Filomat 28:4 (2014), 695–708
DOI 10.2298/FIL1404695S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Families of Differential Equations Associated with the Hermite-Based Appell Polynomials and Other Classes of Hermite-Based Polynomials

H. M. Srivastava[a], M. A. Özarslan[b], Banu Yılmaz[b]

[a]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada
[b]Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Gazimağusa, TR-33310 Mersin, Turkey

Abstract. Recently, Khan *et al.* [S. Khan, G. Yasmin, R. Khan and N. A. M. Hassan, Hermite-based Appell polynomials: Properties and Applications, *J. Math. Anal. Appl.* **351** (2009), 756–764] defined the Hermite-based Appell polynomials by

$$\begin{aligned}\mathcal{G}(x,y,z;t) &:= A(t)\cdot\exp(xt+yt^2+zt^3)\\ &= \sum_{n=0}^{\infty} {}_HA_n(x,y,z)\,\frac{t^n}{n!}\end{aligned}$$

and investigated their many interesting properties and characteristics by using operational techniques combined with the principle of monomiality. Here, in this paper, we find the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials via the factorization method. Furthermore, we derive the corresponding equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials. We also indicate how to deduce the corresponding results for the Hermite-based Genocchi polynomials from those involving the Hermite-based Euler polynomials.

1. Introduction, Definitions and Preliminaries

A polynomial set $\{p_n(x)\}_{n=0}^{\infty}$ is called quasi-monomial if there exist two operators $\hat{P}$ and $\hat{M}$, independent of n, such that

$$\hat{M}\{p_n(x)\} = p_{n+1}(x)$$

and

$$\hat{P}\{p_n(x)\} = np_{n-1}(x) \qquad \big(p_0(x) := 1;\ p_{-1}(x) := 0\big),$$

2010 *Mathematics Subject Classification*. Primary 33C45, 33C55, 34A35; Secondary 11B68, 33C99

Keywords. Appell and Hermite polynomials, Bernoulli and Euler polynomials, Hermite-based Appell polynomials, Hermite-based Bernoulli polynomials, Hermite-based Euler polynomials, Hermite-based Genocchi polynomials, Partial differential equations, Monomiality principle, Quasi-monomial polynomials

Received: 26 June 2013; Accepted: 18 July 2013

Communicated by Dragan S. Djordjević

Email addresses: harimsri@math.uvic.ca (H. M. Srivastava), mehmetali.ozarslan@emu.edu.tr (M. A. Özarslan), banu.yilmaz@emu.edu.tr (Banu Yılmaz)

where it is assumed (as usual) that

$$p_0(x) := 1 \qquad \text{and} \qquad p_{-1}(x) := 0.$$

The operators $\hat{M}$ and $\hat{P}$ are, respectively, called the raising and the lowering operators acting on the polynomials $p_n(x)$. These operators satisfy the following commutation relation:

$$\left[\hat{P}, \hat{M}\right] = \hat{I},$$

where $\hat{I}$ denotes the identity operator. Thus, clearly, the operators $\hat{M}$ and $\hat{P}$ display a Weyl group structure.

Many of the properties of the polynomials $p_n(x)$ can be obtained by using the operators $\hat{M}$ and $\hat{P}$. If the operators $\hat{M}$ and $\hat{P}$ possess a differential character, then the polynomials $p_n(x)$ satisfy the following differential equation:

$$\hat{M}\hat{P}\left\{p_n(x)\right\} = np_n(x).$$

Moreover, since $p_0(x) := 1$, the polynomial set $\{p_n(x)\}_{n=0}^{\infty}$ can be constructed explicitly through the action of the operator $\hat{M}^n$ on $p_0(x)$ as follows:

$$p_n(x) = \hat{M}^n\{1\} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}),$$

where $\mathbb{N}$ denotes (as usual) the set of *positive* integers. Several recent works dealing extensively with the quasi-monomiality principle include (for example) [3], [4], [5], [6], [7], [8] and [16].

