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A Serrin Type Criterion for Incompressible Hydrodynamic Flow of Liquid Crystals in Dimension Three

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Abstract. In the paper, we establish a Serrin type criterion for strong solutions to a simplified density-dependent Ericksen-Leslie system modeling incompressible, nematic liquid crystal materials in dimension three. The density may vanish in an open subset of Ω. As a byproduct, we establish the Serrin type criterion for heat flow of harmonic map whose gradients belong to $L^r_x L^s_t$, where $\frac{2}{s}+\frac{3}{r}\le 1$, for $3<r\le\infty$.

1. Introduction

We consider the following incompressible hydrodynamic flow of nematic liquids crystals in a bounded domain $\Omega\subset\mathbb{R}^3$:

$$\rho_t+\nabla\cdot(\rho u)=0, \tag{1}$$

$$\rho u_t+\rho u\cdot\nabla u+\nabla P=\gamma\Delta u-\lambda\nabla\cdot(\nabla d\otimes\nabla d), \tag{2}$$

$$\nabla\cdot u=0, \tag{3}$$

$$d_t+u\cdot\nabla d=\theta(\Delta d+|\nabla d|^2 d), \tag{4}$$

where $\rho:\Omega\times(0,\infty)\to\mathbb{R}_+$ is the density of the fluid, $u:\Omega\times(0,\infty)\to\mathbb{R}^3$ is the fluid velocity field, $d:\Omega\times(0,\infty)\to S^2$ represents the macroscopic average of the nematic liquid crystal orientation field; P denotes the pressure of the fluid, $\nabla\cdot$ (= div) denotes the divergence operator on $\mathbb{R}^3$; γ, λ and θ are positive constants.

(1)-(4) is a simplified version of Ericksen-Leslie system modeling incompressible, nematic liquid crystal materials. For more details about this model, readers could be referred for instance to [1, 7–10]. For dimension $N=2$ and ρ =const., Lin, Lin and Wang [9] have proved the global existence of Leray-Hopf type weak solutions to (1)-(4) on bounded domains in $\mathbb{R}^2$ (see [4] for $\Omega=\mathbb{R}^2$). Lin and Wang [12] have further proved that such weak solutions are unique. A further discussion for $N=2$ has been done by Xu and Zhang [17], where global regularity and uniqueness of weak solution with small initial data was proved. For $N=3$, Wen and Ding [16] have established the local existence and uniqueness of strong solutions to (1)-(4). whether global weak (or smooth) solutions exist for $N=3$ is still unknown. Recently, Huang

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and Wang [5] obtained a Beale-Kato-Majda criterion for smooth solutions to (1)-(4) in $\mathbb{R}^3$ when ρ =const., namely, if $0 < T_* < \infty$ is the first singular time, then the $L^1_t L^\infty_x$-norm of the vorticity $\nabla \times u$ or the $L^2_t L^\infty_x$-norm of ∇d must become infinity when $t \nearrow T_*$.

We would like to point out that the system (1)-(4) includes two important equations as special cases:

(i) When u is zero, (4) becomes the heat flow of harmonic map (see [11]).

(ii) When d is a constant vector field, (1)-(3) becomes the nonhomogeneous incompressible Naiver-Stokes equations (see [13]).

In this paper, we shall establish a Serrin type criterion for strong solutions to the system (1)-(4), along with the following initial-boundary condition:

$$(\rho, u, d)\Big|_{t=0} = (\rho_0, u_0, d_0), \tag{5}$$

and

$$(u, \frac{\partial d}{\partial \nu})\Big|_{\partial\Omega} = 0, \tag{6}$$

where ν is the unit outward normal vector of $\partial\Omega$.

To state the definition of strong solutions to the initial-boundary-value problem (1)-(6), we give some notations which will be used throughout the paper.

Denote $Q_T = \Omega \times [0, T]$, $\int f\,dx = \int_\Omega f\,dx$, $L^q := L^q(\Omega)$, $W^{k,p} := W^{k,p}(\Omega)$, $H^k := W^{k,2}(\Omega)$.

Definition 1.1. *(Strong solution) For $T > 0$, (ρ, u, d) is called a strong solution to the incompressible nematic liquid crystal system (1)-(6) in $\Omega \times (0, T]$, if*

$$\begin{gathered}
\rho \in C([0,T]; W^{1,r}),\ \rho_t \in C([0,T]; L^r),\ \text{for some } r > 3,\\
u \in C([0,T]; H^2 \cap H^1_0) \cap L^2(0,T; H^3),\\
u_t \in L^2(0,T; H^1_0),\quad \sqrt{\rho}u_t \in L^\infty(0,T; L^2),\ |d| = 1,\ \text{in } \overline{Q}_T,\\
d \in C([0,T]; H^3) \cap L^2(0,T; H^4),\ d_t \in C([0,T]; H^1) \cap L^2(0,T; H^2),\\
d_{tt} \in L^2(0,T; L^2).
\end{gathered}$$

and (ρ, u, d) satisfies (1)-(4) a.e. in $\Omega \times (0, T]$.

The constants γ, λ, θ play no roles in the analysis, we assume $\gamma = \lambda = \theta = 1$ henceforth.

