



## Approximation Properties of Szász Type Operators Involving Charlier Polynomials

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**Abstract.** In this paper, we give some approximation properties of Szász type operators involving Charlier polynomials in the polynomial weighted space and we give the quantitative Voronovskaya- type asymptotic formula.

### 1. Introduction

Orthogonal polynomials are important area of mathematical analysis, mathematical and theoretical physics. Applications to physics, it is interesting to study properties of these polynomials related to properties of the suitable oscillator-like systems called oscillators ([13], [4]). Some of the most frequently used polynomials are Hermite, Charlier and etc.

In mathematical analysis and also in the positive approximation processes, the notion of orthogonal polynomials seldomly appears. Cheney and Sharma [5] established an operator

$$P_n(f; x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t)x^k \quad (1)$$

where  $t \leq 0$  and  $L_k^{(n)}$  denotes the Laguerre polynomials. For the special case  $t = 0$ , the operators given by (1) reduce to the well-known Meyer-König and Zeller operators [10].

In view of the relation between orthogonal polynomials and positive linear operators have been investigated by many researchers. One of them is Varma and Tasdelen 's study. They found positive linear operators which are the generalization of Szász operators [12]

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \quad (2)$$

They consider positive linear operators including Charlier polynomials which is one of the discrete orthogonal polynomials. Charlier polynomials [9] have the generating functions of the form

$$e^t \left(1 - \frac{t}{a}\right)^u = \sum_{k=0}^{\infty} C_k^{(a)}(u) \frac{t^k}{k!}, \quad |t| < a$$

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and the explicit representation

$$C_k^{(a)}(u) = \sum_{r=0}^k \binom{k}{r} (-u)_r \left(\frac{1}{a}\right)_r$$

where  $(\alpha)_k$  is the Pochhammer’s symbol given by

$$(\alpha)_0 = 1, (\alpha)_k = \alpha(\alpha + 1)\dots(\alpha + k - 1) \quad k = 1, 2, \dots$$

We know that for  $a > 0$  and  $u \leq 0$ , Charlier polynomials are positive. These linear positive operators are given by

$$L_n(f, x, a) = e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^a(-(1-a)nx)}{k!} f\left(\frac{k}{n}\right) \tag{3}$$

where  $a > 1$  and  $x \geq 0$ . If we apply respectively  $a \rightarrow \infty$  and  $x - \frac{1}{n}$  instead of  $x$  to the both sides of the operators, we obtain Szász operators given by (2).

Now we consider the well-known Korovkin theorem which has an important role in approximation theory has been established for a sequence of positive linear operators. We know that the Korovkin theorem is valid on  $C[a, b]$ . The first example is Bernstein polynomials which uniformly converges to a function on  $[0, 1]$ . This theorem has been extended in several directions.

We know that if we take a function on  $[0, \infty)$ , then the uniform norm is not valid to evaluate the rate of convergence for unbounded functions. For this reason, we use a weighted modulus of continuity for unbounded functions. In [6] and [7] weighted Korovkin type theorems have been proved by Gadjiev et al. Some weighted Korovkin type theorems can also be found in the space of weighted functions (see [2], [15] etc.).

The paper is organized as follows. In section 2, we evaluate some moments of  $L_n(f)$ . In section 3, we give some approximation properties of these operators in the polynomial weighted space and in Section 4, we give the quantitative Voronovskaya- type asymptotic formula.

## 2. Auxiliary Results

We first present some results which will be used in the proofs of our theorems.

