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Carathéodory's Approximate Solution to Stochastic Differential Delay Equation

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Abstract. The main aim of this paper is to discuss Carathéodory's and Euler-Maruyama's approximate solutions to stochastic differential delay equation. To make the theory more understandable, we impose the non-uniform Lipschitz condition and non-linear growth condition.

1. Introduction

In 2007, Mao [7] considered the following an estimate on difference between the Carathéodory's approximate solution $x_n(t)$ and the unique solution $x(t)$ to the stochastic differential delay equation:

Theorem 1.1. *Let uniform Lipschitz condition and linear growth condition hold. That is there exists a constant $\overline{K}$ such that for all $t \in [t_0, T]$, and all $x, y, \overline{x}, \overline{y} \in R^d$*

$$|F(x, y, t) - F(\overline{x}, \overline{y}, t)|^2 \vee |G(x, y, t) - G(\overline{x}, \overline{y}, t)|^2 \leq \overline{K}(|x - \overline{x}|^2 + |y - \overline{y}|^2);$$

and there is moreover a $K > 0$ such that for all $(x, y, t) \in R^d \times R^d \times [t_0, T]$,

$$|F(x, y, t)|^2 \vee |G(x, y, t)|^2 \leq K(1 + |x|^2 + |y|^2).$$

Then

$$E\Big(\sup_{t_0 \leq t \leq T} |x(t) - x_n(t)|^2\Big) \leq 4c_3 \exp(5c_3(T - t_0))$$
$$\times \Big(\frac{6c_1 + Tc_2}{n} + 2c_1\mu\{t \in [t_0, t_0 + \tau] : 0 < \delta(t) < 1/n\}\Big)$$

where $c_1 = (1/2 + 4E\|\xi\|^2)\exp(6K(T - t_0 + 4)(T - t_0))$, $c_2 = 4K(1 + 2c_1)$, and $c_3 = 4\overline{K}(T - t_0 + 4)$ and μ stands for Lebesgue measure on R.

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For results related to the stochastic differential delay equation, see [2]-[4], [6]-[10], [12], [13], and references therein for details.

In the recent paper [9], by employing non-Lipschitz condition and non-linear growth condition, Ren and Xia established the following results for d-dimensional stochastic functional differential equation.

Theorem 1.2. *Assume that there exists a constant K and a concave function κ such that*
(i) (non-Lipschitz condition) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi, t) - f(\psi, t)|^2 \vee |g(\varphi, t) - g(\psi, t)|^2 \leq \kappa(\|\varphi - \psi\|^2),$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$ and $\int_{0+} du/\kappa(u) = \infty$.
(ii) (non-linear growth condition) $f(0, t), g(0, t) \in L^2$ and for all $t \in [t_0, T]$, it follows that s

$$|f(0, t)|^2 \vee |g(0, t)|^2 \leq K,$$

where $K > 0$ is a constant. Then, there exist a unique solution to the equation

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \quad on \;\; t_0 \leq t \leq T,$$

with initial data.

For various related results, see [1], [5], [7], [11], and references therein for details.

Motivated by above results, we establish in this paper more estimate on difference between the approximate solutions and the unique solution to stochastic differential delay equation that can be obtained from non-uniform Lipschitz condition and non-linear growth condition. When we try to carry over this procedure to the this delay equation, we used Carathéodory and Euler-Maruyama approximation procedure.

2. Preliminary

Let $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$. Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$.

Let $BC((-\infty, 0]; R^d)$ denote the family of bounded continuous R^d-valued functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\leq 0} |\varphi|$. Let $\mathcal{M}^2((-\infty, 0]; R^d)$ denote the family of $\mathcal{F}_{t_0}$-measurable, R^d-valued process $\varphi(t) = \varphi(t, \omega), t \in (-\infty, 0]$ such that $E \int_{\infty}^{0} |\varphi(t)|^2 dt < \infty$.

