) w\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)w(x)dx$$
$$\leq M(a,b)\int_0^1 h_1(t)h_2(1-t)dt + N(a,b)\int_0^1 h_1(t)h_2(t)dt.$$

where

$$M(a,b) = f(a)w(a) + f(b)w(b) \tag{5}$$

and

$$N(a,b) = f(a)w(b) + f(b)w(a). \tag{6}$$

Proof. Since f and w are h_1-preinvex and h_2-preinvex functions respectively and η satisfies Condition C, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) w\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$= f(a+(1-t)\eta(b,a) + \frac{1}{2}\eta(a+t\eta(b,a), a+(1-t)\eta(b,a)))$$
$$\times w(a+(1-t)\eta(b,a) + \frac{1}{2}\eta(a+t\eta(b,a), a+(1-t)\eta(b,a)))$$
$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a))][w(a+t\eta(b,a)) + w(a+(1-t)\eta(b,a))]$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[f(a+t\eta(b,a))w(a+t\eta(b,a))+f(a+(1-t)\eta(b,a))w(a+(1-t)\eta(b,a))]+$$
$$h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\{[h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]M(a,b)+[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]N(a,b)\}.$$

Integrating above inequality with respect to t on [0,1], we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right)w\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)w(x)dx$$
$$\leq 2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[M(a,b)\int_0^1 h_1(t)h_2(1-t)dt+N(a,b)\int_0^1 h_1(t)h_2(t)dt\right].$$

From the above inequality, we can obtain the required result. □

Theorem 3.5. *Let $f: I\to(0,\infty)$ and $w: I\to(0,\infty)$ be h_1-preinvex and h_2-preinvex functions respectively with $a<a+\eta(b,a)$, then we have*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)w(x)dx\leq M(a,b)\int_0^1 h_1(t)h_2(t)dt+N(a,b)\int_0^1 h_1(t)h_2(1-t)dt,$$

where $M(a,b)$ and $N(a,b)$ are given by (5) *and* (6).

Proof. Let f, w be nonnegative h_1-preinvex and ,h_2-preinvex function respectively, then for all $t\in[0,1]$, we have

$$f(a+t\eta(b,a))w(a+t\eta(b,a))$$
$$\leq[h_1(1-t)f(a)+h_1(t)f(b)][h_2(1-t)w(a)+h_2tw(b)]$$
$$=h_1(1-t)h_2(1-t)f(a)w(a)+h_1(t)h_2(1-t)f(b)w(a)+h_1(1-t)h_2(t)f(a)w(b)+h_1(t)h_2(t)f(b)w(b).$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)w(x)dx$$
$$\leq f(a)w(a)\int_0^1 h_1(1-t)h_2(1-t)dt+f(a)w(b)\int_0^1 h_1(1-t)h_2(t)dt$$
$$+f(b)w(a)\int_0^1 h_1(t)h_2(1-t)dt+f(b)w(b)\int_0^1 h_1(t)h_2(t)dt$$
$$=[f(a)w(a)+f(b)w(b)]\int_0^1 h_1(t)h_2(t)dt+[f(a)w(b)+f(b)w(a))]\int_0^1 h_1(t)h_2(1-t)dt$$
$$=M(a,b)\int_0^1 h_1(t)h_2(t)dt+N(a,b)\int_0^1 h_1(t)h_2(1-t)dt.$$

Thus we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)w(x)dx \leq M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt.$$

This completes the proof. □

Next, we prove some Hermite-Hadamard type inequalities for differentiable h-preinvex functions.
We need the following result, which is an extension of a result proved by Dragomir et al. [6].

Lemma 3.6 ([3]). *Let $f : I \to (0,\infty)$ be a differentiable mapping $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$. If $f' \in L_1[a, a+\eta(b,a)]$, then*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a)+f(a+\eta(b,a))}{2} = \frac{\eta(b,a)}{2}\left[\int_0^1 (1-2t)f'(a+t\eta(b,a))dt\right].$$