**Lemma 2.1.** *Let  $L_n$  be defined by (3). Then we have*

$$L_n(1; x, a) = 1, \tag{4}$$

$$L_n(t; x, a) = x + \frac{1}{n}, \tag{5}$$

$$L_n(t^2; x, a) = x^2 + \frac{x}{n} \left(3 + \frac{1}{a-1}\right) + \frac{2}{n^2}, \tag{6}$$

$$L_n(t^3; x, a) = x^3 + \frac{x^2}{n} \left(6 + \frac{3}{a-1}\right) + \frac{x}{n^2} \left(10 + \frac{6}{a-1} + \frac{2}{(a-1)^2}\right) + \frac{5}{n^3}, \tag{7}$$

$$L_n(t^4; x, a) = x^4 + \frac{x^3}{n} \left(10 + \frac{3}{a-1}\right) + \frac{x^2}{n^2} \left(31 + \frac{30}{a-1} + \frac{5}{(a-1)^2}\right) + \frac{x}{n^3} \left(37 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{15}{(a-1)^3}\right) + \frac{15}{n^4}. \tag{8}$$

**Lemma 2.2.** *For the operator  $L_n$  given by (3), we have the following equalities.*

$$L_n((t-x); x, a) = \frac{1}{n},$$

$$L_n((t-x)^2; x, a) = \frac{x}{n} \left( 1 + \frac{1}{a-1} \right) + \frac{2}{n^2},$$

$$L_n((t-x)^4; x, a) = \frac{x^2}{n^2} \left( 3 + \frac{6}{a-1} - \frac{3}{(a-1)^2} \right) + \frac{x}{n^3} \left( 17 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{15}{(a-1)^3} \right) + \frac{15}{n^4}.$$

### 3. Approximation Properties in a Weighted Space

In this section, by using Bohman-Korovkin type theorem proved in [8], we present the direct approximation property of the operator  $L_n$  given by (3).

Let us denote by  $B_2(\mathbb{R}^+)$  the weighted space of real-valued functions  $f$  defined on  $\mathbb{R}^+$  with the property  $|f(x)| \leq M_f(1+x^2)$  for all  $x \in \mathbb{R}^+$ , where  $M_f$  is a constant depending on the function  $f$ . We consider the weighted subspace  $C_2(\mathbb{R}^+)$  of  $B_2(\mathbb{R}^+)$  given by

$$C_2(\mathbb{R}^+) = \{f \in B_2(\mathbb{R}^+) : f \text{ is continuous on } \mathbb{R}^+\}$$

and

$$C_2^2(\mathbb{R}^+) = \{f \in C_2(\mathbb{R}^+) : f', f'' \in C_2(\mathbb{R}^+)\}.$$

We also consider the space of functions

$$C_2^*(\mathbb{R}^+) = \left\{ f \in C_2(\mathbb{R}^+) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} = k \in \mathbb{R} \right\}$$

equipped with the norm

$$\|f\|_2 = \sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{1+x^2}.$$

**Theorem A.** Let  $T_n$  be a sequence of linear positive operators mapping  $C_2(\mathbb{R}^+)$  into  $B_2(\mathbb{R}^+)$  and satisfying the conditions

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|T_n(t^v, x) - x^v|}{1+x^2} = 0, \text{ for } v = 0, 1, 2.$$

Then for any  $f \in C_2^*(\mathbb{R}^+)$ , we have

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|T_n(f, x) - f(x)|}{1+x^2} = 0$$

and there exists a function  $f^* \in C_2(\mathbb{R}) \setminus C_2^*(\mathbb{R})$  such that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|T_n(f^*, x) - f^*(x)|}{1+x^2} \geq 1.$$

For  $f \in C_2^*(\mathbb{R})$ , we will consider the weighted modulus of continuity defined in [15] given by

$$\Omega_2(f, \delta) = \sup_{x \in \mathbb{R}^+, |h| \leq \delta} \frac{|f(x+h) - f(x)|}{(1+x^2)(1+h^2)}.$$

This function has the following properties:

1.  $\Omega_2(f, \delta) \leq 2 \|f\|_2$ ,
2.  $\Omega_2(f, m\delta) \leq 2m(1 + \delta^2)\Omega_2(f, \delta), m \in \mathbb{N}$ ,
3.  $\lim_{\delta \rightarrow 0} \Omega_2(f, \delta) = 0$ .

Note that, we can not find a rate of convergence in terms of usual first modulus of continuity  $\omega_1(f, \delta)$  of a function  $f$ . Because on the infinite interval, the modulus of continuity  $\omega_1(f; \delta)$  does not tend to zero as  $\delta$  tends to zero. For this reason, we consider the weighted modulus of continuity  $\Omega_2(f, \delta)$ .