In [9], considered following d-dimensional stochastic functional differential equations

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \quad on \;\; t_0 \leq t \leq T, \tag{1}$$

where $x_t = \{x(t + \theta) : -\infty < \theta \leq 0\}$ can be regarded as a $BC((-\infty, 0]; R^d)$-value stochastic process, where $f : BC((-\infty, 0]; R^d) \times [t_0, T] \to R^d$ and $g : BC((-\infty, 0]; R^d) \times [t_0, T] \to R^{d\times m}$ be Borel measurable. Moreover, the initial value is followed:

$$\begin{aligned} &x_{t_0} = \xi = \{\xi(\theta) : -\infty \leq \theta \leq 0\} \quad \text{is an}\, \mathcal{F}_{t_0} - \text{measurable} \\ &BC([-\infty, 0]; R^d) - \text{value random variable such that}\, \xi \in \mathcal{M}^2((-\infty, 0]; R^d). \end{aligned} \tag{2}$$

A special but important class of stochastic functional differential equations is the stochastic differential delay equations. Let us begin with the discussion of the following stochastic differential delay equation

$$dx(t) = F(x(t), x(t-\delta(t)), t)dt + G(x(t), x(t-\delta(t)), t)dB(t) \tag{3}$$

on $t \in [t_0, T]$ with initial data (2), where $\delta : [t_0, T] \to [0, \infty)$, $F : R^d \times R^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to R^{d\times m}$ be Borel measurable.

If we define

$$f(\varphi, t) = F(\varphi(0), \varphi(-\delta(t)), t) \quad \text{and} \quad g(\varphi, t) = G(\varphi(0), \varphi(-\delta(t)), t)$$

for $(\varphi, t) \in BC((-\infty, 0]; R^d) \times [t_0, T]$, then equation (3) can be written as equation (1) so one can apply the existence-and-uniqueness theorem established in the previous section to the delay equation (3).

On the other hand, we impose the non-uniform Lipschitz condition and weakened linear growth condition. That is such that for all $t \in [t_0, T]$, and all $x, y, \overline{x}, \overline{y} \in R^d$

$$|F(x, y, t) - F(\overline{x}, \overline{y}, t)|^2 \vee |G(x, y, t) - G(\overline{x}, \overline{y}, t)|^2 \le \kappa(|x - \overline{x}|^2 + |y - \overline{y}|^2); \tag{4}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$ and $\int_{0+} du/\kappa(u) = \infty$, and there is a $K > 0$ such that for all $(x, y, t) \in R^d \times R^d \times [t_0, T]$,

$$|F(0, 0, t)|^2 \vee |G(0, 0, t)|^2 \le K. \tag{5}$$

Let us now prepare a few lemmas in order to show the main result.

Lemma 2.1. *(Doob's martingale inequality) [7] Let $\{X(t)\}_{t\ge 0}$ be an R^d-valued martingale and let $[a, b]$ be a bounded interval on R^+. If $p > 1$ and $X(t) \in L^p(\Omega, R^d)$, then*

$$E\Big(\sup_{a\le t\le b} |X(t)|^p\Big) \le \left(\frac{p}{p-1}\right)^p E(|X(b)|^p).$$

In particular, $E(\sup_{a\le t\le b} |X(t)|^2) \le 4E(|X(b)|^2)$ when $p = 2$.

Lemma 2.2. *(Moment inequality) [7] If $p \ge 2, g \in \mathcal{M}^2([0, T]; R^{d\times m})$ such that $E \int_0^T |g(s)|^p \, ds < \infty$, then*

$$E\left|\int_0^T g(s)\, \mathrm{d}B(s)\right|^p \le \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E \int_0^T |g(s)|^p \, ds.$$

In particular, $E|\int_0^T g(s)\, \mathrm{d}B(s)|^2 \le E \int_0^T |g(s)|^2\, ds$ when $p = 2$.