A polynomial set $\{\mathcal{A}_n(x)\}_{n=0}^{\infty}$ is called an Appell set of polynomials (see, for details, [19, p. 398, Problem 28]; see also [1] and the recent work [14] and the references cited therein) if

$$\frac{d}{dx}\{\mathcal{A}_n(x)\} = n\mathcal{A}_{n-1}(x) \qquad (n \in \mathbb{N}_0;\ \mathcal{A}_{-1}(x) := 0)$$

or, equivalently, if

$$A(t) \cdot e^{xt} = \sum_{n=0}^{\infty} \mathcal{A}_n(x)\, \frac{t^n}{n!}, \tag{1}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \qquad (a_0 \neq 0). \tag{2}$$

The familiar *three-variable* Hermite polynomials $\{H_n(x, y, z)\}_{n=0}^{\infty}$ generated by

$$\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} H_n(x, y, z)\, \frac{t^n}{n!} \tag{3}$$

are quasi-monomials under the action of the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{P}}$ given by

$$\hat{\mathcal{M}} := x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2} \qquad \text{and} \qquad \hat{\mathcal{P}} := \frac{\partial}{\partial x}, \tag{4}$$

satisfy the following differential equation:

$$\left(3z\frac{\partial^3}{\partial x^3} + 2y\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - n\right)\{H_n(x, y, z)\} = 0 \tag{5}$$

and possess the following operational representation:

$$H_n(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{x^n\}. \tag{6}$$

Recently, by suitably combining the generating functions (1) and (3), Khan *et al.* [11] defined the Hermite-based Appell polynomials $\{{}_HA_n(x,y,z)\}_{n=0}^{\infty}$ by means of the following generating function:

$$\mathcal{G}(x,y,z;t) := A(t)\cdot\exp\left(t\hat{\mathcal{M}}\right)\{1\} = \sum_{n=0}^{\infty} {}_HA_n(x,y,z)\,\frac{t^n}{n!}, \tag{7}$$

where the operator $\hat{\mathcal{M}}$ is defined in (4) and the power series $A(t)$ is given by (2). In fact, with the aid of the Berry decoupling identity:

$$e^{\hat{A}+\hat{B}} = \exp\left(\frac{m^2}{12}\right)\cdot\exp\left(-\frac{m}{2}\hat{A}^{\frac{1}{2}} + \hat{A}\right)e^{\hat{B}} \qquad \left(\left[\hat{A},\hat{B}\right] = m\hat{A}^{\frac{1}{2}}\right), \tag{8}$$

the generating function (7) of the Hermite-based Appell polynomials ${}_HA_n(x,y,z)$ can be rewritten in the following form [11, p. 759, Eq. (2.3)]:

$$\mathcal{G}(x,y,z;t) := A(t)\cdot\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HA_n(x,y,z)\,\frac{t^n}{n!}. \tag{9}$$

Some important examples of the Appell polynomials $\mathcal{A}_n(x)$ defined by the generating function (1) include the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α, which are usually defined by means of the following generating functions (see, for details, [9, Vol. III, p. 253 *et seq.*], [12, Section 2.8] and [17, p. 61 *et seq.*]; see also [15], [18, p. 81 *et seq.*] and [20] and the references to several related earlier works cited therein):

$$\left(\frac{t}{e^t-1}\right)^{\alpha}\cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\,\frac{t^n}{n!} \qquad \left(|t| < 2\pi;\ 1^{\alpha} := 1\right), \tag{10}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha}\cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\,\frac{t^n}{n!} \qquad \left(|t| < \pi;\ 1^{\alpha} := 1\right) \tag{11}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha}\cdot e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\,\frac{t^n}{n!} \qquad \left(|t| < \pi;\ 1^{\alpha} := 1\right), \tag{12}$$

so that, obviously, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given, respectively, by

$$B_n(x) := B_n^{(1)}(x), \quad E_n(x) := E_n^{(1)}(x) \quad \text{and} \quad G_n(x) := G_n^{(1)}(x) \qquad (n \in \mathbb{N}_0). \tag{13}$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n of order n, we have

$$B_n := B_n(0) = B_n^{(1)}(0), \quad E_n := 2^n E_n\left(\frac{1}{2}\right) = 2^n E_n^{(1)}\left(\frac{1}{2}\right)$$
$$\text{and} \qquad G_n := G_n(0) = G_n^{(1)}(0) \qquad (n \in \mathbb{N}_0), \tag{14}$$

respectively.