The existence of local strong solutions could be obtained in the paper [16], which might be slightly modified. More precisely,

Theorem 1.2. *Assume that $\inf \rho_0 \geq 0$, $\rho_0 \in W^{1,r}$, for some $r > 3$, $u_0 \in H^2 \bigcap H^1_0$, $\nabla d_0 \in H^2$ and $|d_0| = 1$ in $\overline{\Omega}$, in addition, the following compatiblity conditions are valid*

$$\Delta u_0 - \nabla P_0 - \nabla \cdot (\nabla d_0 \otimes \nabla d_0) = \sqrt{\rho_0} g \ \text{ and } \nabla \cdot u_0 = 0,\ \text{ in } \Omega, \tag{7}$$

for some $(P_0, g) \in H^1 \times L^2$. Then there exist a positive time $T_0 > 0$ and a unique strong solution (ρ, u, d) of (1)-(6) in $\Omega \times (0, T_0]$.

The main result here is stated as follows:

Theorem 1.3. *Let (ρ, u, d) be a strong solution to (1)-(6). If $0 < T_* < +\infty$ is the maximum time of existence of the strong solutions, then*

$$\int_0^{T_*} \left(\|u\|^s_{L^r} + \|\nabla d\|^s_{L^r} \right) dt = \infty, \tag{8}$$

where $\frac{2}{s} + \frac{3}{r} \leq 1$, for $3 < r \leq \infty$.

Remark 1.4. *If d is a constant vector field, then (8) is the well-known Serrin type criterion for incompressible Navier-Stokes equations, see [2, 14].*

Remark 1.5. *If ρ and u are zero, then (8) with $r = 3$, $s = \infty$ has been established by Wang [15] for the heat flow of harmonic map.*

2. Proof of Theorem 1.3

Let $0 < T_* < \infty$ be the maximum time for the existence of strong solution (ρ, u, d) to (1)-(6). Namely, (ρ, u, d) is a strong solution to (1)-(6) in $\Omega \times (0, T]$ for any $0 < T < T_*$, but not a strong solution in $\Omega \times (0, T_*]$. Suppose that (8) were false, i.e.

$$M_0 := \int_0^{T_*} \left(\|u\|_{L^r}^s + \|\nabla d\|_{L^r}^s \right) dt < \infty. \tag{1}$$

The goal is to show that under the assumption (1), there is a bound $C > 0$ depending only on M_0, ρ_0, u_0, d_0, Ω, and T_* such that

$$\sup_{0 \le t < T_*} \left[\| \sqrt{\rho} u_t \|_{L^2} + \|\rho\|_{W^{1,r}} + \|u\|_{H^2} + \|d\|_{H^3} \right] \le C. \tag{2}$$

With (2), we can then show without much difficulty that T_* is not the maximum time, which is the desired contradiction.

Throughout the rest of the paper, we denote by C a generic constant depending only on ρ_0, u_0, d_0, T_*, M_0, Ω. We denote by

$$A \lesssim B$$

if there exists a generic constant C such that $A \le CB$. For two 3×3 matrices $M = (M_{ij}), N = (N_{ij})$, denote the scalar product between M and N by

$$M : N = \sum_{i,j=1}^{3} M_{ij} N_{ij}.$$

For $d : \Omega \to S^2$, denote by $\nabla d \otimes \nabla d$ as the 3×3 matrix given by

$$(\nabla d \otimes \nabla d)_{ij} = \langle \nabla_i d, \nabla_j d \rangle, \ 1 \le i, j \le 3.$$

The proof is divided into several steps, and we proceed as follows.

Step 1. We shall first establish upper-lower bounds of ρ. More precisely, we have

Lemma 2.1. *Let $0 < T_* < +\infty$ be the maximum time for the strong solution (ρ, u, d) to (1)-(6). If (7) and (1) hold, then for a.e. $(x, t) \in \Omega \times [0, T_*)$, we have*

$$0 \le \inf_{x \in \overline{\Omega}} \rho_0 \le \rho \le \sup_{x \in \overline{\Omega}} \rho_0. \tag{3}$$

Proof. The proof is quite classical by using the characteristic methods (see for instance [6]). □

Step 2. We next establish the global energy inequality for strong solutions, namely,

Lemma 2.2. *Let $0 < T_* < +\infty$ be the maximum time for the strong solution (ρ, u, d) to (1)-(6). If (7) and (1) hold, then for a.e. $t \in [0, T_*)$, we have*

$$\begin{aligned} &\frac{1}{2} \int_\Omega \left(\rho |u|^2 + |\nabla d|^2 \right)(t)\, dx + \int_0^t \int_\Omega \left(|\nabla u|^2 + \left| \Delta d + |\nabla d|^2 d \right|^2 \right) dx\, ds \\ &= \frac{1}{2} \int_\Omega \left(\rho_0 |u_0|^2 + |\nabla d_0|^2 \right) dx. \end{aligned} \tag{4}$$

Proof. Multiplying (2) and (4) by u and $\Delta d + |\nabla d|^2 d$, respectively, integrating by parts over $\Omega \times [0, t]$, and using $|d| = 1$, we can easily get (4). □

Step 3. Estimates of $(\nabla u, \nabla^2 d)$ in $L_t^\infty L_x^2$.