3. Approximate Solutions

Let us first discuss the Carathéodory approximation procedure. Consider the stochastic differential delay equation (3) with initial data (2). It is in this spirit we define the Carathéodory approximation as follows: For each integer $n \ge 1$, define $x_n(t)$ on $(-\infty, T]$ by

$$x_n(t_0 + \theta) = \xi(\theta) \quad \text{for} - \infty < \theta \le 0$$

and

$$\begin{aligned} x_n(t) &= \xi(0) + \int_{t_0}^{t} I_{D_n^c} F(x_n(s-1/n), x_n(s-\delta(s)), s)ds \\ &\quad + \int_{t_0}^{t} I_{D_n} F(x_n(s-1/n), x_n(s-\delta(s)-1/n), s)ds \\ &\quad + \int_{t_0}^{t} I_{D_n^c} G(x_n(s-1/n), x_n(s-\delta(s)), s)dB(s) \\ &\quad + \int_{t_0}^{t} I_{D_n} G(x_n(s-1/n), x_n(s-\delta(s)-1/n), s)dB(s) \end{aligned} \tag{6}$$

for $t_0 \le t \le T$, where

$$D_n = \{t \in [t_0, T] : \delta(t) < 1/n\} \quad \text{for} \quad D_n^c = [t_0, T] - D_n.$$

Since our goal is to study exponential estimates on difference between the approximate solutions and the uniqueness solutons, we assume that there exists a unique solution $x(t)$ to equation (3) under non Lipschitz condition and non-linear growth condition. We also assume that all the Lebesgue and Itô integrals employed further are well defined.

We start with following an exponential estimate.

Lemma 3.1. *Let (4) and (5) hold. Then, for all $n \ge 1$, we have*

$$E\Big(\sup_{-\infty < s \le t} |x_n(s)|^2\Big) \le \Big(\frac{1}{2} + 6E\|\xi\|^2 + KC_1(T - t_0)\Big)e^{2aC_1(t-t_0)} \tag{7}$$

for all $t \ge t_0$, where $C_1 = 10(T - t_0 + 4)$.

Proof. By Hölder's inequality, Doob's martingale inequality and Lemma 2.2, we can derive from (6) that for $t_0 \le t \le T$,

$$\begin{aligned} &E\Big(\sup_{t_0 \le s \le t} |x_n(s)|^2\Big) \\ &\le 5E|\xi(0)|^2 + 5(T - t_0)E\int_{t_0}^{t} I_{D_n^c}(s)|F(x_n(s-1/n), x_n(s-\delta(s)), s)|^2 ds \\ &+5(T - t_0)E\int_{t_0}^{t} I_{D_n}(s)|F(x_n(s-1/n), x_n(s-\delta(s)-1/n), s)|^2 ds \\ &+5 \cdot 4E\int_{t_0}^{t} I_{D_n^c}(s)|G(x_n(s-1/n), x_n(s-\delta(s)), s)|^2 ds \\ &+5 \cdot 4E\int_{t_0}^{t} I_{D_n}(s)|G(x_n(s-1/n), x_n(s-\delta(s)-1/n), s)|^2 ds. \end{aligned}$$

By the condition (4) and (5), we obtain

$$\begin{aligned} &E\Big(\sup_{t_0 \le s \le t} |x_n(s)|^2\Big) \\ &\le 5E|\xi(0)|^2 + C_1\int_{t_0}^{t} I_{D_n^c}(s)[\kappa(|x_n(s-1/n)|^2 + |x_n(s-\delta(s))|^2) + K]ds \\ &+C_1\int_{t_0}^{t} I_{D_n}(s)[\kappa(|x_n(s-1/n)|^2 + |x_n(s-\delta(s)-1/n)|^2) + K]ds, \end{aligned}$$

where $C_1 = 10(T - t_0 + 4)$. Given that $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constants a such that $\kappa(u) \leq a(1 + u)$ for all $u \geq 0$. Therefore

$$E\Big(\sup_{t_0\leq s\leq t}|x_n(s)|^2\Big)$$
$$\leq 5E\|\xi\|^2 + KC_1(T - t_0) + aC_1\int_{t_0}^{t}\bigg(1 + 2E\Big(\sup_{-\infty<r\leq s}|x_n(r)|^2\Big)\bigg)ds.$$

