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Stability of the Pexiderized Quadratic Functional Equation in Paranormed Spaces

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Abstract. The aim of the present paper is to investigate the Hyers-Ulam stability of the Pexiderized quadratic functional equation, namely of $f(x+y) + f(x-y) = 2g(x) + 2h(y)$ in paranormed spaces. More precisely, first we examine the stability for odd and even functions and then we apply our results to prove the Hyers-Ulam stability of the quadratic functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ in paranormed spaces for a general function.

1. Introduction

The stability problem for functional equations originated from a question of Ulam [21] concerning the stability of group homomorphisms and affirmatively answered by Hyers [6] for Banach spaces. Subsequently, Hyers' result was generalized by Aoki [1] for additive mappings and Rassias [18] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influence in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Rassias [17] considered the Cauchy difference controlled by a product of different powers of norms. The above results have been generalized by Forti [3] and Găvruta [5] who permitted the Cauchy difference to become arbitrarily unbounded. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see [4, 9, 10, 19] and references therein).

The functional equation

$$f(x+y) + f(x-y) = 2g(x) + 2h(y) \tag{1}$$

is known as a Pexiderized quadratic functional equation. In the case $f = g = h$ the equation (1) reduces to quadratic functional equation. In various spaces, several results for the generalized Hyers-Ulam stability of functional equations (1) have been investigated by several researchers [2, 7, 8, 12, 20, 22]. Recently, several interesting results regarding the generalized Hyers-Ulam stability of many functional equations have been proved (cf. [11, 13–16]) in paranormed spaces.

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The main purpose of this paper is to establish the Hyers-Ulam stability of the Pexiderized quadratic functional equation (1) in paranormed spaces. The paper is organized as follows: In section 1, we present a brief introduction and introduce related definitions. In section 2, we prove the Hyers-Ulam stability of the functional equation (1) in paranormed spaces for odd functions case. In section 3, we prove the Hyers-Ulam stability of the functional equation (1) in paranormed spaces for even functions case. In section 4, we apply our results to prove the Hyers-Ulam stability of the quadratic functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ in paranormed spaces for a general function case.

Next, we recall some basic facts concerning Fréchet spaces used in this paper.

Definition 1.1. (cf. [11, 13]) *Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that*
(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ *(triangle inequality);*
(4) *If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).*
In this case, the pair (X, P) is called a paranormed space if P is a paranorm on the vector space X.

The paranorm is called total if, in addition, we have $P(x) = 0$ implies $x = 0$. A *Fréchet space* is a total and complete paranormed space. Throughout this paper, assume that (X, P) is a *Fréchet space* and $(Y, \|\cdot\|)$ is a Banach space. It is easy to see that if P is a paranorm on X, then $P(nx) \leq nP(x)$ for all $x \in X$ and $n \in \mathbb{N}$.

2. Stability of the Functional Equation (1): Odd Functions Case

In this section, we prove some results related to the Hyers-Ulam stability of the Pexiderized quadratic functional equation (1) in paranormed spaces when f, g and h are odd functions.

Theorem 2.1. *Let r, θ be positive real numbers with $r > 1$. Suppose that f, g and h are odd functions from Y to X such that*

$$P(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - g(x) - h(y)) \leq \theta(\|x\|^r + \|y\|^r) \tag{2}$$

for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x) - A(x)) \leq \frac{8}{2^r - 2}\theta\|x\|^r, \tag{3}$$

$$P(g(x) + h(x) - A(x)) \leq \frac{2(2^r + 2)}{2^r - 2}\theta\|x\|^r \tag{4}$$

for all $x \in Y$.

Proof. Interchanging x with y in (2), we get

$$P(\frac{1}{2}f(x+y) - \frac{1}{2}f(x-y) - g(y) - h(x)) \leq \theta(\|x\|^r + \|y\|^r) \tag{5}$$

for all $x, y \in Y$. It follows from (2) and (5) that

$$P(f(x+y) - g(x) - h(y) - g(y) - h(x)) \leq 2\theta(\|x\|^r + \|y\|^r) \tag{6}$$