Lemma 2.3. *Let $0 < T_* < +\infty$ be the maximum time for the strong solution (ρ, u, d) to (1)-(6). If (7) and (1) hold, then for a.e. $t \in [0, T_*)$, we have*

$$\int_\Omega \left(|\nabla u|^2 + |\nabla^2 d|^2\right)(t)dx + \int_0^t \int_\Omega \left(\rho|u_t|^2 + |\nabla^3 d|^2 + |\nabla d_t|^2\right) dxdt \leq C. \tag{5}$$

Proof. Multiplying (2) by u_t, and integrating by parts over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx + \int \rho|u_t|^2\,dx = -\int \rho(u\cdot\nabla)u\cdot u_t\,dx - \int \nabla\cdot(\nabla d\otimes\nabla d)\cdot u_t\,dx = \sum_{i=1}^{2} I_i. \tag{6}$$

For I_1, by Cauchy inequality and (3), we have

$$I_1 \leq \frac{1}{4}\int \rho|u_t|^2\,dx + C\int |u|^2|\nabla u|^2\,dx. \tag{7}$$

Using Hölder inequality and interpolation inequality, we have

$$\int |u|^2|\nabla u|^2\,dx \leq C\|u\|_{L^r}^2 \|\nabla u\|_{L^{\frac{2r}{r-2}}}^2, \tag{8}$$

and

$$\|\nabla u\|_{L^{\frac{2r}{r-2}}} \leq \|\nabla u\|_{L^2}^{1-\frac{3}{r}} \|\nabla u\|_{L^6}^{\frac{3}{r}} \leq C\|\nabla u\|_{L^2}^{1-\frac{3}{r}} \|\nabla u\|_{H^1}^{\frac{3}{r}}. \tag{9}$$

(8) and (9), together with Cauchy inequality, yield

$$\int |u|^2|\nabla u|^2\,dx \leq C\|u\|_{L^r}^2 \|\nabla u\|_{L^2}^{2(1-\frac{3}{r})} \|\nabla u\|_{H^1}^{\frac{6}{r}} \leq C_\epsilon(\|u\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 + \epsilon\|\nabla u\|_{H^1}^2, \tag{10}$$

for any $\epsilon \in (0, 1)$, where $\frac{2}{s} + \frac{3}{r} \leq 1$, for $3 < r \leq \infty$, which implies $\frac{2r}{r-3} \leq s$. If $r = \infty$, then (10) is obvious.

Since

$$-\Delta u + \nabla P = -\rho u_t - \rho u\cdot\nabla u - \nabla\cdot(\nabla d\otimes\nabla d),$$

we apply the H^2-estimate for the Stokes equations (see for instance [3]), together with the similar arguments as (10), we have

$$\begin{aligned}\|\nabla u\|_{H^1}^2 \lesssim& \|\rho u_t\|_{L^2}^2 + \|\rho u\cdot\nabla u\|_{L^2}^2 + \int |\nabla d|^2|\nabla^2 d|^2\,dx\\ \lesssim& \|\sqrt{\rho}u_t\|_{L^2}^2 + \|u\cdot\nabla u\|_{L^2}^2 + (\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 + \|\nabla^2 d\|_{H^1}^2,\end{aligned} \tag{11}$$

where we have used (3). Substituting (11) into (10), and taking $\epsilon \in (0, 1)$ sufficiently small, we obtain

$$\begin{aligned}\int |u|^2|\nabla u|^2\,dx \leq& C_\epsilon(\|u\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 + C\epsilon\|\sqrt{\rho}u_t\|_{L^2}^2 + C_\epsilon(\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2\\ &+ C\epsilon\|\nabla^2 d\|_{H^1}^2.\end{aligned} \tag{12}$$

Substituting (12) into (7), we have

$$\begin{aligned}I_1 \leq& \frac{1}{4}\int \rho|u_t|^2\,dx + C_\epsilon(\|u\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 + C\epsilon\|\sqrt{\rho}u_t\|_{L^2}^2\\ &+ C_\epsilon(\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 + C\epsilon\|\nabla^2 d\|_{H^1}^2.\end{aligned} \tag{13}$$

For I_2, using Cauchy inequality, we have

$$\begin{aligned} I_2 =& \int \nabla d \otimes \nabla d : \nabla u_t \, dx \\ =& \frac{d}{dt}\int \nabla d \otimes \nabla d : \nabla u \, dx - \int \nabla d_t \otimes \nabla d : \nabla u \, dx - \int \nabla d \otimes \nabla d_t : \nabla u \, dx \\ \le& \frac{d}{dt}\int \nabla d \otimes \nabla d : \nabla u \, dx + C_\epsilon \int |\nabla d|^2 |\nabla u|^2 \, dx + \epsilon \|\nabla d_t\|_{L^2}^2. \end{aligned} \tag{14}$$