Note that

$$\frac{1}{2} + E\Big(\sup_{-\infty<s\leq t}|x_n(s)|^2\Big)$$
$$\leq \frac{1}{2} + 6E\|\xi\|^2 + KC_1(T - t_0) + 2aC_1\int_{t_0}^{t}\bigg(\frac{1}{2} + E\Big(\sup_{-\infty<r\leq s}|x_n(r)|^2\Big)\bigg)ds.$$

An application of the Gronwall inequality implies that

$$\frac{1}{2} + E\Big(\sup_{-\infty<s\leq t}|x_n(s)|^2\Big) \leq \bigg(\frac{1}{2} + 6E\|\xi\|^2 + KC_1(T - t_0)\bigg)e^{2aC_1(t-t_0)},$$

and the desired inequality follows immediately. The proof is complete. □

In other words, the estimate for $E|x_n(t)|^2$ can be done via the estimate for the second moment. For instance, we have the following lemma.

Lemma 3.2. *Let (4) and (5) hold. Then, we have*

$$E\Big(\sup_{-\infty<s\leq t}|x(s)|^2\Big)$$
$$\leq C_2 := \bigg(\frac{1}{2} + 4E\|\xi\|^2 + 6K(T - t_0 + 4)(T - t_0)\bigg)e^{12a(T-t_0+4)(t-t_0)} \tag{8}$$

for all $t \geq t_0$. Moreover, for any $t_0 \leq s < t \leq T$ with $t - s < 1$,

$$E|x(t) - x(s)|^2 \leq C_3(t - s), \tag{9}$$

where $C_3 = 8(K + a(1 + 2C_2))$.

Proof. *The proof of (8) is similar to that of Lemma 3.1. By Hölder's inequality, Doob's martingale inequality and Lemma 2.2, we can derive that*

$$E\Big(\sup_{t_0\leq s\leq t}|x(s)|^2\Big)$$
$$\leq 3E|\xi(0)|^2 + 6(T - t_0 + 4)\int_{t_0}^{t}[\kappa(|x(s)|^2 + |x(s - \delta(s))|^2) + K]ds.$$

By the definition of $\kappa(\cdot)$, we can find a positive constants a such that $\kappa(u) \leq a(1 + u)$ for all $u \geq 0$. Therefore

$$E\Big(\sup_{t_0\leq s\leq t}|x(s)|^2\Big)$$
$$\leq C_4 + 6a(T - t_0 + 4)\int_{t_0}^{t}\bigg(1 + 2E\Big(\sup_{-\infty<r\leq s}|x(r)|^2\Big)\bigg)ds,$$

where $C_4 = 3E\|\xi\|^2 + 6K(T-t_0+4)(T-t_0)$. *Note that*

$$\begin{aligned}&\frac{1}{2} + E\Big(\sup_{-\infty<s\le t}|x(s)|^2\Big)\\ &\le \frac{1}{2} + E\|\xi\|^2 + C_4 + 12a(T-t_0+4)\int_{t_0}^{t}\bigg(\frac{1}{2} + E\Big(\sup_{-\infty<r\le s}|x(r)|^2\Big)\bigg)ds.\end{aligned}$$

An application of the Gronwall inequality implies that

$$\begin{aligned}&\frac{1}{2} + E\Big(\sup_{-\infty<s\le t}|x(s)|^2\Big)\\ &\le \bigg(\frac{1}{2} + 4E\|\xi\|^2 + 6K(T-t_0+4)(T-t_0)\bigg)e^{12a(T-t_0+4)(t-t_0)},\end{aligned}$$

and the desired inequality follows immediately. We need to show (9) but this is straightforward:

$$\begin{aligned}&E|x(t)-x(s)|^2\\ &\le 4K(t-s+1)(t-s) + 4a(t-s+1)E\int_s^t[1+2C_2]ds\\ &\le 8[K+a(1+2C_2)](t-s).\end{aligned}$$

The proof is complete. □

We can now prove one of the main results in this paper.