for all $x, y \in Y$. Letting $y = 0$ in (6), we get

$$P(f(x) - g(x) - h(x)) \leq 2\theta\|x\|^r \tag{7}$$

for all $x \in Y$. From (6) and (7), we conclude that

$$P(f(x+y) - f(x) - f(y)) \leq 4\theta(\|x\|^r + \|y\|^r) \tag{8}$$

for all $x, y \in Y$. Putting $y = x$ in (8), we obtain

$$P(f(2x) - 2f(x)) \leq 8\theta \|x\|^r \tag{9}$$

for all $x \in Y$. Thus

$$P(f(x) - 2f(\frac{x}{2})) \leq \frac{8}{2^r}\theta \|x\|^r$$

for all $x \in Y$. Hence

$$P(2^m f(\frac{x}{2^m}) - 2^n f(\frac{x}{2^n})) \leq \sum_{j=m}^{n-1} \frac{8 \cdot 2^j}{2^{rj+r}}\theta \|x\|^r \tag{10}$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in Y$. It follows from (10) that the sequence $\left\{2^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence in X for all $x \in Y$. Since X is complete, the sequence $\left\{2^n f(\frac{x}{2^n})\right\}$ converges for all $x \in Y$. So one can define a mapping $A : Y \to X$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n}) \tag{11}$$

for all $x \in Y$. Moreover, letting $m = 0$ and passing the limit as $n \to \infty$ in (10), we get (3).

Now, we show that A is additive. It follows from (8) and (11) that

$$\begin{aligned} P(A(x+y) - A(x) - A(y)) &= \lim_{n\to\infty} P(2^n(f(\frac{x+y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n}))) \\ &\leq \lim_{n\to\infty} 2^n P(f(\frac{x+y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n})) \\ &\leq \lim_{n\to\infty} \frac{2^n}{2^{nr}} \cdot 4\theta(\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in Y$. Hence $A(x + y) = A(x) + A(y)$ for all $x, y \in Y$ and the mapping $A : Y \to X$ is additive.

By (3) and (7), we have

$$\begin{aligned} P(g(x) + h(x) - A(x)) &= P(f(x) - A(x) + g(x) + h(x) - f(x)) \\ &\leq P(f(x) - A(x)) + P(g(x) + h(x) - f(x)) \\ &\leq (\frac{8}{2^r - 2} + 2)\theta \|x\|^r \\ &= \frac{2(2^r + 2)}{2^r - 2}\theta \|x\|^r \end{aligned} \tag{12}$$

for all $x \in Y$. Thus we obtained (4). To prove the uniqueness of A, assume that A' be another additive mapping from Y to X, which satisfies (3). Then

$$\begin{aligned} P(A(x) - A'(x)) &= P(2^n(A(\frac{x}{2^n}) - A'(\frac{x}{2^n}))) \leq 2^n(P(A(\frac{x}{2^n}) - f(\frac{x}{2^n})) + P(A'(\frac{x}{2^n}) - f(\frac{x}{2^n}))) \\ &\leq \frac{16 \cdot 2^n}{(2^r - 2)2^{nr}}\theta \|x\|^r \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. So we can conclude that $A(x) = A'(x)$ for all $x \in Y$. This completes the proof of the theorem. □

Corollary 2.2. *Let r, s, θ be positive real numbers with $\lambda = r + s > 1$. Suppose that f, g and h are odd functions from Y to X such that*

$$P(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - g(x) - h(y)) \leq \begin{cases} \theta \|x\|^r \|y\|^s, \\ \theta(\|x\|^r \|y\|^s + \|x\|^{r+s} + \|y\|^{r+s}) \end{cases} \tag{13}$$

for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x) - A(x)) \le \begin{cases} \frac{2}{2^\lambda - 2}\theta\|x\|^\lambda, \\ \frac{10}{2^\lambda - 2}\theta\|x\|^\lambda, \end{cases} \tag{14}$$

$$P(g(x) + h(x) - A(x)) \le \begin{cases} \frac{2}{2^\lambda - 2}\theta\|x\|^\lambda, \\ \frac{2(2^\lambda + 3)}{2^\lambda - 2}\theta\|x\|^\lambda \end{cases} \tag{15}$$

for all $x \in Y$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let r be a positive real number with $r < 1$. Suppose that f, g and h are odd functions from X to Y such that*

$$\|\frac{1}{2}f(x + y) + \frac{1}{2}f(x - y) - g(x) - h(y)\| \le P(x)^r + P(y)^r \tag{16}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{8}{2 - 2^r}P(x)^r, \tag{17}$$

$$\|g(x) + h(x) - A(x)\| \le \frac{2(6 - 2^r)}{2 - 2^r}P(x)^r \tag{18}$$

for all $x \in X$.