Proof. Consider

$$\begin{aligned}
\int_0^1 (1-2t)f'(a+t\eta(b,a))dt &= \left|\frac{f(a+t\eta(b,a))}{\eta(b,a)}(1-2t)\right|_0^1 + 2\int_0^1 \frac{f(a+t\eta(b,a))}{\eta(b,a)}dt \\
&= -\frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{f(a)}{\eta(b,a)} + \frac{2}{\eta(b,a)}\int_0^1 f(a+t\eta(b,a))dt \\
&= -\frac{f(a)+f(a+\eta(b,a))}{\eta(b,a)} + \frac{2}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx \\
&= \frac{2}{\eta(b,a)}\left[-\frac{f(a)+f(a+\eta(b,a))}{2} + \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right].
\end{aligned}$$

Suitable rearrangements complete the proof. □

Theorem 3.7. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, a+\eta(b,a) \in I^\circ$ with $a < a+\eta(b,a)$. If $|f'|$ is a h-preinvex on $[a, a+\eta(b,a)]$, then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2}\,[|f'(a)| + |f'(b)|]\int_0^1 |1-2t|h(t)dt.$$

Proof. Using Lemma 3.6, it follows that

$$\begin{aligned}
\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| &= \left|\frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'(a+t\eta(b,a))dt\right| \\
&\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t||f'(a+t\eta(b,a))|dt
\end{aligned}$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left[h(1-t)|f'(a)|+h(t)|f'(b)|\right]dt$$

$$\leq \frac{\eta(b,a)}{2}\left[|f'(a)|+|f'(b)|\right]\int_0^1 |1-2t|h(t)dt.$$

This completes the proof. □

Theorem 3.8. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I°, $a, a+\eta(b,a) \in I^\circ$ with $a < a+\eta(b,a)$. If $|f'|^q$ is a h-preinvex on $[a, a+\eta(b,a)]$ where $q \geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left[\left[|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}\right]\int_0^1 h(t)dt\right]^{\frac{p-1}{p}}$$

Proof. Using Lemma 3.6, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| = \left|\frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'(a+t\eta(b,a))dt\right|$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t||f'(a+t\eta(b,a))|dt.$$

Using Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}, \tag{7}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now using h-preinvexity of f, we have

$$\int_0^1 |f'(a+t\eta(b,a))|^q \leq \int_0^1 [h(1-t)|f'(a)|^q + h(t)|f'(b)|^q]dt, \tag{8}$$

where

$$\int_0^1 |1-2t|^p dt = \frac{1}{p+1}. \tag{9}$$

Using (7) (8) and (9), we have the required result. □

Theorem 3.9. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I°, $a, a+\eta(b,a) \in I^\circ$ with $a < a+\eta(b,a)$. If $|f'|^q$ is a h-preinvex on $[a, a+\eta(b,a)]$ where $q \geq 1$, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[|f'(a)|^q+|f'(b)|^q\right]\int_0^1 h(t)dt\right]^{\frac{1}{q}}.$$

Proof. Using Lemma 3.6, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| = \left|\frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'(a+t\eta(b,a))dt\right|$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t||f'(a+t\eta(b,a))|dt.$$

Due to the power-mean inequality, we have

$$\int_0^1 |1-2t||f'(a+t\eta(b,a))|dt \leq \left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t||f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}. \quad (10)$$

By the h-preinvexity of f, we have

$$\int_0^1 |1-2t||f'(a+t\eta(b,a))|^q dt \leq \int_0^1 |1-2t|[h(1-t)|f'(a)|^q + h(t)|f'(b)|^q]dt$$
$$= [|f'(a)|^q + |f'(b)|^q]\int_0^1 |1-2t|h(t)dt, \quad (11)$$

where

$$\int_0^1 |1-2t|dt = \frac{1}{2} \quad (12)$$

Using (10), (11) and (12), we have the required result. □

Now using the technique of [14], we prove the following result which play a key role in proving our next results.

Lemma 3.10. *Let $f : I \to (0,\infty)$ be a differentiable mapping $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$. If $f' \in L_1[a, a+\eta(b,a)]$, then*

$$\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx = \int_0^{\frac{1}{2}} tf'(a+t\eta(b,a))dt + \int_{\frac{1}{2}}^1 (t-1)f'(a+t\eta(b,a))dt$$

Proof. Consider

$$\int_0^{\frac{1}{2}} tf'(a+t\eta(b,a))dt + \int_{\frac{1}{2}}^1 (t-1)f'(a+t\eta(b,a))dt$$
$$= \left|\frac{f(a+t\eta(b,a))}{\eta(b,a)}t\right|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{f(a+t\eta(b,a))}{\eta(b,a)}dt + \left|\frac{f(a+t\eta(b,a))}{\eta(b,a)}(t-1)\right|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 \frac{f(a+t\eta(b,a))}{\eta(b,a)}dt$$
$$= \frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx.$$