Theorem 3.3. *Let (4) and (5) hold. Then, we have*

$$E\Big(\sup_{t_0\le t\le T}|x(t)-x_n(t)|^2\Big) \le \Big(aC_5(T-t_0) + \widehat{J_1} + \widehat{J_2}\Big)e^{5aC_5(T-t_0)} \tag{10}$$

where

$$\begin{aligned}&\widehat{J_1} = 2aC_5[4C_2 + TC_3]\frac{1}{n},\\ &\widehat{J_2} = 4aC_5\Big([2C_2 + TC_3]\frac{1}{n} + 2C_2\mu\{t\in[t_0, t_0+1+1/n] : 0 < \delta(t) < 1/n\}\Big),\end{aligned}$$

C_2, C_3 *are defined in Lemma 3.2,* $C_5 = 4(T-t_0+4)$ *an* μ *stands for the Lebesque measure on* R.

Proof. *By Hölder's inequality, Doob's martingale inequality and Lemma 2.2, we can derive that*

$$\begin{aligned}&E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\\ &\le 4(T-t_0)E\int_{t_0}^{t} I_{D_n^c}(s)|F_x(\mathbf{s}) - F_{x_n}(\mathbf{s})|^2ds\\ &\quad +4(T-t_0)E\int_{t_0}^{t} I_{D_n}(s)|F_x(\mathbf{s}) - \widehat{F}_{x_n}(\mathbf{s})|^2ds\\ &\quad +4\cdot 4E\int_{t_0}^{t} I_{D_n^c}(s)|G_x(\mathbf{s}) - G_{x_n}(\mathbf{s})|^2ds\\ &\quad +4\cdot 4E\int_{t_0}^{t} I_{D_n}(s)|G_x(\mathbf{s}) - \widehat{G}_{x_n}(\mathbf{s})|^2ds,\end{aligned}$$

where $F_x(s) = F(x(s), x(s-\delta(s)), s)$, $F_{x_n}(s) = F(x_n(s-1/n), x_n(s-\delta(s)), s)$, $\widehat{F}_{x_n}(s) = F(x_n(s-1/n), x_n(s-\delta(s)-1/n), s)$, $G_x(s) = G(x(s), x(s-\delta(s)), s)$, $G_{x_n}(s) = G(x_n(s-1/n), x_n(s-\delta(s)), s)$, *and* $\widehat{G}_{x_n}(s) = G(x_n(s-1/n), x_n(s-\delta(s)-1/n), s)$. *By the condition (4), (5) and the definition of* $\kappa(\cdot)$*, we obtain*

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\\
&\le aC_5(T-t_0)+5aC_5\int_{t_0}^{t}E\Big(\sup_{t_0\le r\le s}|x(r)-x_n(r)|^2\Big)ds+J_1+J_2,
\end{aligned}
$$

where

$$J_1 = 2aC_5\int_{t_0}^{T}E|x(s)-x(s-1/n)|^2ds$$

and

$$J_2 = 2aC_5\int_{t_0}^{T}I_{D_n}(s)E|x(s-\delta(s))-x_n(s-\delta(s)-1/n)|^2ds.$$

An application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\le (aC_5(T-t_0)+J_1+J_2)\,e^{5aC_5(T-t_0)}. \tag{11}$$

But, using Lemma 3.2, we can estimate

$$J_1\le 8aC_2C_5\frac{1}{n}+2aC_3C_5T\frac{1}{n}=2aC_5[4C_2+TC_3]\frac{1}{n}. \tag{12}$$

Also, setting $D_0=\{t\in[t_0,T]:\delta(t)=0\}$,

$$J_2 \;\le\; 4aC_5\Big([2C_2+TC_3]\frac{1}{n}+2C_2\mu\{[t_0,t_0+1+1/n]\cap(D_n-D_0)\}\Big). \tag{13}$$

Substituting (12) and (13) into (11) yields the required result (10). The proof is complete. □

