Filomat 31:7 (2017), 2151–2162
DOI 10.2298/FIL1707151W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Finite Iterative Algorithms for the Generalized Reflexive and Anti-Reflexive Solutions of the Linear Matrix Equation $AXB = C$

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Abstract. In this paper, an iterative method is presented to solve the linear matrix equation $AXB = C$ over the generalized reflexive (or anti-reflexive) matrix X ($A \in R^{p\times n}, B \in R^{m\times q}, C \in R^{p\times q}, X \in R^{n\times m}$). By the iterative method, the solvability of the equation $AXB = C$ over the generalized reflexive (or anti-reflexive) matrix can be determined automatically. When the equation $AXB = C$ is consistent over the generalized reflexive (or anti-reflexive) matrix X, for any generalized reflexive (or anti-reflexive) initial iterative matrix X_1 , the generalized reflexive (anti-reflexive) solution can be obtained within finite iterative steps in the absence of roundoff errors. The unique least-norm generalized reflexive (or anti-reflexive) iterative solution of $AXB = C$ can be derived when an appropriate initial iterative matrix is chosen. A sufficient and necessary condition for whether the equation $AXB = C$ is inconsistent is given. Furthermore, the optimal approximate solution of $AXB = C$ for a given matrix X_0 can be derived by finding the least-norm generalized reflexive (or anti-reflexive) solution of a new corresponding matrix equation $A\overline{X}B = \overline{C}$. Finally, several numerical examples are given to support the theoretical results of this paper.

1. Introduction

As well known, numerical methods for linear and nonlinear matrix equations play an important role in many science and engineering computation,such as control theory, signal and image processing, photogrammetry, etc. There are several kinds of linear and nonlinear matrix equations, which have been studied deeply, such as Riccati equations[13, 22–26], Sylvester equations[17**?** –21], $AXB = C$[12, 14, 27], and so on.

In this paper, we will consider the following linear matrix equation

$$AXB = C, \tag{1}$$

2010 *Mathematics Subject Classification*. 65F10, 65F50
Keywords. Matrix equation; generalized reflexive matrix; generalized anti-reflexive matrix.
Received: 26 November 2014; Accepted: 15 February 2015
Communicated by Dragana Cvetković Ilić
This work is supported by NNSF of China No. 11461046, NSF of Jiangxi Province with Nos.20161ACB21005, and 20151BAB201009, and the Scientific Research Foundation of Graduate School of Nanchang University with No.YC2016-S040. This work is also supported by Program for Innovative Research Team in UIBE (XK2014127, CXTD5-05) and by Collaborative Innovation Center for Chinese Multinational Enterprise (No. 201504YY006A).
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which has been considered by many authors, where $A \in R^{p\times n}, B \in R^{m\times q}, C \in R^{p\times q}, X \in R^{n\times m}$. In [1], Penrose provided a sufficient and necessary condition for the consistency of this equation above and, for the consistent case, gave a representation of its general solution. In [2–4], the necessary and sufficient conditions for the existence of symmetric solutions and symmetric positive semidefinite solutions have been presented, as well as explicit formulae using generalized inverses. Wang and Chang [5] studied least squares symmetric solutions to the equation using the generalized singular value decomposition, and a sufficient and necessary condition for its solvability and a representation of its general solution were also established therein. In [6], Yuan and Dai considered the generalized reflexive solutions of the matrix Eq. (1) and an associated optimal problem. By using of the generalized singular value decomposition, a necessary and sufficient condition for the Eq. (1) to have a solution over generalized reflexive matrices has been presented. In [15, 16], the authors studied generalized (P, Q)-reflexive solution of the linear systems of matrix equations and the least-norm generalized (P,Q)-reflexive solution of matrix equations $A_iXB_i = C_i$. Recently, M. Dehghan and M. Hajarian presented iterative methods for the reflexive and anti-reflexive solutions for some kinds of matrix equations ([9–11]).

Throughout the paper we will use the following notations:

R^n will denote the real $n-$ vector space and the set of $n\times m$ matrices by $R^{n\times m}$. For a matrix $A \in R^{m\times n}$, $\|A\|$ represents its Frobenius norm, $R(A)$ represents its column space, $tr(A)$ represents its trace and $vec(\cdot)$ represents the vec operator, i.e., $vec(A) = (a_1^T, a_2^T, \cdots, a_n^T)^T$ for the matrix $A = (a_1, a_2, \cdots, a_n) \in R^{m\times n}, a_i \in R^m, i = 1, 2, \cdots, n.A \otimes B$ stands for the Kronecker product of matrices A and B. In [7], the definition and some properties of generalized reflexive (anti-reflexive) matrix have been presented.

Definition 1 A matrix $P \in R^{n\times n}$ is called a generalized reflection matrix if $P^T = P$ and $P^2 = I$. A matrix $A \in R^{n\times m}$ is said to be a generalized reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix $P \in R^{n\times n}, Q \in R^{m\times m}$, if $A = PAQ(A = -PAQ)$. We denote the set of all generalized reflexive (or anti-reflexive) matrices by $R_r^{n\times m}(P, Q)$(or $R_a^{n\times m}(P, Q)$).