Proof. The proof of Theorem 2.3 is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let r, s be positive real numbers with $\lambda = r + s < 1$. Suppose that f, g and h are odd functions from X to Y such that*

$$\|\frac{1}{2}f(x + y) + \frac{1}{2}f(x - y) - g(x) - h(y)\| \le \begin{cases} P(x)^r P(y)^s, \\ P(x)^r P(y)^s + (P(x)^{r+s} + P(y)^{r+s}) \end{cases} \tag{19}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \begin{cases} \frac{2}{2-2^\lambda}P(x)^\lambda, \\ \frac{10}{2-2^\lambda}P(x)^\lambda, \end{cases} \tag{20}$$

$$\|g(x) + h(x) - A(x)\| \le \begin{cases} \frac{2}{2-2^\lambda}P(x)^\lambda, \\ \frac{2(7-2^\lambda)}{2-2^\lambda}P(x)^\lambda \end{cases} \tag{21}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.3. □

3. Stability of the Functional Equation (1): Even Functions Case

In this section, we prove some results related to the Hyers-Ulam type stability of the Pexiderized quadratic functional equation (1) in paranormed spaces when f, g and h are even functions.

Theorem 3.1. *Let r, θ be positive real numbers with $r > 2$. Suppose that f, g and h are even functions from Y to X such that $f(0) = g(0) = h(0) = 0$ and*

$$P(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - g(x) - h(y)) \leq \theta(||x||^r + ||y||^r) \tag{22}$$

for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q : Y \to X$ such that

$$P(Q(x) - f(x)) \leq \frac{8}{2^r - 4}\theta||x||^r, \tag{23}$$

$$P(Q(x) - g(x)) \leq \frac{2^r + 4}{2^r - 4}\theta||x||^r, \tag{24}$$

$$P(Q(x) - h(x)) \leq \frac{2^r + 4}{2^r - 4}\theta||x||^r \tag{25}$$

for all $x \in Y$.

Proof. Interchanging x with y in (22), we have

$$P(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - g(y) - h(x)) \leq \theta(||x||^r + ||y||^r) \tag{26}$$

for all $x, y \in Y$. Putting $x = 0$ in (22), we get

$$P(f(y) - h(y)) \leq \theta||y||^r \tag{27}$$

for all $y \in Y$. For $y = 0$ in (22) becomes

$$P(f(x) - g(x)) \leq \theta||x||^r \tag{28}$$

for all $x \in Y$. Combining (22), (26), (27) and (28), we obtain

$$P(f(x+y) + f(x-y) - 2f(x) - 2(y)) \leq 4\theta(||x||^r + ||y||^r) \tag{29}$$

for all $x, y \in Y$. Letting $y = x$ in (29), we have

$$P(f(2x) - 4f(x)) \leq 8\theta||x||^r \tag{30}$$

for all $x \in Y$. Thus

$$P(f(x) - 4f(\frac{x}{2})) \leq \frac{8}{2^r}\theta||x||^r \tag{31}$$

for all $x \in Y$. Hence

$$P(4^m f(\frac{x}{2^m}) - 4^n f(\frac{x}{2^n})) \leq \sum_{j=m}^{n-1} \frac{8 \cdot 4^j}{2^{rj+r}}\theta||x||^r \tag{32}$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in Y$. It follows from (32) that the sequence $\left\{4^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence in X for all $x \in Y$. Since X is complete, the sequence $\left\{4^n f(\frac{x}{2^n})\right\}$ converges for all $x \in Y$. Hence one can define the mapping $Q : Y \to X$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) \tag{33}$$

for all $x \in Y$. Moreover, letting $m = 0$ and passing the limit as $n \to \infty$ in (32), we get (23).