Let us now turn to the Euler-Maruyama approximation procedure. We first give the definition of the Euler-Maruyama approximation sequence. For each integer $n\ge 1$, define $x_n(t)$ on $(-\infty, T]$ by

$$x_n(t_0+\theta)=\xi(\theta)\quad \text{for} -\infty<\theta\le 0$$

and

$$
\begin{aligned}
x_n(t) \;=\; & x_n(t_0+k/n)\\
&+\int_{t_0+k/n}^{t}F(x_n(t_0+k/n), x_n(t_0+k/n-\delta(s)), s)ds\\
&+\int_{t_0+k/n}^{t}G(x_n(t_0+k/n), x_n(t_0+k/n-\delta(s)), s)dB(s)
\end{aligned} \tag{14}
$$

for $t_0+k/n<t\le[t_0+(k+1)/n]\wedge T$, $k=0,1,2,\cdots$. Moreover, if we define $\widehat{x}_n(t_0)=x_n(t_0)$, $\widetilde{x}_n(t_0)=x_n(t_0-\delta(t_0))$,

$$\widehat{x}_n(t)=x_n(t_0+k/n),\quad \text{and}\quad \widetilde{x}_n(t)=x_n(t_0+k/n-\delta(t))$$

for $t_0+k/n<t\le[t_0+(k+1)/n]\wedge T$, $k=0,1,2,\cdots$, it then follows from (14) that

$$x_n(t)=\xi(0)+\int_{t_0}^{t}F(\widehat{x}_n(s),\widetilde{x}_n(s),s)ds+\int_{t_0}^{t}G(\widehat{x}_n(s),\widetilde{x}_n(s),s)dB(s) \tag{15}$$

In the sequel of this section $x_n(t)$ always means the Euler-Maruyama approximation rather than the Carathéodory one. The following lemma shows that the Euler-Maruyama approximation sequence is bounded in L^2.

Lemma 3.4. *Let (4) and (5) hold. Then, for all $n \geq 1$, we have*

$$E\Big(\sup_{-\infty<s\leq t}|x_n(s)|^2\Big) \leq \Big(\frac{1}{2} + 4E\|\xi\|^2 + KC_6(T-t_0)\Big)e^{2aC_6(T-t_0)} \tag{16}$$

for all $t \geq t_0$, where $C_6 = 6(T - t_0 + 4)$.

Proof. *It is easy to see from (15) that for $t_0 \leq t \leq T$,*

$$\begin{aligned}&E\Big(\sup_{t_0\leq s\leq t}|x_n(s)|^2\Big)\\ &\leq 3E|\xi(0)|^2 + C_6E\int_{t_0}^t[\kappa(|\widehat{x}_n(s)|^2 + |\widetilde{x}_n(s)|^2) + K]ds,\end{aligned}$$

where $C_6 = 6(T - t_0 + 4)$. Recalling the definition of $\kappa(\cdot)$, $\widehat{x}_n(s)$, and $\widetilde{x}_n(s)$, we then see that

$$\begin{aligned}&E\Big(\sup_{t_0\leq s\leq t}|x_n(s)|^2\Big)\\ &\leq 3E\|\xi\|^2 + KC_6(T-t_0) + aC_6\int_{t_0}^t\Big(1 + 2E\Big(\sup_{-\infty<r\leq s}|x_n(r)|^2\Big)\Big)ds.\end{aligned}$$

Consequently

$$\begin{aligned}&\frac{1}{2} + E\Big(\sup_{-\infty<s\leq t}|x_n(s)|^2\Big)\\ &\leq \frac{1}{2} + 4E\|\xi\|^2 + KC_6(T-t_0) + 2aC_6\int_{t_0}^t\Big(\frac{1}{2} + E\Big(\sup_{-\infty<r\leq s}|x_n(r)|^2\Big)\Big)ds.\end{aligned}$$

An application of the Gronwall inequality implies that

$$\frac{1}{2} + E\Big(\sup_{-\infty<s\leq t}|x_n(s)|^2\Big) \leq \Big(\frac{1}{2} + 4E\|\xi\|^2 + KC_6(T-t_0)\Big)e^{2aC_6(t-t_0)},$$

and the desired inequality follows immediately. The proof is complete. □

The following theorem shows that the Euler-Maruyama approximate solution converges to the unique solution of equation (3) and gives and estimate for the difference between the approximate solution $x_n(t)$ and the accurate solution $x(t)$.