According to the definition above, we can prove
Lemma 2[15] For an arbitrary matrix $A \in R^{n\times m}$, we have

$$A + PAQ \in R_r^{n\times m}(P, Q), A - PAQ \in R_a^{n\times m}(P, Q)$$

Lemma 3[15] If $A \in R_r^{n\times m}(P, Q), B \in R_a^{n\times m}(P, Q)$, then we have $tr(A^TB) = 0$.

In this paper, we consider the following problems.

Problem I. For given matrices $A \in R^{p\times n}, B \in R^{m\times q}, C \in R^{p\times q}$, find matrix $X \in R_r^{n\times m}(P, Q)$ such that $AXB = C$.
Problem II. When Problem I is consistent, let S_R denote the set of generalized reflexive solutions of Problem I, for a given matrix $Y \in R^{n\times m}$, find $X_0 \in S_R$ such that

$$\|X_0 - Y\|^2 = \min_{X\in S_R} \|X - Y\|^2.$$

Problem III. For given matrices $A \in R^{p\times n}, B \in R^{m\times q}, C \in R^{p\times q}$, find matrix $X \in R_a^{n\times m}(P, Q)$ such that $AXB = C$.
Problem IV. When Problem I is consistent, let S_A denote the set of generalized anti-reflexive solutions of Problem I, for a given matrix $Y \in R^{n\times m}$, find $X_0 \in S_A$ such that

$$\|X_0 - Y\|^2 = \min_{X\in S_A} \|X - Y\|^2.$$

In fact, Problem II and Problem IV are to find the optimal approximately generalized reflexive and anti-reflexive solution to a given matrix $Y \in R^{n\times m}$.

As to Problem I and Problem II, a sufficient and necessary condition for its solvability has been presented by Yuan and Dai in [6]. In this paper, an iterative method is presented to solve the linear matrix equation $AXB = C$ over the generalized reflexive (anti-reflexive) matrix X. By this method, the solvability of the equation $AXB = C$ over the generalized reflexive (anti-reflexive) matrix can be determined automatically. When the equation $AXB = C$ is consistent over the generalized reflexive (anti-reflexive) matrix X, for any generalized reflexive (anti-reflexive) initial iterative matrix X_1 , the generalized reflexive (anti-reflexive) solution can be obtained within finite iterative steps in the absence of roundoff errors. The unique least-norm generalized reflexive (anti-reflexive) iterative solution of $AXB = C$ can be derived when an appropriate initial iterative matrix is chosen. A sufficient and necessary condition for whether the equation $AXB = C$ is inconsistent is given. Furthermore, the optimal approximate solution of $AXB = C$ for a given matrix X_0 can be derived by finding the least-norm generalized reflexive (anti-reflexive) solution of a new corresponding matrix equation $A\overline{X}B = \overline{C}$. Finally, several numerical examples are given to support the theoretical results of this paper.

2. An Algorithm for Solving Problem I and Problem II.

Firstly, we present an iterative algorithm for solving Problem I.

Algorithm 1: (An iterative algorithm for solving Problem I)

Step 1. Input matrices $A \in R^{p\times n}, B \in R^{m\times q}, C \in R^{p\times q}$;

Step 2. Choose any matrix $X_1 \in R_r^{n\times m}(P, Q)$, where $P \in R^{n\times n}, Q \in R^{m\times m}$ are two arbitrary generalized reflection matrices;

Step 3. Compute $R_1 = C - AX_1B, P_1 = \frac{1}{2}(A^TR_1B^T + PA^TR_1B^TQ), k = 1$;

Step 4. Compute

$$X_{k+1} = X_k + \frac{||R_k||^2}{||P_k||^2}P_k,$$

$$R_{k+1} = C - AX_{k+1}B = R_k - \frac{||R_k||^2}{||P_k||^2}AP_kB,$$

$$P_{k+1} = \frac{1}{2}(A^TR_{k+1}B^T + PA^TR_{k+1}B^TQ) + \frac{||R_{k+1}||^2}{||R_k||^2}P_k;$$

Step 5. If $R_{k+1} = 0$, or $R_{k+1} \neq 0$ and $P_{k+1} = 0$, stop; otherwise, let $k = k + 1$, go to Step 4.

Obviously, we know that $X_k, P_k \in R_r^{n\times m}(P, Q)$, where $k = 1, 2, \cdots$

Before we analyze the properties of Algorithm 1, we will firstly introduce the following definition.

Definition 4 Let $P, Q \in R^{m\times n}$, the matrices P, Q are called orthogonal to each other, if $tr(P^TQ) = 0$.

Lemma 5 For the sequences R_i, P_i which are produced by Algorithm 1 and $s \geq 2$, we have that

$$tr(R_j^TR_i) = 0, tr(P_j^TP_i) = 0 (i \neq j; i, j = 1, 2, \cdots, s) \quad (2)$$

Proof: We can complete the proof by induction.