**Remark 3.1.** Since any linear and positive operator is monotone, the relations (4) and (6) guarantee that  $L_n(f) \in C_2(\mathbb{R}^+)$  for each  $f \in C_2(\mathbb{R}^+)$ .

**Lemma 3.2.** There exists a positive constant  $M_1(a)$  such that

$$\|L_n(1 + x^2; x, a)\|_2 \leq M_1(a), n \in \mathbb{N}. \quad (9)$$

Moreover for every  $f \in C_2(\mathbb{R}^+)$ , we have

$$\|L_n(f)\|_2 \leq M_1(a) \|f\|_2, n \in \mathbb{N}. \quad (10)$$

Thus  $L_n$  is a linear positive operator from  $C_2(\mathbb{R})$  into  $C_2(\mathbb{R})$ .

*Proof.* First we shall prove the inequality (9).

$$\begin{aligned} & \frac{1}{1+x^2} L_n(1+x^2; x, a) \\ &= \frac{1}{1+x^2} \left( 1+x^2 + \frac{x}{n} \left( 3 + \frac{1}{a-1} + \frac{2}{n^2} \right) \right) \\ &= \frac{1}{1+x^2} + \frac{1}{1+x^2} \left( x^2 + \frac{x}{n} \left( 3 + \frac{1}{a-1} + \frac{2}{n^2} \right) \right) \\ &\leq M_1(a), \end{aligned} \quad (11)$$

$M_1(a)$  is a positive constant depending on  $a$ . On the other hand we have

$$\|L_n(f)\|_2 \leq \|L_n(1+x^2)\|_2 \|f\|_2$$

for every  $f \in C_2(\mathbb{R}^+)$ . By applying (11), we have

$$\|L_n(f)\|_2 \leq M_1(a) \|f\|_2, n \in \mathbb{N}.$$

Thus the proof is completed.  $\square$

**Lemma 3.3.** There exists a positive constant  $M_2(a)$  such that

$$\|L_n((1+t^2)(t-x)^2; \cdot)\| \leq \frac{M_2(a)}{n}, n \in \mathbb{N}.$$

*Proof.* From the linearity of  $L_n$ , we can write

$$L_n((1+t^2)(t-x)^2; x, a) = L_n((t-x)^2; x, a) + L_n((t-x)^2 t^2; x, a).$$

From Lemma 2.2, we can write

$$\begin{aligned} & \frac{1}{1+x^2} L_n((t-x)^2 t^2; x, a) \\ &= \frac{1}{1+x^2} \left\{ \frac{x^3}{n} \left( 1 + \frac{10}{a-1} \right) + \frac{x^2}{n^2} \left( 13 + \frac{18}{a-1} + \frac{1}{(a-1)^2} \right) \right. \\ & \quad \left. + \frac{x}{n^3} \left( 27 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{15}{(a-1)^3} \right) + \frac{15}{n^4} \right\} \\ &= \frac{1}{1+x^2} \frac{x}{n} \left\{ x^2 \left( 1 + \frac{10}{a-1} \right) + \frac{x}{n} \left( 13 + \frac{18}{a-1} + \frac{1}{(a-1)^2} \right) \right. \\ & \quad \left. + \frac{1}{n^2} \left( 27 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{15}{(a-1)^3} \right) \right\} + \frac{1}{1+x^2} \frac{15}{n^4} \\ &= \frac{1}{1+x^2} \frac{x}{n} T_2(a), \end{aligned}$$

where  $T_2(a)$  is a polynomial of degree 2. Therefore we have

$$\frac{1}{1+x^2} L_n((t-x)^2 t^2; x, a) \leq \frac{x}{n} M_2(a). \tag{12}$$

□

**Theorem 3.4.** For each  $f \in C_2^*(\mathbb{R}^+)$ , we have

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|L_n(f, x, a) - f(x)|}{1+x^2} = 0.$$