Theorem 3.5. *Let (4) and (5) hold. Then, the difference between the Euler-Maruyama approximate solution $x_n(t)$ and the accurate solution $x(t)$ of equation (3) can be estimate as*

$$E\Big(\sup_{t_0\leq t\leq T}|x(t) - x_n(t)|^2\Big) \leq \Big(\frac{1}{2}aC_5(T-t_0) + \widehat{J_3} + \widehat{J_4}\Big)e^{2aC_5(T-t_0)} \tag{17}$$

where

$$\widehat{J_3} = aC_3C_5[T-t_0]\frac{1}{n}, \quad \widehat{J_4} = 4aC_2C_5[T-t_0],$$

C_2, C_3, and C_5 are defined in Lemma 3.2 and Theorem 3.3.

Proof. *By Hölder's inequality, Doob's martingale inequality and Lemma 2.2, we can derive that*

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\\
&\le 2(T-t_0)E\int_{t_0}^{t}|F(x(s),x(s-\delta(s),s)-F(\widehat{x_n}(s),\widetilde{x}_n(s),s)|^2ds\\
&\quad +2\cdot 4E\int_{t_0}^{t}|G(x(s),x(s-\delta(s)),s)-G(\widehat{x_n}(s),\widetilde{x}_n(s),s)|^2ds,
\end{aligned}
$$

By the condition (4), (5), and $\kappa(\cdot)$*, we then see that*

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\le 2a(T-t_0)(T-t_0+4)\\
&+2a(T-t_0+4)\int_{t_0}^{t}E|x(s)-\widehat{x}_n(s)|^2+E|x(s-\delta(s))-\widetilde{x}_n(s)|^2ds.
\end{aligned}
\tag{18}
$$

Define $\widehat{x}(t_0)=x(t_0)$, $\widetilde{x}(t_0)=x(t_0-\delta(t_0))$,

$$
\widehat{x}(t)=x(t_0+k/n),\quad \text{and}\quad \widetilde{x}(t)=x(t_0+k/n-\delta(t))
$$

for $t_0+k/n<t\le[t_0+(k+1)/n]\wedge T$, $k=0,1,2,\cdots$*, it then follows from (18) that*

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\\
&\le \frac{1}{2}aC_5(T-t_0)+2aC_5\int_{t_0}^{t}E\Big(\sup_{t_0\le r\le s}|x(r)-x_n(r)|^2\Big)ds+J_3+J_4,
\end{aligned}
$$

where

$$
J_3=aC_5\int_{t_0}^{T}E|x(s)-\widehat{x}(s)|^2ds
$$

and

$$
J_4=aC_5\int_{t_0}^{T}E|x(s-\delta(s))-\widetilde{x}(s)|^2ds.
$$

An application of the Gronwall inequality implies that

$$
E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^2\Big)\le (aC_5(T-t_0)/2+J_3+J_4)\,e^{2aC_5(T-t_0)}. \tag{19}
$$

But, using Lemma 3.2, we can estimate

$$
\begin{aligned}
J_2 &= aC_5\sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T}E|x(s)-x(t_0+k/n)|^2ds\\
&\le aC_3C_5\frac{1}{n}[T-t_0].
\end{aligned}
\tag{20}
$$

Also,

$$
\begin{aligned}
J_4 &= aC_5\int_{t_0}^{T}E|x(s-\delta(s))-x(t_0+k/n-\delta(s))|^2ds\\
&\le 4aC_2C_5[T-t_0].
\end{aligned}
\tag{21}
$$

Substituting (20) and (21) into (19) yields the required result (17). The proof is complete. □

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