To estimate the second term of the right hand side of (14), we apply the similar arguments as (10). Then

$$\int |\nabla d|^2 |\nabla u|^2 \, dx \le C_\epsilon (\|\nabla d\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 + \epsilon \|\nabla u\|_{H^1}^2. \tag{15}$$

Substituting (11) and (12) into (15), we obtain

$$\begin{aligned} \int |\nabla d|^2 |\nabla u|^2 \, dx \le& C_\epsilon (\|\nabla d\|_{L^r}^s + \|u\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 + C\epsilon \|\sqrt{\rho} u_t\|_{L^2}^2 \\ &+ C_\epsilon (\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 + C\epsilon \|\nabla^2 d\|_{H^1}^2. \end{aligned} \tag{16}$$

Putting (13), (14) and (16) into (6), choosing ϵ sufficiently small, we obtain

$$\begin{aligned} \frac{d}{dt}\int |\nabla u|^2 \, dx + \int \rho |u_t|^2 \, dx \le& 2\frac{d}{dt}\int \nabla d \otimes \nabla d : \nabla u \, dx + C_\epsilon (\|u\|_{L^r}^s + \|\nabla d\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 \\ &+ C_\epsilon (\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 + C\epsilon \|\nabla^2 d\|_{H^1}^2 + C\epsilon \|\nabla d_t\|_{L^2}^2. \end{aligned} \tag{17}$$

Integrating (17) over $[0, t]$, for $0 < t < T_*$, and applying Cauchy inequality, we obtain

$$\begin{aligned} &\int |\nabla u|^2 \, dx + \int_0^t \int \rho |u_t|^2 \, dx ds \\ \le& \frac{1}{2}\|\nabla u\|_{L^2}^2 + C\int |\nabla d|^4 \, dx + C_\epsilon \int_0^t (\|u\|_{L^r}^s + \|\nabla d\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 \, ds \\ &+ C_\epsilon \int_0^t (\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 \, ds + C\epsilon \int_0^t \|\nabla^2 d\|_{H^1}^2 \, ds + C\epsilon \int_0^t \|\nabla d_t\|_{L^2}^2 \, ds + C. \end{aligned}$$

Thus,

$$\begin{aligned} &\int |\nabla u|^2 \, dx + \int_0^t \int \rho |u_t|^2 \, dx ds \\ \le& C\int |\nabla d|^4 \, dx + C_\epsilon \int_0^t (\|u\|_{L^r}^s + \|\nabla d\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 \, ds \\ &+ C_\epsilon \int_0^t (\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 \, ds + C\epsilon \int_0^t \|\nabla^2 d\|_{H^1}^2 \, ds + C\epsilon \int_0^t \|\nabla d_t\|_{L^2}^2 \, ds + C. \end{aligned} \tag{18}$$

Next, we shall make some estimates about d. To do these, differentiating (4) with respect to x, we have

$$\nabla d_t - \nabla \Delta d = -\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2 d). \tag{19}$$

Multiplying (19) by $4|\nabla d|^2 \nabla d$, and integrating by parts over Ω, we have

$$\begin{aligned} &\frac{d}{dt}\int |\nabla d|^4 \, dx + 4\int (|\nabla d|^2 |\nabla^2 d|^2 + 2|\nabla d|^2 \big|\nabla |\nabla d|\big|^2) \, dx \\ =& 4\int \nabla(|\nabla d|^2 d)|\nabla d|^2 \nabla d \, dx - 4\int \nabla(u \cdot \nabla d)|\nabla d|^2 \nabla d \, dx = \sum_{i=1}^{2} II_i. \end{aligned} \tag{20}$$

Since

$$\nabla(|\nabla d|^2 d) = |\nabla d|^2 \nabla d + \nabla(|\nabla d|^2) d \text{ and } \nabla d \cdot d = 0,$$

for II_1, we have

$$II_1 = 4\int |\nabla d|^6\,dx \lesssim \int |\nabla d|^2 |\Delta d|^2\,dx, \tag{21}$$

where we have used the fact

$$|d| = 1, \text{ and } |\nabla d|^2 = -\Delta d \cdot d \le |\Delta d|. \tag{22}$$

Using the similar arguments as (10), we have

$$\int |\nabla d|^2 |\Delta d|^2\,dx \le C_\epsilon(\|\nabla d\|_{L^r}^s + 1)\|\Delta d\|_{L^2}^2 + \epsilon\|\Delta d\|_{H^1}^2. \tag{23}$$

Thus,

$$II_1 \le C_\epsilon(\|\nabla d\|_{L^r}^s + 1)\|\Delta d\|_{L^2}^2 + \epsilon\|\Delta d\|_{H^1}^2. \tag{24}$$

For II_2, using (22) again, together with integration by parts, $\operatorname{div} u = 0$ and Cauchy inequality, we have

$$\begin{aligned} II_2 &\lesssim \int |\nabla d|^4 |\nabla u|\,dx - \int u \cdot \nabla(|\nabla d|^4)\,dx \\ &\lesssim \int |\nabla d|^2 |\Delta d||\nabla u|\,dx \\ &\lesssim \int |\nabla d|^2 |\Delta d|^2\,dx + \int |\nabla d|^2 |\nabla u|^2\,dx. \end{aligned} \tag{25}$$