Next, we show that Q is quadratic. It follows from (29) and (33) that

$$\begin{aligned}
&P(Q(x+y)+Q(x-y)-2Q(x)-2Q(y))\\
&\quad= \lim_{n\to\infty} P(4^n(f(\frac{x+y}{2^n})+f(\frac{x-y}{2^n})-2f(\frac{x}{2^n})-2f(\frac{y}{2^n})))\\
&\quad\le \lim_{n\to\infty} 4^n P(f(\frac{x+y}{2^n})+f(\frac{x-y}{2^n})-2f(\frac{x}{2^n})-2f(\frac{y}{2^n}))\\
&\quad\le \lim_{n\to\infty} \frac{4^n}{2^{nr}}\cdot 4\theta(\|x\|^r+\|y\|^r)=0
\end{aligned}$$

for all $x, y \in Y$. Hence $Q(x+y)+Q(x-y) = 2Q(x)+2Q(y)$ for all $x, y \in Y$ and the mapping $Q : Y \to X$ is quadratic.

By (23) and (28), we have

$$\begin{aligned}
P(Q(x)-g(x)) &= P(Q(x)-f(x)+f(x)-g(x))\\
&\le P(Q(x)-f(x))+P(f(x)-g(x))\\
&\le (\frac{8}{2^r-4}+1)\theta\|x\|^r\\
&= \frac{2^r+4}{2^r-4}\theta\|x\|^r \qquad (34)
\end{aligned}$$

for all $x \in Y$. Thus we obtained (24). Similarly, we show that the above inequality also holds for h. The uniqueness assertion can be done on the same lines as in Theorem 2.1. This completes the proof of the theorem. □

Corollary 3.2. *Let r, s, θ be positive real numbers with $\lambda = r+s > 2$. Suppose f, g and h are even functions from Y to X such that $f(0) = g(0) = h(0) = 0$ and (13) for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q : Y \to X$ such that*

$$P(Q(x)-f(x)) \le \begin{cases} \frac{2}{2^\lambda-4}\,\theta\|x\|^\lambda, \\ \frac{10}{2^\lambda-4}\,\theta\|x\|^\lambda, \end{cases} \qquad (35)$$

$$P(Q(x)-g(x)) \le \begin{cases} \frac{2}{2^\lambda-4}\,\theta\|x\|^\lambda, \\ \frac{2^\lambda+6}{2^\lambda-4}\,\theta\|x\|^\lambda \end{cases} \qquad (36)$$

$$P(Q(x)-h(x)) \le \begin{cases} \frac{2}{2^\lambda-4}\,\theta\|x\|^\lambda, \\ \frac{2^\lambda+6}{2^\lambda-4}\,\theta\|x\|^\lambda \end{cases} \qquad (37)$$

for all $x \in Y$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let r be a positive real number with $r < 2$. Suppose that f, g and h are even functions from X to Y such that $f(0) = g(0) = h(0) = 0$ and satisfy*

$$\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-g(x)-h(y)\| \le P(x)^r+P(y)^r \qquad (38)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|Q(x)-f(x)\| \le \frac{8}{4-2^r}P(x)^r, \qquad (39)$$

$$\|Q(x)-g(x)\| \le \frac{12-2^r}{4-2^r}P(x)^r \qquad (40)$$

$$\|Q(x)-h(x)\| \le \frac{12-2^r}{4-2^r}P(x)^r \qquad (41)$$

for all $x \in X$.

Proof. The proof Theorem 3.3 is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let r, s be positive real numbers with $\lambda = r + s < 2$. Suppose that f, g and h are even functions from X to Y such that $f(0) = g(0) = h(0) = 0$ and satisfy (19) for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|Q(x) - f(x)\| \leq \begin{cases} \frac{4}{4-2^\lambda} P(x)^\lambda, \\ \frac{10}{4-2^\lambda} P(x)^\lambda, \end{cases} \tag{42}$$

$$\|Q(x) - g(x)\| \leq \begin{cases} \frac{4}{4-2^\lambda} P(x)^\lambda, \\ \frac{14-2^\lambda}{4-2^\lambda} P(x)^\lambda \end{cases} \tag{43}$$

$$\|Q(x) - h(x)\| \leq \begin{cases} \frac{4}{4-2^\lambda} P(x)^\lambda, \\ \frac{14-2^\lambda}{4-2^\lambda} P(x)^\lambda \end{cases} \tag{44}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.3. □

4. Applications of Stability Results: A General Function Case

In this section, we apply our results to prove the Hyers-Ulam stability of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ in paranormed spaces for a general function case.