For $s = 2$, we have

$$\begin{aligned}
tr(R_2^TR_1) &= tr((R_1 - \tfrac{||R_1||^2}{||P_1||^2}AP_1B)^TR_1) \\
&= ||R_1||^2 - \tfrac{||R_1||^2}{||P_1||^2}tr((AP_1B)^TR_1) \\
&= ||R_1||^2 - \tfrac{||R_1||^2}{||P_1||^2}tr(P_1^TA^TR_1B^T) \\
&= ||R_1||^2 - \tfrac{||R_1||^2}{||P_1||^2}tr((\tfrac{1}{2}P_1^T(A^TR_1B^T + PA^TR_1B^TQ) + \tfrac{1}{2}P_1^T(A^TR_1B^T - PA^TR_1B^TQ)) - PA^TR_1B^TQ)) \\
&= ||R_1||^2 - \tfrac{||R_1||^2}{||P_1||^2}tr(\tfrac{1}{2}P_1^T(A^TR_1B^T + PA^TR_1B^TQ)) \\
&= ||R_1||^2 - \tfrac{||R_1||^2}{||P_1||^2}||P_1||^2 = 0,
\end{aligned}$$

$$\begin{aligned}
tr(P_2^TP_1) &= tr((\tfrac{1}{2}(A^TR_2B^T+PA^TR_2B^TQ)+\tfrac{\|R_2\|^2}{\|R_1\|^2}P_1)^TP_1)\\
&= tr(\tfrac{1}{2}(A^TR_2B^T+PA^TR_2B^TQ)^TP_1)+\tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2\\
&= tr((A^TR_2B^T)^TP_1)+\tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2\\
&= tr(R_2^TAP_1B)+\tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2\\
&= \tfrac{\|P_1\|^2}{\|R_1\|^2}tr(R_2^T(R_1-R_2))+\tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2\\
&= \tfrac{\|P_1\|^2}{\|R_1\|^2}(0-\|R_2\|^2)+\tfrac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2=0.
\end{aligned}$$

So we know (2) holds for $s=2$. Now assuming (2) holds for $s=k$, then we have

$$\begin{aligned}
tr(R_{k+1}^TR_k) &= tr((R_k-\tfrac{\|R_k\|^2}{\|P_k\|^2}AP_kB)^TR_k)\\
&= \|R_k\|^2-\tfrac{\|R_k\|^2}{\|P_k\|^2}tr((AP_kB)^TR_k)\\
&= \|R_k\|^2-\tfrac{\|R_k\|^2}{\|P_k\|^2}tr(P_k^TA^TR_kB^T)\\
&= \|R_k\|^2-\tfrac{\|R_k\|^2}{\|P_k\|^2}tr((\tfrac{1}{2}P_k^T(A^TR_kB^T+PA^TR_kB^TQ)+\tfrac{1}{2}P_k^T(A^TR_kB^T-PA^TR_kB^TQ))\\
&= \|R_k\|^2-\tfrac{\|R_k\|^2}{\|P_k\|^2}tr(\tfrac{1}{2}P_k^T(A^TR_kB^T+PA^TR_kB^TQ))\\
&= \|R_k\|^2-\tfrac{\|R_k\|^2}{\|P_k\|^2}tr(P_k^T(P_k-\tfrac{\|R_k\|^2}{\|R_{k-1}\|^2}P_{k-1}))\\
&= \|R_k\|^2-\tfrac{\|R_k\|^2}{\|P_k\|^2}\|P_k\|^2=0,
\end{aligned}$$

$$\begin{aligned}
tr(P_{k+1}^TP_k) &= tr((\tfrac{1}{2}(A^TR_{k+1}B^T+PA^TR_{k+1}B^TQ)+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k)^TP_k)\\
&= tr(\tfrac{1}{2}(A^TR_{k+1}B^T+PA^TR_{k+1}B^TQ)^TP_k)+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\|P_k\|^2\\
&= tr((A^TR_{k+1}B^T)^TP_k)+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\|P_k\|^2\\
&= tr(R_{k+1}^TAP_kB)+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\|P_k\|^2\\
&= \tfrac{\|P_k\|^2}{\|R_k\|^2}tr(R_{k+1}^T(R_k-R_{k+1}))+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\|P_k\|^2\\
&= \tfrac{\|P_k\|^2}{\|R_k\|^2}(0-\|R_{k+1}\|^2)+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\|P_k\|^2=0.
\end{aligned}$$

Also, when $i=1,2,\cdots,k-1$, we have that

$$\begin{aligned}
tr(R_{k+1}^TR_i) &= tr((R_k-\tfrac{\|R_k\|^2}{\|P_k\|^2}AP_kB)^TR_i)\\
&= -\tfrac{\|R_k\|^2}{\|P_k\|^2}tr((AP_kB)^TR_i)\\
&= -\tfrac{\|R_k\|^2}{\|P_k\|^2}tr(P_k^TA^TR_iB^T)\\
&= -\tfrac{\|R_k\|^2}{\|P_k\|^2}tr((\tfrac{1}{2}P_k^T(A^TR_iB^T+PA^TR_iB^TQ)+\tfrac{1}{2}P_k^T(A^TR_iB^T-PA^TR_iB^TQ))\\
&= -\tfrac{\|R_k\|^2}{\|P_k\|^2}tr(\tfrac{1}{2}P_k^T(A^TR_iB^T+PA^TR_iB^TQ))\\
&= -\tfrac{\|R_k\|^2}{\|P_k\|^2}tr(P_k^T(P_i-\tfrac{\|R_i\|^2}{\|R_{i-1}\|^2}P_{i-1}))=0,
\end{aligned}$$