This completes the proof. □

Theorem 3.11. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, a + \eta(b, a) \in I^\circ$ with $a < a + \eta(b, a)$. If $|f'|$ is a h-preinvex on $[a, a + \eta(b, a)]$, then we have*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \eta(b,a)\left[|f'(a)|\left(\int_0^{\frac{1}{2}} th(1-t)dt + \int_{\frac{1}{2}}^1 (1-t)h(1-t)dt\right) + |f'(b)|\left(\int_0^{\frac{1}{2}} th(t)dt + \int_{\frac{1}{2}}^1 (1-t)h(t)dt\right)\right].$$

Proof. Using Lemma 3.10 we have

$$\left|\frac{1}{[\eta(b,a)]}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \eta(b,a)\left[\int_0^{\frac{1}{2}} tf'(a+t\eta(b,a))dt + \int_{\frac{1}{2}}^1 (1-t)f'(a+t\eta(b,a))dt\right]$$
$$\leq \eta(b,a)\left[\int_0^{\frac{1}{2}} t[h(1-t)|f'(a)| + h(t)|f'(b)|]dt + \int_{\frac{1}{2}}^1 (1-t)[h(1-t)|f'(a)| + h(t)|f'(b)|]dt\right]$$
$$= \eta(b,a)\left[|f'(a)|\left(\int_0^{\frac{1}{2}} th(1-t)dt + \int_{\frac{1}{2}}^1 (1-t)h(1-t)dt\right) + |f'(b)|\left(\int_0^{\frac{1}{2}} th(t)dt + \int_{\frac{1}{2}}^1 (1-t)h(t)dt\right)\right].$$

This proves the result. □

Theorem 3.12. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, a + \eta(b, a) \in I^\circ$ with $a < a + \eta(b, a)$. If $|f'|^q$ is a h-preinvex on $[a, a + \eta(b, a)]$ where $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left\{\int_0^{\frac{1}{2}}[h(1-t)|f'(a)|^q + h(t)|f'(b)|^q]dt\right\}^{\frac{1}{q}} + \left\{\int_{\frac{1}{2}}^1 [h(1-t)|f'(a)|^q + h(t)|f'(b)|^q]dt\right\}^{\frac{1}{q}}\right]$$

Proof. Using Lemma 3.10 and Hölder's inequality, we have

$$\left|\frac{1}{[\eta(b,a)]}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$

$$\leq \eta(b,a)\left[\int_0^{\frac{1}{2}} tf'(a+t\eta(b,a))dt + \int_{\frac{1}{2}}^{1}(1-t)f'(a+t\eta(b,a))dt\right]$$

$$\leq \eta(b,a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1}(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right] \tag{13}$$

Using h-preinvexity of f, we have

$$\int_0^{\frac{1}{2}} |f'(a+t\eta(b,a))|^q dt \leq \int_0^{\frac{1}{2}} [h(1-t)|f'(a)|^q + h(t)|f'(b)|^q]dt. \tag{14}$$

$$\int_{\frac{1}{2}}^{1} |f'(a+t\eta(b,a))|^q dt \leq \int_{\frac{1}{2}}^{1} [h(1-t)|f'(a)|^q + h(t)|f'(b)|^q]dt. \tag{15}$$

And

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^{1}(1-t)^p dt = \frac{1}{2^{p+1}(p+1)}. \tag{16}$$

Using (13), (14), (15) and (16), we have the required result. □

Remark 3.13. *Now we point out some special cases which are included in our main results.*

1. *If $\eta(b,a) = b-a$ and $h(t) = t$, $h(t) = t^s$, $h(t) = t^{-1}$ and $h(t) = 1$, then our results reduce to the results for classical convexity, s-convexity, Q-convexity and P-convexity respectively.*
2. *If $h(t) = t$, then our results reduce to the results for classical preinvexity.*
3. *If $h(t) = t$, $h(t) = t^s$, $h(t) = t^{-1}$ and $h(t) = 1$, then our results reduce to the results for classical preinvexity, s-preinvexity, Q-preinvexity and P-preinvexity respectively, which also appears to be new.*

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