Putting (16), (23) into (25), and then substituting the resulting inequality and (24) into (20), we obtain

$$\begin{aligned} &\frac{d}{dt}\int |\nabla d|^4\,dx + 4\int (|\nabla d|^2 |\nabla^2 d|^2 + 2|\nabla d|^2 \big|\nabla|\nabla d|\big|^2)\,dx \\ \le& C_\epsilon(\|\nabla d\|_{L^r}^s + 1)\|\nabla^2 d\|_{L^2}^2 + C\epsilon\|\nabla^2 d\|_{H^1}^2 + C_\epsilon(\|\nabla d\|_{L^r}^s + \|u\|_{L^r}^s + 1)\|\nabla u\|_{L^2}^2 \\ &+ C\epsilon\|\sqrt{\rho} u_t\|_{L^2}^2. \end{aligned} \tag{26}$$

Multiplying (19) by $\nabla\Delta d$, integrating by parts over Ω and using $\frac{\partial d_t}{\partial \nu} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 dx + \|\nabla\Delta d\|_{L^2}^2 = \int \Big(\nabla(u\cdot\nabla d) - \nabla(|\nabla d|^2 d)\Big) : \nabla\Delta d\,dx \\ \le& \epsilon\|\Delta d\|_{H^1}^2 + C_\epsilon \int (|\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2)\,dx + C_\epsilon \int |\nabla d|^2 |\nabla^2 d|^2\,dx + C_\epsilon \int |\nabla d|^6\,dx \\ \le& \epsilon C\|\nabla\Delta d\|_{L^2}^2 + C_\epsilon(\|\nabla d\|_{L^r}^s + \|u\|_{L^r}^s + 1)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + C\epsilon\|\sqrt{\rho} u_t\|_{L^2}^2, \end{aligned}$$

where we have used (16), (21) and (23), together with the arguments as (10) dealing with the term $\int |u|^2 |\nabla^2 d|^2$ (replacing the second u on the left side of (10) by ∇d). Choosing ϵ sufficiently small, we have

$$\begin{aligned} &\frac{d}{dt}\int |\Delta d|^2 dx + \frac{3}{2}\|\nabla\Delta d\|_{L^2}^2 \\ \le& C_\epsilon(\|\nabla d\|_{L^r}^s + \|u\|_{L^r}^s + 1)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + C\epsilon\|\sqrt{\rho} u_t\|_{L^2}^2. \end{aligned} \tag{27}$$

Multiplying (19) by ∇d_t, integrating by parts over Ω, and using $\frac{\partial d_t}{\partial \nu} = 0$ on $\partial\Omega$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2 dx + \|\nabla d_t\|_{L^2}^2 = \int \Big(\nabla(|\nabla d|^2 d) - \nabla(u\cdot\nabla d)\Big)\nabla d_t\,dx \\ \le& \epsilon\|\nabla d_t\|_{L^2}^2 + C\int \Big(|\nabla d|^2 |\nabla^2 d|^2 + |\nabla d|^6 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2\Big)\,dx \\ \le& \epsilon\|\nabla d_t\|_{L^2}^2 + \epsilon C\|\nabla\Delta d\|_{L^2}^2 + C_\epsilon(\|\nabla d\|_{L^r}^s + \|u\|_{L^r}^s + 1)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) + C\epsilon\|\sqrt{\rho} u_t\|_{L^2}^2. \end{aligned} \tag{28}$$

Putting (26), (27) and (28) together, choosing ϵ sufficiently small, we have

$$\begin{aligned}&\frac{d}{dt}\int\left(|\Delta d|^2+|\nabla d|^4\right)dx+\int\left(|\nabla\Delta d|^2+|\nabla d_t|^2+|\nabla d|^2|\nabla^2 d|^2\right)dx\\&\lesssim\left(\|\nabla d\|_{L^r}^s+\|u\|_{L^r}^s+1\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)+C\epsilon\|\sqrt{\rho}u_t\|_{L^2}^2.\end{aligned}\tag{29}$$

Integrating (29) from 0 to t, for $0<t<T_*$, and using the standard elliptic estimates for Neumann problem and (4), we obtain

$$\begin{aligned}&\int\left(|\nabla^2 d|^2+|\nabla d|^4\right)(t)\,dx+\int_0^t\int\left(|\nabla^3 d|^2+|\nabla d_t|^2+|\nabla d|^2|\nabla^2 d|^2\right)dxds\\&\le C\int_0^t\left(\|\nabla d\|_{L^r}^s+\|u\|_{L^r}^s+1\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)ds+C\epsilon\int_0^t\|\sqrt{\rho}u_t\|_{L^2}^2\,ds+C.\end{aligned}\tag{30}$$

Multiplying (30) by $2C$ and adding the resulting inequality into (18), choosing ϵ sufficiently small, we have

$$\begin{aligned}&\int\left(|\nabla u|^2+|\nabla^2 d|^2\right)(t)\,dx+\int_0^t\int\left(\rho|u_t|^2+|\nabla^3 d|^2+|\nabla d_t|^2\right)dxds\\&\le C\int_0^t\left(\|\nabla d\|_{L^r}^s+\|u\|_{L^r}^s+1\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)ds+C.\end{aligned}\tag{31}$$