$$\begin{aligned}
tr(P_{k+1}^TP_i) &= tr((\tfrac{1}{2}(A^TR_{k+1}B^T+PA^TR_{k+1}B^TQ)+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k)^TP_i)\\
&= tr(\tfrac{1}{2}(A^TR_{k+1}B^T+PA^TR_{k+1}B^TQ)^TP_i)\\
&= tr(\tfrac{1}{2}(A^TR_{k+1}B^T+PA^TR_{k+1}B^TQ)^T\\
&= +\tfrac{1}{2}(A^TR_{k+1}B^T-PA^TR_{k+1}B^TQ))^TP_i)\\
&= tr(\tfrac{1}{2}(A^TR_{k+1}B^TP_i)=tr(R_{k+1}^TAP_iB)\\
&= \tfrac{\|P_i\|^2}{\|R_i\|^2}tr(R_{k+1}^T(R_i-R_{i+1}))=0.
\end{aligned}$$

Therefore, (2) holds for $s=k+1$ and we complete the proof.

Lemma 6 Suppose X be an arbitrary solution of Problem I and X_k,R_k,P_k be the sequences in Algorithm 1, then we have

$$tr((X-X_k)P_k^T)=\|R_k\|^2. \tag{3}$$

Proof: For $s = 1$, we have

$$\begin{aligned} tr((X - X_1)P_1^T) &= tr(\tfrac{1}{2}(X - X_1)(A^T R_1 B + PA^T R_1 B^T Q)^T) \\ &= tr((X - X_1)(A^T R_1 B^T)^T) \\ &= tr((X - X_1)BR_1^T A) \\ &= tr(A(X - X_1)BR_1^T) \\ &= tr((C - AX_1 B)R_1^T) = \|R_1\|^2. \end{aligned}$$

Now we assume that (3) holds for $s = k - 1$, then we can get

$$\begin{aligned} tr((X - X_k)P_{k-1}^T) &= tr((X - X_{k-1} - \tfrac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2} P_{k-1})P_{k-1}^T) \\ &= tr((X - X_{k-1})P_{k-1}^T) - \tfrac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2}\|P_{k-1}\|^2 \\ &= \|R_{k-1}\|^2 - \tfrac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2}\|P_{k-1}\|^2 \\ &= 0, \end{aligned}$$

$$\begin{aligned} tr((X - X_k)P_k^T) &= tr((X - X_k)(\tfrac{1}{2}(A^T R_k B^T + PA^T R_k B^T Q) + \tfrac{\|R_k\|^2}{\|R_{k-1}\|^2} P_{k-1})^T) \\ &= tr(\tfrac{1}{2}(X - X_k)(A^T R_k B^T + PA^T R_k B^T Q)^T) + \tfrac{\|R_k\|^2}{\|R_{k-1}\|^2} tr((X - X_k)P_{k-1}^T) \\ &= tr((X - X_k)(A^T R_k B^T)^T) \\ &= tr(R_k^T A(X - X_k)B) = \|R_k\|^2, \end{aligned}$$

that is, (3) holds for $s = k$ and the proof is completed.

By Lemma 5 and Lemma 6, we can get the following theorem.

Theorem 7 Suppose that Problem I is consistent, then for an arbitrary initial matrix $X_1 \in R_r^{n\times m}(P,Q)$, a generalized reflexive solution of Problem I can be obtained by Algorithm 1 within finite iterative steps in the absence of roundoff errors.

Proof: If $R_i \neq 0$, we have that $P_i \neq 0, i = 1, 2, \cdots, pq$ from Lemma 6. Hence, we can compute X_{pq+1} and R_{pq+1} by Algorithm 1. Now according to Lemma 5, we know

$$tr(R_{pq+1}^T R_i) = 0, i = 1, 2, \cdots, pq,$$

and

$$tr(R_j R_i^T) = 0, i, j = 1, 2, \cdots, pq, i \neq j.$$

As $R_i \in R^{p\times q}$, the set of $R_1, R_2, \cdots, R_{pq}$ forms an orthogonal basis of the matrix space $R^{p\times q}$, which implies $R_{pq+1} = 0$ and X_{pq+1} is a generalized reflexive solution of Problem I.

Now we prove the solution of Problem I can be obtained by Algorithm 1 within finite iterative steps in the absence of roundoff errors.

Let $s = \min(pq, mn)$, when Problem I is consistent, if $mn \leq pq$ and $R_i \neq 0, i = 1, 2, \cdots, mn$, then $P_i \neq 0, i = 1, 2, \cdots, mn$, and so $X_{mn+1}, R_{mn+1}, P_{mn+1}$ can be computed by Algorithm 1. As $P_i \in R^{n\times m}$, we can get $P_{mn+1} = 0$, and then $R_{mn+1} = 0$, that is, X_{mn+1} is a solution of Problem I. If $mn > pq$, by the first part of the proof, we can know X_{pq+1} is a solution of Problem I.