*Proof.* It is easy to see that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|L_n(1, x, a) - 1|}{1+x^2} = 0.$$

From Lemma 2.1 we get

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|L_n(t, x, a) - x|}{1+x^2} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Also, using Lemma 2.1 again, we can write

$$\sup_{x \in \mathbb{R}^+} \frac{|L_n(t^2, x, a) - x^2|}{1+x^2} \leq \sup_{x \in \mathbb{R}^+} \frac{x}{1+x^2} \frac{1}{n} \left( 3 + \frac{1}{a-1} \right) + \frac{2}{n^2},$$

which implies that for  $a > 1$

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|L_n(t^2, x, a) - x^2|}{1+x^2} = 0.$$

Since the conditions of Theorem A of [15] are satisfied, we obtain for any  $f \in C_2^*(\mathbb{R}^+)$

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^+} \frac{|L_n(f, x, a) - f(x)|}{1+x^2} = 0.$$

So this completes the proof. □

**Theorem 3.5.** For  $f \in C_2^*(\mathbb{R}^+)$ ,  $n \in \mathbb{N}$  we have

$$\sup_{x \in \mathbb{R}^+} \frac{|L_n(f, x, a) - f(x)|}{1 + x^2} \leq K\Omega_2(f, n^{-1/2}),$$

where  $K$  is a constant that is independent of  $f$  and  $n$ .

*Proof.* From the properties of  $\Omega_2$  it is obvious that for any  $\lambda > 0$ ,

$$\Omega_2(f, \lambda\delta) \leq 2(\lambda + 1)(1 + \delta^2)\Omega_2(f, \delta).$$

For  $\delta > 0$ , if we use the definition of  $\Omega_2$  and the last inequality with  $\lambda = \frac{|t-x|}{\delta}$  we have

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + x^2)(1 + (t - x)^2)\Omega_2(f, |t - x|) \\ &\leq 2(1 + x^2)(1 + (t - x)^2)(1 + \delta^2)\left(1 + \frac{|t - x|}{\delta}\right)\Omega_2(f, \delta) \\ &\leq 8(1 + x^2)\left(1 + \frac{(t - x)^4}{\delta^4}\right). \end{aligned} \tag{13}$$

Applying  $L_n$  to (13), by the linearity and monotonicity of  $L_n$  we obtain

$$|L_n(f, x, a) - f(x)| \leq 8(1 + x^2)\Omega_2(f, \delta)\left(1 + \frac{1}{\delta^4}L_n((t - x)^4, x, a)\right)$$

From Lemma 2.2, we can write for  $a > 1$

$$L_n((t - x)^4, x, a) = (x^2 + x + 1)\mathcal{O}\left(\frac{1}{n^2}\right).$$

If we choose  $\delta = n$ , we have

$$\sup_{x \in \mathbb{R}} \frac{|L_n(f, x, a) - f(x)|}{(1 + x^2)^3} \leq K\Omega_2(f, n^{-1/2}).$$

□

**Theorem 3.6.** There exist an absolute constant  $M_3(a)$  such that

$$\frac{1}{1 + x^2} |L_n(f; x, a) - f(x)| \leq \frac{1}{n} \left( \|f'\|_2 + \|f''\|_2 \left\{ x \left( 1 + \frac{1}{a-1} \right) + \frac{2}{n} + M_2(a) \right\} \right),$$

where  $f \in C_2^2(\mathbb{R}^+)$  and  $x \in [0, \infty)$ .