The specific choice of the power series $A(t)$ in each of these examples becomes obvious when we compare the generating function (1) with the generating functions (10), (11) and (12) *and* their special cases when $\alpha = 1$. We choose here to introduce the Hermite-based Bernoulli polynomials ${}_HB_n(x, y, z)$, the Hermite-based Euler polynomials ${}_HE_n(x, y, z)$ and the Hermite-based Genocchi polynomials ${}_HG_n(x, y, z)$ by means of the following generating functions:

$$\left(\frac{t}{e^t - 1}\right)\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HB_n(x, y, z)\,\frac{t^n}{n!} \qquad (|t| < 2\pi), \tag{15}$$

$$\left(\frac{2}{e^t + 1}\right)\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HE_n(x, y, z)\,\frac{t^n}{n!} \qquad (|t| < \pi) \tag{16}$$

and

$$\left(\frac{2t}{e^t + 1}\right)\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} {}_HG_n(x, y, z)\,\frac{t^n}{n!} \qquad (|t| < \pi), \tag{17}$$

respectively.

In their special cases when $z = 0$, the generating functions (15), (16) and (17) would reduce immediately to the generating functions of the *two-dimensional* Bernoulli, Euler and Genocchi polynomials. In particular, the special case of the generating function (15) when $z = 0$ was investigated by Bretti and Ricci [2] who also derived the differential, integro-differential and partial differential equations of the *two-dimensional* Appell polynomials (see, for details, [2]). The above-mentioned investigation for the extended two-dimensional Appell polynomials was presented by Yılmaz and Özarslan [21]. Earlier, in the year 2002, He and Ricci [10] made use of the *factorization method* in order to derive the differential equations for the one-variable Appell polynomials. Moreover, the Hermite-based Apostol-Bernoulli polynomials, the Hermite-based Apostol-Euler polynomials and the Hermite-based Apostol-Genocchi polynomials, as well as their unification, were introduced and studied recently by Özarslan [13].

The main idea of the so-called factorization method is to find the lowering operator L_n^- and the raising operators L_n^+ and then use such relationships as follows:

$$L_{n+1}^- L_n^+ \{{}_HA_n(x, y, z)\} = {}_HA_n(x, y, z). \tag{18}$$

The object of this paper is to find the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials ${}_HA_n(x, y, z)$ defined by (9) via the factorization method. We also derive the corresponding equations for the Hermite-based Bernoulli polynomials ${}_HB_n(x, y, z)$ defined by (15) and the Hermite-based Euler polynomials ${}_HE_n(x, y, z)$ defined by (16). It is fairly straightforward to obtain, in a similar manner, the analogous results for the Hermite-based Genocchi polynomials ${}_HG_n(x, y, z)$ defined by (17). Alternatively, of course, one can freely use the following relationships:

$${}_HG_n(x, y, z) = n\ {}_HE_{n-1}(x, y, z) \tag{19}$$

and

$${}_HE_n(x, y, z) = \left(\frac{1}{n + 1}\right)\ {}_HG_{n+1}(x, y, z), \tag{20}$$

which obviously follow immediately by comparing the generating functions (16) and (17).

We choose to organize our paper as follows: In Section 2, we find the recurrence relations of the three-variable Hermite-based Appell polynomials ${}_HA_n(x,y,z)$ defined by (9) and we obtain the lowering and the raising operators of the three-variable Hermite-based Appell polynomials ${}_HA_n(x,y,z)$ with respect to variables x, y and z. Furthermore, we give the corresponding recurrence relations as well as the lowering and the raising operators for the three-dimensional Hermite-based Bernoulli polynomials ${}_HB_n(x,y,z)$ defined by (15) and the three-dimensional Hermite-based Euler polynomials ${}_HE_n(x,y,z)$ defined by (16). In Section 3, we find the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials ${}_HA_n(x,y,z)$ and also list the corresponding results for the Hermite-Bernoulli polynomials ${}_HB_n(x,y,z)$ and the Hermite-Euler polynomials ${}_HE_n(x,y,z)$ as their special cases.

2. Recurrence Relations and Shift Operators

In this section, we begin by deriving the recurrence relations and the shift operators for the Hermite-based Appell polynomials ${}_HA_n(x,y,z)$ defined by the generating function (9).