Theorem 8 Problem I is inconsistent if and only if there exists a positive integer k such that $R_k \neq 0$ and $P_k = 0$ in the process of the iteration.

Proof: Firstly, by Lemma 6 we know, if exists k such that $R_k \neq 0$ and $P_k = 0$, then we know Problem I is inconsistent.

Now we prove the necessity. If Problem I is inconsistent, then $R_k \neq 0$ for any positive integer k. According to the proof of Theorem 7, we know there must exist a positive integer k such that $P_k = 0$. Otherwise, if

$P_k \neq 0$ for all positive integer k, there must have a solution of Problem I, which contradict to the inconsistency of Problem I.

Lemma 9[8] Assume that the consistent system of linear equations $My = b$ has a solution $y_0 \in R(M^T)$, then y_0 is the least-norm solution of the system of linear equations.

Theorem 10 If Problem I is consistent and $X_1 = A^T HB^T + PA^T HB^T Q$ is the initial iterative matrix, where H is arbitrary, or especially let $X_1 = 0$, then the generalized reflexive solution X^* generated by Algorithm 1 is the least-norm generalized reflexive solution of the problem I.

Proof: Firstly, by Algorithm 1 and Theorem 7, we can obtain the solution of Problem I denoted by X^* within finite steps. Obviously, X^* can be represented by $X^* = A^T MB^T + PA^T MB^T Q$.

Then we will prove that X^* is just the least-norm solution of Problem I.

Consider the system of matrix equations

$$\begin{bmatrix} B^T \otimes A \\ B^T Q \otimes AP \end{bmatrix} X = \begin{bmatrix} C \\ C \end{bmatrix}. \tag{4}$$

Noting that

$$\begin{aligned} x^* &= vec(A^T MB^T + PA^T MB^T Q) \\ &= (B^T \otimes A + B^T Q \otimes AP)y \end{aligned}$$

and

$$\begin{bmatrix} B^T \otimes A \\ B^T Q \otimes AP \end{bmatrix}^T \begin{bmatrix} y \\ y \end{bmatrix} \in R\left(\begin{bmatrix} B^T \otimes A \\ B^T Q \otimes AP \end{bmatrix}^T\right),$$

by Lemma 9 we know x^* is the least-norm solution of Eq. (4), so X^* is the least-norm solution of Eq. (3) since vec operator is isomorphic, i.e., X^* is the least-norm solution of Problem I.

Now we consider Problem II.

For an arbitrary matrix $Y \in R^{n\times m}$ and $X \in S_r$, by Lemma 3 we can get

$$||X - Y||^2 = ||X - \frac{1}{2}(Y + PYQ) - \frac{1}{2}(Y - PYQ)||^2 = ||X - \frac{1}{2}(Y + PYQ)||^2 + ||\frac{1}{2}(Y - PYQ)||^2.$$

When Problem I is consistent, the set of solutions of Problem I denoted by S_r is no empty. Hence, to find $X_0 \in S_r$ such that $\min ||X - Y||^2$ is equivalent to find $X_0 \in S_r$ such that

$$\min ||X - \frac{1}{2}(Y + PYQ)||^2. \tag{5}$$

It is easy to verify that

$$A(X - \frac{1}{2}(Y + PYQ))B = C - \frac{1}{2}A(Y + PYQ)B.$$

Let $\overline{X} = X - \frac{1}{2}(Y + PYQ), \overline{C} = C - \frac{1}{2}A(Y + PYQ))B$, then (5) is equivalent to find the least-norm generalized reflexive solution $\overline{X}^*$ of the following matrix equation

$$A\overline{X}B = \overline{C}. \tag{6}$$

According to Theorem 10, if we take the initial iterative matrix $\overline{X}_1 = A^T HB^T + PA^T HB^T Q$ for (6), where H is arbitrary, or especially let $\overline{X}_1 = 0$, then the least-norm solution $\overline{X}^*$ of (6) can be obtained by Algorithm 1, and so is the solution of Problem II, i.e., $X_0 = \overline{X}^* + \frac{1}{2}(Y + PYQ)$.

Now, we can consider Problem III and Problem IV. According to Lemma 2 and Lemma 3, one algorithm can be proposed easily for solving the generalized anti-reflexive solution of the matrix equation $AXB = C$ and the corresponding theoretical results can also be deduced.

Algorithm 2: (An iterative algorithm for solving Problem III)

Step 1. Input matrices $A \in R^{p\times n}, B \in R^{m\times q}, C \in R^{p\times q}$;

Step 2. Choose any matrix $X_1 \in R_a^{n\times m}(P,Q)$, where $P \in R^{n\times n}, Q \in R^{m\times m}$ are two arbitrary generalized reflection matrices;

Step 3. Compute $R_1 = C - AX_1B, P_1 = \frac{1}{2}(A^TR_1B^T - PA^TR_1B^TQ), k = 1$;

Step 4. Compute

$$X_{k+1} = X_k + \frac{\|R_k\|^2}{\|P_k\|^2}AP_k,$$

$$R_{k+1} = C - AX_{k+1}B = R_k - \frac{\|R_k\|^2}{\|P_k\|^2}AP_kB,$$

$$P_{k+1} = \frac{1}{2}(A^TR_{k+1}B^T - PA^TR_{k+1}B^TQ) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k;$$

Step 5. If $R_{k+1} = 0$, or $R_{k+1} \neq 0$ and $P_{k+1} = 0$, stop; otherwise, let $k = k+1$, go to Step 4.