By Gronwall inequality and (1), we get (5). □

As an immediate consequence of Lemma 2.3, we have

Corollary 2.4. *Let $0<T_*<+\infty$ be the maximum time for the strong solution (ρ,u,d) to (1)-(6). If (7) and (1) hold, then for a.e. $t\in[0,T_*)$, we have*

$$\int_\Omega |d_t|^2(t)\,dx+\int_0^t\|\nabla u\|_{H^1}^2(s)\,ds\le C.\tag{32}$$

Proof. It follows from (4) and (5) that

$$\int_\Omega |d_t|^2\le C.$$

By (11), (12), (1) and (5), we get the last part of (32). □

Step 4. Estimates of $(\nabla^2 u,\nabla^3 d)$ in $L_t^\infty L_x^2$.

Lemma 2.5. *Let $0<T_*<+\infty$ be the maximum time for the strong solution (ρ,u,d) to (1)-(6). If (7) and (1) hold, then for a.e. $t\in[0,T_*)$, we have*

$$\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla d_t\|_{L^2}+\|\nabla d\|_{H^2}+\int_0^t\int_\Omega(|\nabla u_t|^2+|d_{tt}|^2)dxdt\le C.\tag{33}$$

Proof. Differentiating the equation (2) with respect to t, we get

$$\begin{aligned}&\rho u_{tt}+\rho_t u_t+\rho u\cdot\nabla u_t+\rho_t u\cdot\nabla u+\rho u_t\cdot\nabla u+\nabla P_t\\&=\Delta u_t-\nabla\cdot(\nabla d_t\otimes\nabla d+\nabla d\otimes\nabla d_t).\end{aligned}\tag{34}$$

Multiplying (34) by u_t, integrating by parts over Ω, and using (1), (3), Sobolev inequality, and Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \int |\nabla u_t|^2dx \\
&\lesssim \int (\rho|u||\nabla u_t||u_t| + \rho|u||\nabla u|^2|u_t| + \rho|u|^2|\nabla^2 u||u_t| + \rho|u|^2|\nabla u||\nabla u_t| \\
&\quad + \rho|u_t|^2|\nabla u| + |\nabla d_t||\nabla d||\nabla u_t|)\, dx \\
&= \sum_{j=1}^{6} III_j.
\end{aligned} \tag{35}$$

For III_1 and III_5, we have

$$\begin{aligned}
III_1 + III_5 &\lesssim \|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2} + \|\sqrt{\rho}\|_{L^\infty}\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^3}\|u_t\|_{L^6} \\
&\lesssim \|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla u_t\|_{L^2} \lesssim \|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}} \\
&\le \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2,
\end{aligned}$$

where we have used Hölder inequality, Sobolev inequality, (3), (5), the interpolation inequality and Young inequality.

Similarly, we have

$$\begin{aligned}
III_2 + III_3 + III_4 \lesssim& \|\rho\|_{L^\infty}\|u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^3}^2 + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6} \\
&+ \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
\lesssim& \|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} + \|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2} \le \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2,
\end{aligned}$$

and

$$III_6 \lesssim \|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6} \le \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^3}^2 \le \frac{1}{6}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}\|\nabla d_t\|_{H^1}.$$

Substituting these estimates of III_i into (35), for $i = 1, 2, ..., 6$, we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u_t|^2\, dx + \int |\nabla u_t|^2\, dx \\
&\lesssim \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla d_t\|_{L^2}\|\nabla d_t\|_{H^1} \\
&\lesssim \|\sqrt{\rho}u_t\|_{L^2}^2 + \|u\|_{L^r}^s + \|\nabla d\|_{L^r}^s + \|\nabla^2 d\|_{H^1}^2 + \|\nabla d_t\|_{L^2}\|\nabla d_t\|_{H^1} + 1,
\end{aligned} \tag{36}$$

where we have used (5), (11) and (12).

Differentiating (4) with respect to t, multiplying the resulting equation by d_{tt}, integrating by parts over Ω, and using $\frac{\partial d_t}{\partial \nu} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\nabla d_t|^2\, dx + \int |d_{tt}|^2\, dx &= \int \langle \partial_t\left(|\nabla d|^2 d - u\cdot\nabla d\right), d_{tt}\rangle\, dx \\
&\lesssim \|d_{tt}\|_{L^2}\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|d_{tt}\|_{L^2}\|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6} \\
&\quad + \|d_{tt}\|_{L^2}\|u\|_{L^6}\|\nabla d_t\|_{L^3} + \|d_{tt}\|_{L^2}\|u_t\|_{L^6}\|\nabla d\|_{L^3} \\
&\le \frac{1}{2}\|d_{tt}\|_{L^2}^2 + C(\|\nabla d_t\|_{L^2}\|\nabla d_t\|_{H^1} + \|\nabla u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1),
\end{aligned}$$

where we have used Sobolev inequality, Hölder inequality, the interpolation inequality, (4), (5) and (32). This implies

$$\frac{d}{dt}\int |\nabla d_t|^2dx + \int |d_{tt}|^2dx \le C(\|\nabla d_t\|_{L^2}\|\nabla d_t\|_{H^1} + \|\nabla u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1). \tag{37}$$