*Proof.* Using the Taylor formula

$$f(t) = f(x) + f'(x)(t - x) + \int_x^t \int_x^s f''(u) du ds, \quad t > 0,$$

which implies that

$$f(t) = f(x) + f'(x)(t - x) + \int_x^t (t - u)f''(u) du,$$

applying this formula to the operator, we have

$L_n(f(t); x, a) = f(x) + f'(x)L_n((t-x); x, a) + L_n\left(\int_x^t (t-u)f''(u)du; x, a\right)$ . Using the inequality  $\left|\int_x^t (t-u)f''(u)du\right| \leq \frac{1}{2}\|f''\|_2(2+t^2+x^2)(t-x)^2$ , it follows

$$\begin{aligned} & \frac{1}{1+x^2} |L_n(f; x, a) - f(x)| \\ &= \frac{|f'(x)|}{1+x^2} |L_n((t-x); x, a)| + \frac{1}{1+x^2} \left| L_n\left(\int_x^t \int_x^s f''(u)duds; x, a\right) \right| \\ &\leq \frac{1}{n} \|f'\|_2 + \frac{1}{2(1+x^2)} \|f''\|_2 L_n((t-x)^2((2+t^2+x^2)); x, a) \\ &\leq \frac{1}{n} \|f'\|_2 + \frac{1}{2} \|f''\|_2 L_n((t-x)^2; x, a) + \frac{1}{1+x^2} \frac{1}{2} \|f''\|_2 L_n((1+t^2)(t-x)^2; x, a). \end{aligned}$$

From Lemma 3.3, we obtain

$$\begin{aligned} & \frac{1}{1+x^2} |L_n(f; x, a) - f(x)| \\ &\leq \frac{1}{n} \|f'\|_2 + \frac{1}{2} \|f''\|_2 L_n((t-x)^2; x, a) + \frac{1}{1+x^2} \frac{1}{2} \|f''\|_2 L_n((t-x)^2; x, a) \\ &\quad + \frac{1}{1+x^2} \frac{1}{2} \|f''\|_2 L_n((t-x)^2 t^2; x, a) \\ &\leq \frac{1}{n} \|f'\|_2 + \frac{1}{1+x^2} \frac{1}{2} \|f''\|_2 \left\{ \frac{x}{n} \left(1 + \frac{1}{a-1}\right) + \frac{2}{n^2} \right\} + \|f''\|_2 \frac{M_2(a)}{n} \\ &\leq \frac{1}{n} \left( \|f'\|_2 + \|f''\|_2 \left\{ x \left(1 + \frac{1}{a-1}\right) + \frac{2}{n} + M_2(a) \right\} \right). \end{aligned}$$

This follows that similar to [11], we apply the Steklov function  $f_h$  of  $f \in C_p$ ,

$$f_h(x) := \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} (2f(x+s+t) - f(x+2(s+t)))dsdt$$

for  $x, h > 0$ . Let  $C_{2,*}(\mathbb{R}^+)$  is the set of real valued function  $f$ , continuous on  $\mathbb{R}^+$  and  $\frac{f(x)}{1+x^2}$  is uniformly continuous. It is known [11] that if  $f \in C_{2,*}(\mathbb{R}^+)$ , then

$$\|f - f_h\|_2 \leq \omega_2(f; h)$$

and

$$\|f_h''\|_2 \leq 9h^{-2}\omega_2(f; h), \quad h > 0$$

where

$$\omega_2(f; h) = \sup_{0 < h \leq \delta} \frac{|f(x+2h) - 2f(x+h) + f(x)|}{1+x^2}.$$

Also we consider another type Steklov function  $f_k$  of  $f \in C_{2,*}(\mathbb{R}^+)$  :

$$f_k(x) = \frac{1}{k} \int_0^k f(x+t) dt.$$

This formula implies that

$$\|f - f_k\|_2 \leq \omega(f; k)$$

and

$$\|f'_k\|_2 \leq k^{-1} \omega(f; k), \quad k > 0.$$

Using this inequalities, we have the following approximation theorem.  $\square$

**Theorem 3.7.** Let  $f \in C_{2,*}(\mathbb{R}^+)$ , the set of real valued function  $f$ , continuous on  $\mathbb{R}^+$  and  $\frac{f(x)}{1+x^2}$  is uniformly continuous and let  $x \in \mathbb{R}^+$ . We have

$$\frac{1}{1+x^2} |L_n(f; x, a) - f(x)| \leq \omega_2\left(f; \frac{1}{\sqrt{n}}\right) \left\{ x \left(1 + \frac{1}{a-1}\right) + \frac{2}{n} + M_2(a) + (M_1(a) + 1) \right\} + \frac{1}{\sqrt{n}} \omega\left(f; \frac{1}{\sqrt{n}}\right).$$