Obviously, we know that $X_k, P_k \in R_a^{n\times m}(P,Q)$, where $k = 1, 2, \cdots$

By Algorithm 2, we can get the following lemmas.

Lemma 11 For the sequences R_i, P_i which are produced by Algorithm 2 and $s \geq 2$, we have that

$$tr(R_j^TR_i) = 0, tr(P_j^TP_i) = 0 (i \neq j; i, j = 1, 2, \cdots, s) \tag{7}$$

Lemma 12 Suppose X be an arbitrary solution of Problem III and X_k, R_k, P_k be the sequences in Algorithm 2, then we have

$$tr((X - X_k)P_k^T) = \|R_k\|^2. \tag{8}$$

Proof: The proof is the same as Lemma 6.

By Lemma 11 and Lemma 12, we can also get the following theorems, and the proofs of which are similar with Theorem 7, Theorem 8, and Theorem 10 by suitable modifications.

Theorem 13 Suppose that Problem III is consistent, then for an arbitrary initial matrix $X_1 \in R_a^{n\times m}(P,Q)$, a generalized anti-reflexive solution of Problem III can be obtained by Algorithm 2 within finite iterative steps in the absence of roundoff errors.

Theorem 14 Problem III is inconsistent if and only if there exists a positive integer k such that $R_k \neq 0$ and $P_k = 0$ in the process of the iteration.

Theorem 15 If Problem III is consistent and $X_1 = A^THB^T - PA^THB^TQ$ is the initial iterative matrix, where H is arbitrary, or especially let $X_1 = 0$, then the generalized reflexive solution X^* generated by Algorithm 2 is the least-norm generalized anti-reflexive solution of the problem III.

Then we can present an finite iterative method for solving Problem IV, by using Algorithm 2.

For an arbitrary matrix $Y \in R^{n\times m}$ and $X \in S_a$, then we can have

$$\|X - Y\|^2 = \|X - \frac{1}{2}(Y + PYQ) - \frac{1}{2}(Y - PYQ)\|^2$$

$$= \|X - \frac{1}{2}(Y - PYQ)\|^2 + \|\frac{1}{2}(Y + PYQ)\|^2.$$

When Problem III is consistent, the set of solutions of Problem III denoted by S_a is no empty. Hence, to find $X_0 \in S_a$ such that $\min \|X - Y\|^2$ is equivalent to find $X_0 \in S_a$ such that

$$\min \|X - \frac{1}{2}(Y - PYQ)\|^2. \tag{9}$$

As

$$A(X - \frac{1}{2}(Y - PYQ))B = C - \frac{1}{2}A(Y - PYQ)B,$$

let

$$\overline{X} = X - \frac{1}{2}(Y - PYQ), \overline{C} = C - \frac{1}{2}A(Y - PYQ))B$$

, then (9) is equivalent to find the least-norm generalized anti-reflexive solution $\overline{X}^*$ of the following matrix equation

$$A\overline{X}B = \overline{C}. \tag{10}$$

By Theorem 15, if we take the initial iterative matrix $\overline{X}_1 = A^T HB^T - PA^T HB^T Q$ for (10), where H is arbitrary, or especially let $\overline{X}_1 = 0$, then the least-norm solution $\overline{X}^*$ of (10) can be obtained by Algorithm 2, and so is the solution of Problem IV, i.e., $X_0 = \overline{X}^* + \frac{1}{2}(Y - PYQ)$.

3. Numerical Examples

In this section, we present some examples to demonstrate our results. The stopping criterion used is that the Frobenius norm of R_k or P_k less than ϵ, where $\epsilon = 10^{-10}$.

Example 1. We consider the matrix equation $AXB = C$, with

$$A = \begin{pmatrix} 1 & 0 & 10 & -32 \\ 9 & 13 & 0 & 62 \\ 11 & -32 & 7 & 0 \\ 12 & 0 & 21 & 3 \\ 12 & 8 & 35 & 2 \\ 13 & 0 & 45 & 23 \end{pmatrix}, \quad B = \begin{pmatrix} 12 & -11 & 3 \\ 12 & 0 & 2 \\ 21 & 9 & 13 \\ 45 & 12 & 35 \\ 15 & 36 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} -762 & -6430 & 3178 \\ 4896 & 0 & 816 \\ 7179 & -4501 & 3510 \\ 5841 & -13503 & 7914 \\ 7179 & -22505 & 12764 \\ 11925 & -28935 & 16860 \end{pmatrix}.$$

(a) Find the generalized reflexive solution and the least-norm generalized reflexive solution of the matrix equation with respect to P, Q

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(b) Let S_r denote the set of all generalized reflexive solutions of the matrix equation $AXB = C$ with respect to P, Q. For a given matrix $Y \in R^{n\times m}$, find $X_0 \in S_r$ such that $\|X_0 - Y\|^2 = \min_{X\in S_r} \|X - Y\|^2$.