Now we need to estimate $\|\nabla d_t\|_{H^1}$. In fact, by applying the standard H^2-estimate on the equation (4) under the boundary condition (6), together with (4), (5), (32) and the interpolation inequality, we have

$$\begin{aligned}\|\nabla d_t\|_{H^1} \lesssim & \|\nabla d_t\|_{L^2} + \|d_{tt}\|_{L^2} + \|\partial_t(u\cdot\nabla d)\|_{L^2} + \|\partial_t(|\nabla d|^2 d)\|_{L^2}\\ \lesssim & \|\nabla d_t\|_{L^2} + \|d_{tt}\|_{L^2} + \|u_t\|_{L^6}\|\nabla d\|_{L^3} + \|u\|_{L^6}\|\nabla d_t\|_{L^3}\\ & + \|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6}\\ \lesssim & \|d_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla d_t\|_{L^2}^{\frac12}\|\nabla d_t\|_{H^1}^{\frac12} + \|\nabla d_t\|_{L^2} + 1\\ \le & \frac12\|\nabla d_t\|_{H^1} + C\left(\|d_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla d_t\|_{L^2} + 1\right).\end{aligned}$$

Thus

$$\|\nabla d_t\|_{H^1} \lesssim \|d_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla d_t\|_{L^2} + 1. \tag{38}$$

Substituting (38) into (37), and using Cauchy inequality, we obtain

$$\frac{d}{dt}\int |\nabla d_t|^2\,dx + \int |d_{tt}|^2\,dx \le \frac12\|d_{tt}\|_{L^2}^2 + C\left(\|\nabla u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1\right).$$

Thus

$$\frac{d}{dt}\int |\nabla d_t|^2 dx + \frac12\int |d_{tt}|^2 dx \le C\left(\|\nabla u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1\right). \tag{39}$$

Multiplying (36) by $2C$ and adding the resulting inequality into (39), applying (38), using (1), (5), and then employing Gronwall inequality, we obtain

$$\sup_{0\le t\le T}\int (\rho|u_t|^2 + |\nabla d_t|^2)\,dx + \int_0^T\int_\Omega (|\nabla u_t|^2 + |d_{tt}|^2)\,dx\,dt \le C. \tag{40}$$

To estimate $\nabla^3 d$ in $L_t^\infty L_x^2(\Omega\times[0,T])$, applying (5), (40) and the standard H^3-estimate on the equation (19) under the boundary condition (6), we have

$$\begin{aligned}\|\nabla d\|_{H^2}^2 \lesssim & \|\nabla d_t\|_{L^2}^2 + \|\nabla(u\cdot\nabla d)\|_{L^2}^2 + \|\nabla(|\nabla d|^2 d)\|_{L^2}^2 + 1\\ \lesssim & \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right) + \|\nabla d\|_{L^6}^6 + \int |u|^2|\nabla^2 d|^2 dx + 1\\ \lesssim & 1 + \|\nabla d\|_{L^\infty}^2 + \int |u|^2|\nabla^2 d|^2\,dx.\end{aligned} \tag{41}$$

For the last term on the right hand side of (41), using the interpolation inequality, and applying (5), we have

$$\begin{aligned}\int |u|^2|\nabla^2 d|^2 dx \le & \|u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2 \lesssim \|\nabla u\|_{L^2}^2\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{H^1}\\ \le & \epsilon\|\nabla d\|_{H^2}^2 + C_\epsilon.\end{aligned} \tag{42}$$

Substituting (42) into (41), and choosing ϵ sufficiently small, we have

$$\|\nabla d\|_{H^2}^2 \lesssim 1 + \|\nabla d\|_{L^\infty}^2. \tag{43}$$

Using the interpolation inequality again, together with (43), (4) and Young inequality, we have

$$\|\nabla d\|_{H^2}^2 \le C + C\|\nabla d\|_{L^2}^{\frac12}\|\nabla d\|_{H^2}^{\frac32} \le \frac12\|\nabla d\|_{H^2}^2 + C,$$

which clearly yields that

$$\|\nabla d\|_{H^2} \le C.$$

The proof is now complete. □

Corollary 2.6. *Under the same assumptions as in Lemma 2.5, we have for a.e. $t \in [0, T_*)$*

$$\|\nabla u\|_{H^1}(t) + \int_0^t \|u\|_{W^{2,6}}^2(s)\, ds \le C, \text{ and } \int_0^t \left(\|\nabla^2 d_t\|_{L^2}^2 + \|\nabla^4 d\|_{L^2}^2\right)(s)\, ds \le C. \tag{44}$$