Theorem 4.1. *Let r, θ be positive real numbers with $r > 2$. Suppose that f is a mapping from Y to X such that $f(0) = 0$ and satisfies*

$$P(\frac{1}{2} f(x + y) + \frac{1}{2} f(x - y) - f(x) - f(y)) \leq \theta(\|x\|^r + \|y\|^r) \tag{45}$$

for all $x, y \in Y$. Then there are unique mappings $A, Q : Y \to X$ such that A is additive, Q is quadratic and

$$P(f(x) - A(x) - Q(x)) \leq \left(\frac{8}{2^r - 2} + \frac{8}{2^r - 4} \right) \theta \|x\|^r \tag{46}$$

for all $x \in Y$.

Proof. Since f satisfies inequality (45), and passing to the odd part f^o and the even part f^e of f. Hence we have

$$P(\frac{1}{2} f^o(x + y) + \frac{1}{2} f^o(x - y) - f^o(x) - f^o(y)) \leq \theta(\|x\|^r + \|y\|^r)$$
$$P(\frac{1}{2} f^e(x + y) + \frac{1}{2} f^e(x - y) - f^e(x) - f^e(y)) \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in Y$. From the proofs of Theorems 2.1 and 3.1, we obtain a unique additive mapping A and a unique quadratic mapping Q satisfying

$$P(f^o(x) - A(x)) \leq \tfrac{8}{2^r-2} \theta \|x\|^r \quad \text{and} \quad P(f^e(x) - Q(x)) \leq \tfrac{8}{2^r-4} \theta \|x\|^r$$

for all $x \in Y$. Therefore, we have

$$P(f(x) - A(x) - Q(x)) \leq \left(\frac{8}{2^r - 2} + \frac{8}{2^r - 4} \right) \theta \|x\|^r$$

for all $x \in Y$, as desired. This completes the proof of the theorem. □

Corollary 4.2. *Let r, s, θ be positive real numbers with $\lambda = r + s > 2$. Suppose that f be a mapping from Y to X such that $f(0) = 0$ and satisfies*

$$P(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)) \leq \begin{cases} \theta\|x\|^r\|y\|^s, \\ \theta(\|x\|^r\|y\|^s + \|x\|^{r+s} + \|y\|^{r+s}) \end{cases} \tag{47}$$

for all $x, y \in Y$. Then there are unique mappings $A, Q : Y \to X$ such that A is additive, Q is quadratic and

$$P(f(x) - A(x) - Q(x)) \leq \begin{cases} (\frac{2}{2^\lambda-2} + \frac{2}{2^\lambda-4})\theta\|x\|^\lambda, \\ (\frac{10}{2^\lambda-2} + \frac{10}{2^\lambda-4})\theta\|x\|^\lambda, \end{cases} \tag{48}$$

for all $x \in Y$.

Proof. The proof is similar to the proof of Theorem 4.1 and the result follows from Corollaries 2.2 and 3.2. □

Theorem 4.3. *Let r be a positive real numbers with $r < 1$. Suppose that f is a mapping from X to Y such that $f(0) = 0$ and satisfies*

$$\|\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\| \leq P(x)^r + P(y)^r \tag{49}$$

for all $x, y \in X$. Then there are unique mappings $A, Q : X \to Y$ such that A is additive, Q is quadratic and

$$\|f(x) - A(x) - Q(x)\| \leq (\frac{8}{2-2^r} + \frac{8}{4-2^r})P(x)^r \tag{50}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 4.1 and the result follows from Theorems 2.3 and 3.3. □

Corollary 4.4. *Let r, s be positive real numbers with $\lambda = r + s < 1$. Suppose that f is a mapping from X to Y such that $f(0) = 0$ and satisfies*

$$\|\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\| \leq \begin{cases} P(x)^rP(y)^s, \\ P(x)^rP(y)^s + P(x)^{r+s} + P(y)^{r+s} \end{cases} \tag{51}$$

for all $x, y \in X$. Then there are unique mappings $A, Q : X \to Y$ such that A is additive, Q is quadratic and

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} (\frac{2}{2-2^\lambda} + \frac{2}{4-2^\lambda})P(x)^\lambda, \\ (\frac{10}{2-2^\lambda} + \frac{10}{4-2^\lambda})P(x)^\lambda, \end{cases} \tag{52}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 4.3 and the result follows from Corollaries 2.4 and 3.4. □

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