Firstly, we can compute the generalized reflexive solution and least-norm generalized reflexive solution of the matrix equation $AXB = C$ by using Algorithm 1.

Choose an arbitrary initial iterative matrix X_1 which is a generalized reflexive matrix, such as

$$X_1 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 4 & 0 & 3 & 6 & -1 \\ 0 & -7 & 0 & 0 & 0 \end{pmatrix}.$$

By Algorithm 1, we can get

$$X_{11} = \begin{pmatrix} 0 & 13.000 & 0 & 0 & 0 \\ 0 & -11.000 & 0 & 0 & 0 \\ 2.4622 & 0 & 0.4590 & 9.9042 & -20.5249 \\ 0 & -7.0000 & 0 & 0 & 0 \end{pmatrix},$$
$$\|R_{11}\| = 1.4060e - 011.$$

So we obtain a generalized reflexive solution of the matrix equation $AXB = C$ as follows

$$X = \begin{pmatrix} 0 & 13.000 & 0 & 0 & 0 \\ 0 & -11.000 & 0 & 0 & 0 \\ 2.4622 & 0 & 0.4590 & 9.9042 & -20.5249 \\ 0 & -7.0000 & 0 & 0 & 0 \end{pmatrix}.$$

If we take the initial iterative matrix as $X_1 = A^T MB^T + PA^T MB^T Q$ with

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 0 \\ 4 & 3 & -1 \\ -11 & 3 & 4 \\ 9 & 8 & 5 \end{pmatrix},$$

then we have

$$X_{15} = \begin{pmatrix} 0 & 13.000 & 0 & 0 & 0 \\ 0 & -11.000 & 0 & 0 & 0 \\ 2.5116 & 0 & 0.1875 & 10.0008 & -20.4742 \\ 0 & -7.0000 & 0 & 0 & 0 \end{pmatrix},$$
$$\|R_{15}\| = 8.8499e - 011.$$

that is, we obtain the least-norm generalized reflexive solution

$$X^* = \begin{pmatrix} 0 & 13.000 & 0 & 0 & 0 \\ 0 & -11.000 & 0 & 0 & 0 \\ 2.5116 & 0 & 0.1875 & 10.0008 & -20.4742 \\ 0 & -7.0000 & 0 & 0 & 0 \end{pmatrix}.$$

If we take $X_1 = 0$, then we have

$$X^* = X_{10} = \begin{pmatrix} 0 & 13.000 & 0 & 0 & 0 \\ 0 & -11.000 & 0 & 0 & 0 \\ 2.5116 & 0 & 0.1875 & 10.0008 & -20.4742 \\ 0 & -7.0000 & 0 & 0 & 0 \end{pmatrix},$$
$$\|R_{10}\| = 8.8499e - 011.$$

If we take

$$Y = \begin{pmatrix} 1 & 3 & -4 & 5 & 9 \\ 0 & 2 & -3 & 4 & 0 \\ 3 & 2 & 11 & 9 & 3 \\ 0 & 9 & 7 & -3 & -5 \end{pmatrix},$$

then

$$\overline{C} = C - \tfrac{1}{2}A(Y + PYQ)B = \begin{pmatrix} -4512 & -9250 & -922 \\ -2436 & 0 & -406 \\ 2532 & -6475 & 303 \\ -9972 & -19425 & -2019 \\ -18756 & -32375 & -3721 \\ -23292 & -41625 & -4647 \end{pmatrix},$$
$$\overline{X} = X - \tfrac{1}{2}(Y + PYQ).$$

By Algorithm 1, taking the initial iterative matrix $\overline{X}_1 = 0$, we can obtain the least-norm generalized reflexive solution $\overline{X}^*$ of the matrix equation $A\overline{X}B = \overline{C}$ as follows

$$\overline{X}^* = X_{11} = \begin{pmatrix} 0 & 10.000 & 0 & 0 & 0 \\ 0 & -13.000 & 0 & 0 & 0 \\ -1.5072 & 0 & -5.2119 & -0.9921 & -24.5213 \\ 0 & -2.0000 & 0 & 0 & 0 \end{pmatrix},$$
$$\|R_{11}\| = 1.6819e-011.$$

So we can obtain a solution of Problem II, i.e.,

$$X_0 = \overline{X}^* + \frac{1}{2}(Y + PYQ) = \begin{pmatrix} 0 & 13.000 & 0 & 0 & 0 \\ 0 & -11.000 & 0 & 0 & 0 \\ 1.4928 & 0 & 5.7881 & 8.0079 & -21.5213 \\ 0 & 7.0000 & 0 & 0 & 0 \end{pmatrix}$$