Proof. By (11), we have

$$\begin{aligned}\|\nabla u\|_{H^1}^2 \lesssim& \|\rho u_t\|_{L^2}^2 + \|\rho u \cdot \nabla u\|_{L^2}^2 + \int |\nabla d|^2 |\nabla^2 d|^2\, dx\\ \lesssim& \|u\|_{L^6}^2 \|\nabla u\|_{L^3}^2 + \|\nabla d\|_{L^6}^2 \|\nabla^2 d\|_{L^3}^2 + 1\\ \le& C\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} + C \le \frac{1}{2}\|\nabla u\|_{H^1}^2 + C,\end{aligned}$$

where we have used Hölder inequality, (3), the interpolation inequality, (5), (33) and Cauchy inequality. Thus,

$$\sup_{0\le t\le T} \|\nabla u\|_{H^1} \le C. \tag{45}$$

Similarly, we have

$$\begin{aligned}\|u\|_{W^{2,6}} \lesssim& \|\rho u_t\|_{L^6} + \|\rho u \cdot \nabla u\|_{L^6} + \|\nabla \cdot (\nabla d \otimes \nabla d)\|_{L^6}\\ \lesssim& \|u_t\|_{L^6} + \|u\|_{L^\infty}\|\nabla u\|_{L^6} + \|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^6}\\ \lesssim& \|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2 + 1,\end{aligned}$$

thus

$$\int_0^T \|u\|_{W^{2,6}}^2\, dt \le C,$$

where we have used (33) and (45).

It follows from (38) and (33) that

$$\int_0^T \|\nabla^2 d_t\|_{L^2}^2 dt \le C. \tag{46}$$

Applying the standard H^4-estimate to (4), we have

$$\begin{aligned}\|\nabla^4 d\|_{L^2}^2 \lesssim& \|\nabla^2 d_t\|_{L^2}^2 + \|\nabla^2(u\cdot\nabla d)\|_{L^2}^2 + \|\nabla^2(|\nabla d|^2 d)\|_{L^2}^2 + 1\\ \lesssim& \|\nabla^2 d_t\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla^3 d\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\nabla^2 u\|_{L^2}^2 + \|\nabla u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2 + 1\\ \lesssim& \|\nabla^2 d_t\|_{L^2}^2 + 1.\end{aligned}$$

Integrating this inequality over $[0, T]$, and using (46), we get

$$\int_0^T \|\nabla^4 d\|_{L^2}^2 dt \le C.$$

□

Step 5. Estimate of $\nabla\rho$ in $L_t^\infty L_x^r$.

Lemma 2.7. *Let $0 < T_* < +\infty$ be the maximum time for the strong solution (ρ, u, d) to (1)-(6). If (7) and (1) hold, then for a.e. $t \in [0, T_*)$, we have*

$$\|\nabla\rho\|_{L^r}(t) + \int_0^t \|\nabla u\|_{H^2}^2\, ds \le C. \tag{47}$$

Proof. Using (3), we change (1) into this equation

$$\rho_t + u \cdot \nabla\rho = 0. \tag{48}$$

Differentiating (48) with respect to x, we have

$$\nabla\rho_t + \nabla u \cdot \nabla\rho + u \cdot \nabla\nabla\rho = 0. \tag{49}$$

Multiplying (49) by $r|\nabla\rho|^{r-2}\nabla\rho$, integrating by parts over Ω, and using the interpolation inequality and (5), we have

$$\frac{d}{dt}\|\nabla\rho\|_{L^r}^r \lesssim \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^r}^r \lesssim \|\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{W^{1,6}}^{\frac{3}{4}}\|\nabla\rho\|_{L^r}^r \lesssim \|\nabla u\|_{W^{1,6}}^{\frac{3}{4}}\|\nabla\rho\|_{L^r}^r. \tag{50}$$

By Gronwall inequality, together with (44), we have

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^r} \le C. \tag{51}$$

By (2) together with the H^3-estimate for the Stokes equations, we have

$$\begin{aligned}
\|\nabla u\|_{H^2}^2 \lesssim & \|\nabla(\rho u_t)\|_{L^2}^2 + \|\nabla(\rho u\cdot\nabla u)\|_{L^2}^2 + \|\nabla\nabla\cdot(\nabla d\otimes\nabla d)\|_{L^2}^2 \\
\lesssim & \|\nabla u_t\|_{L^2}^2 + \|\nabla\rho\|_{L^3}^2\|u_t\|_{L^6}^2 + \|\nabla\rho\|_{L^3}^2\|u\|_{L^\infty}^2\|\nabla u\|_{L^6}^2 + \|\rho\|_{L^\infty}^2\|\nabla u\|_{L^4}^4 \\
& + \|\rho\|_{L^\infty}^2\|u\|_{L^\infty}^2\|\nabla^2 u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\nabla^3 d\|_{L^2}^2 + \|\nabla^2 d\|_{L^4}^4 \\
\lesssim & \|\nabla u_t\|_{L^2}^2 + 1,
\end{aligned}$$

where we have used Hölder inequality, Sobolev inequality, (3), (33), (44) and (51). This, together with (33), gives

$$\int_0^t \|\nabla u\|_{H^2}^2 \le C.$$

The proof is now complete. □

Step 6. Completion of proof of Theorem 1.3:

With the above established estimates, we obtain (2). This implies that T_* is not the maximum time of existence of strong solutions, which contradicts the definition of T_*. Therefore, (1) is false. The proof of Theorem 1.3 is now complete. □

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