*Proof.* For  $f \in C_{2,*}(\mathbb{R}^+)$  and  $h > 0$ ,

$$|L_n(f; x, a) - f(x)| \leq |L_n((f - f_h); x, a) - (f - f_h)(x)| + |L_n(f_h; x, a) - f_h(x)|$$

and therefore

$$\begin{aligned} & \frac{1}{1+x^2} |L_n(f; x, a) - f(x)| \\ & \leq \|f - f_h\|_2 \left( \frac{1}{1+x^2} L_n(1+t^2; x, a) + 1 \right) + \frac{1}{n} \left( \|f'_h\|_2 + \|f''_h\|_2 \left\{ x \left(1 + \frac{1}{a-1}\right) + \frac{2}{n} + M_2(a) \right\} \right) \\ & = \omega_2(f; h) (M_1(a) + 1) + \frac{1}{n} \left( \|f'_h\|_2 + \|f''_h\|_2 \left\{ x \left(1 + \frac{1}{a-1}\right) + \frac{2}{n} + M_2(a) \right\} \right) \\ & = \omega_2(f; h) \frac{1}{h^2 n} \left\{ x \left(1 + \frac{1}{a-1}\right) + \frac{2}{n} + M_2(a) + (M_1(a) + 1) \right\} + \frac{1}{nh} \omega(f; h). \end{aligned}$$

Thus choosing  $h = \frac{1}{\sqrt{n}}$ , the proof is completed.  $\square$

#### 4. Voronovskaya Type Theorem

In this part, we give the Voronovskaya type theorem for the operator  $L_n$  given by (3).

**Theorem 4.1.** Assume that  $f''$  exists at a fixed point  $x \in [0, \infty)$  and  $f, f' \in C_2(\mathbb{R})$ . Then the following equality holds

$$\lim_{n \rightarrow \infty} n [L_n(f; x, a) - f(x)] = f'(x) + x \left(1 + \frac{1}{a-1}\right) \frac{f''(x)}{2}$$

for every  $x \in [0, \infty)$ .

*Proof.* Let  $x \in [0, \infty)$  be fixed. By Taylor's formula we have

$$f(t) = f(x) + (t-x)f'(x) + (t-x)^2 \frac{f''(x)}{2} + k(t; x)(t-x)^2$$

for  $k(\cdot) \in C_2(\mathbb{R}^+)$  and  $\lim_{t \rightarrow x} k(t; x) = 0$ . Applying  $L_n$  to Taylor's formula, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} n [L_n(f; x, a) - f(x)] &= nL_n((t-x); x, a) f'(x) + nL_n((t-x)^2; x, a) \frac{f''(x)}{2} \\ &\quad + nL_n(k(t; x)(t-x)^2; x, a). \end{aligned}$$

By Cauchy-Schwartz inequality, we have

$$nL_n(k(t; x)(t-x)^2; x, a) \leq \sqrt{L_n(k^2(t; x); x, a)} \sqrt{n^2 L_n((t-x)^4; x, a)}.$$

We observe that  $k^2(x; x) = 0$ . It is known that

$$\lim_{n \rightarrow \infty} L_n(f; x, a) = f(x)$$

holds from [14]. Therefore

$$\lim_{n \rightarrow \infty} L_n(k^2(t; x); x, a) = k^2(x; x) = 0.$$

Moreover, we know that

$$\lim_{n \rightarrow \infty} \sqrt{n^2 L_n((t-x)^4; x, a)} = \lim_{n \rightarrow \infty} x \left( 3 + \frac{6}{a-1} + \frac{3}{(a-1)^2} \right)^{1/2}.$$

So we obtain the following

$$\lim_{n \rightarrow \infty} n [L_n(k(t; x)(t-x)^2; x, a)] = 0$$

which completes the proof.  $\square$

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