Example 2. We consider the matrix equation $AXB = C$, with

$$A = \begin{pmatrix} -13 & 64 & 9 & 0 \\ 3 & 13 & 44 & 0 \\ -7 & -11 & 24 & 12 \\ 7 & 0 & 0 & 1 \\ 34 & 13 & -7 & 5 \\ 0 & 5 & 0 & -19 \end{pmatrix}, \ B = \begin{pmatrix} 1 & -11 & 7 \\ -8 & 2 & 0 \\ -5 & -7 & 14 \\ 0 & 110 & 2 \\ 11 & 7 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 5437 & -4795 & 2201 \\ -7240 & -464 & 3495 \\ -5966 & -802 & 1601 \\ 10 & 4180 & 148 \\ 2904 & 21340 & 597 \\ 508 & -7518 & -411 \end{pmatrix}.$$

(a) Find the generalized anti-reflexive solution and the least-norm generalized anti-reflexive solution of the matrix equation with respect to P, Q

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Let S_a denote the set of all generalized anti-reflexive solutions of the matrix equation $AXB = C$ with respect to P, Q. For a given matrix $Y \in R^{n\times m}$, find $X_0 \in S_a$ such that $\|X_0 - Y\|^2 = \min_{X\in S_a} \|X - Y\|^2$.

At first, we compute the generalized anti-reflexive solution and least-norm generalized anti-reflexive solution of the matrix equation $AXB = C$ by using Algorithm 2.

Choose an arbitrary initial iterative matrix X_1 which is a generalized reflexive matrix, such as

$$X_1 = \begin{pmatrix} 3 & 0 & 0 & -4 & 0 \\ 0 & 13 & 35 & 0 & -44 \\ 0 & 22 & 15 & 0 & 9 \\ 6 & 0 & 0 & -8 & 0 \end{pmatrix}.$$

By Algorithm 2, we can get

$$X_{16} = \begin{pmatrix} 1.0000 & 0 & 0 & 5.0000 & 0 \\ 0 & -3.0000 & 2.0000 & 0 & 9.0000 \\ 0 & 12.0000 & 5.0000 & 0 & -7.0000 \\ 3.0000 & 0 & 0 & 4.0000 & 0 \end{pmatrix},$$
$$\|R_{16}\| = 3.2184e-011.$$

So we obtain a generalized anti-reflexive solution of the matrix equation $AXB = C$ as follows

$$X = \begin{pmatrix} 1.0000 & 0 & 0 & 5.0000 & 0 \\ 0 & -3.0000 & 2.0000 & 0 & 9.0000 \\ 0 & 12.0000 & 5.0000 & 0 & -7.0000 \\ 3.0000 & 0 & 0 & 4.0000 & 0 \end{pmatrix}.$$

If we take $X_1 = 0$, then we can get the least-norm generalized anti-reflexive solution

$$X^* = X_{15} = \begin{pmatrix} 1.0000 & 0 & 0 & 5.0000 & 0 \\ 0 & -3.0000 & 2.0000 & 0 & 9.0000 \\ 0 & 12.0000 & 5.0000 & 0 & -7.0000 \\ 3.0000 & 0 & 0 & 4.0000 & 0 \end{pmatrix},$$
$$\|R_{15}\| = 6.1397e - 011.$$

If we take

$$Y = \begin{pmatrix} 1 & 3 & -4 & 5 & 9 \\ 0 & 2 & -3 & 4 & 0 \\ 3 & 2 & 11 & 9 & 3 \\ 0 & 9 & 7 & -3 & -5 \end{pmatrix},$$

then

$$\overline{C} = C - \tfrac{1}{2}A(Y - PYQ)B = \begin{pmatrix} 6153 & 1269 & 3724 \\ -4106 & 806 & -2786 \\ -4266 & 8958 & -2366 \\ 3 & 737 & 35 \\ 2386 & 3828 & 1673 \\ 513 & -13913 & -315 \end{pmatrix},$$
$$\overline{X} = X - \tfrac{1}{2}(Y - PYQ).$$

By Algorithm 2, taking the initial iterative matrix $\overline{X}_1 = 0$, we can obtain the least-norm generalized reflexive solution $\overline{X}^*$ of the matrix equation $A\overline{X}B = \overline{C}$ as follows

$$\overline{X}^* = X_{16} = \begin{pmatrix} -0.0000 & 0 & 0 & -0.0000 & 0 \\ 0 & -5.0000 & 5.0000 & 0 & 9.0000 \\ 0 & 10.0000 & -6.0000 & 0 & -7.0000 \\ 3.0000 & 0 & 0 & 7.0000 & 0 \end{pmatrix},$$
$$\|R_{16}\| = 8.9389e - 011.$$

So we can obtain a solution of Problem II, i.e.,

$$X_0 = \overline{X}^* + \frac{1}{2}(Y - PYQ) = \begin{pmatrix} -1.0000 & 0 & 0 & 5.0000 & 0 \\ 0 & -3.0000 & 2.0000 & 0 & 9.0000 \\ 0 & 12.0000 & 5.0000 & 0 & -7.0000 \\ 3.0000 & 0 & 0 & 4.0000 & 0 \end{pmatrix}.$$

Example 3. We consider the matrix equation $AXB = C$, with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then by Algorithm 1 we have $\|R_7\| = 2.5709, \|P_7\| = 8.1098e-11$. By Algorithm 2 we can get $\|R_7\| = 4.3608, \|P_7\| = 1.5968e-09$. Therefore, the matrix equation has no generalized reflexive and anti-reflexive solution.

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