Theorem 1. *The Hermite-based Appell polynomials* ${}_HA_n(x,y,z)$ *defined by the generating function* (9) *satisfy the following recurrence relations*:

$$\begin{aligned}{}_HA_{n+1}(x,y,z) &= (x+\alpha_0)\,{}_HA_n(x,y,z) + \sum_{k=1}^{n}\binom{n}{k}\alpha_k\,{}_HA_{n-k}(x,y,z)\\ &\quad + 2ny\,{}_HA_{n-1}(x,y,z) + 3n(n-1)z\,{}_HA_{n-2}(x,y,z),\end{aligned} \tag{21}$$

where

$${}_HA_{-1}(x,y,z) := 0 \qquad \text{and} \qquad {}_HA_{-2}(x,y,z) := 0 \tag{22}$$

and the coefficients $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *are given by the following expansion*:

$$\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty}\alpha_k\,\frac{t^k}{k!}. \tag{23}$$

The shift operators are given by

$${}_x\mathcal{L}_n^- := \frac{1}{n}D_x, \tag{24}$$

$${}_y\mathcal{L}_n^- := \frac{1}{n}D_x^{-1}D_y, \tag{25}$$

$${}_z\mathcal{L}_n^- := \frac{1}{n}D_x^{-2}D_z, \tag{26}$$

$${}_x\mathcal{L}_n^+ := x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k + 2yD_x + 3zD_x^2, \tag{27}$$

$${}_y\mathcal{L}_n^+ := x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-k}D_y^k + 2yD_x^{-1}D_y + 3zD_x^{-2}D_y^2 \tag{28}$$

and

$${}_z\mathcal{L}_n^+ := x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-2k}D_z^k + 2yD_x^{-2}D_z + 3zD_x^{-4}D_z^2, \tag{29}$$

where

$$D_x := \frac{\partial}{\partial x},\;\; D_y := \frac{\partial}{\partial y},\;\; D_z := \frac{\partial}{\partial z} \qquad \text{and} \qquad D_x^{-1} := \int_0^x f(\xi)\,d\xi. \tag{30}$$

Proof. Upon differentiating both sides of the generating relation (9) with respect to t, we have

$$\frac{\partial}{\partial t}\{\mathcal{G}(x,y,z;t)\} = \mathcal{G}(x,y,z;t)\left(\frac{A'(t)}{A(t)} + x + 2yt + 3zt^2\right). \tag{31}$$

We now substitute the corresponding series forms for $\mathcal{G}(x,y,z;t)$ from (9) and for the quotient:

$$\frac{A'(t)}{A(t)}$$

from (23). By equating the coefficients of t^n in the equation resulting from (31), we obtain the recurrence relation (21) asserted by Theorem 1.

We now use this recurrence relation (21) to find the shift operators ${}_x\mathcal{L}_n^+$, ${}_y\mathcal{L}_n^+$ and ${}_z\mathcal{L}_n^+$ with respect to x, y and z. First of all, in order to find the shift operator ${}_x\mathcal{L}_n^-$, we differentiate both sides of the generating relation (9) with respect to x and equate the coefficients of t^n, so that we have

$$\frac{\partial}{\partial x}\{{}_HA_n(x,y,z)\} = n\ {}_HA_{n-1}(x,y,z).$$

Thus, clearly, the operator given by (24) satisfies the following relation:

$${}_x\mathcal{L}_n^-\{{}_HA_n(x,y,z)\} = {}_HA_{n-1}(x,y,z).$$

By taking the derivative with respect to y in the generating relation (9), we have

$$\frac{\partial}{\partial y}\{{}_HA_n(x,y,z)\} = n(n-1)\ {}_HA_{n-2}(x,y,z) = n\frac{\partial}{\partial x}\{{}_HA_{n-1}(x,y,z)\},$$

so that

$$D_x^{-1}D_y\{{}_HA_n(x,y,z)\} = n\ {}_HA_{n-1}(x,y,z),$$

and, therefore, we get (25).

Upon differentiating both sides of the generating relation (9) with respect to z, we have

$$\frac{\partial}{\partial z}\{{}_HA_n(x,y,z)\} = n(n-1)(n-2)\ {}_HA_{n-3}(x,y,z) = n\frac{\partial^2}{\partial x^2}\{{}_HA_{n-1}(x,y,z)\},$$

so that

$$D_x^{-2}D_z\{{}_HA_n(x,y,z)\} = n\ {}_HA_{n-1}(x,y,z),$$

which yields (26).

Next, in order to obtain the raising operator ${}_x\mathcal{L}_n^+$, we use the following relations:

$$\begin{aligned} {}_HA_{n-k}(x,y,z) &= \left({}_x\mathcal{L}_{n-k+1}^-\ {}_x\mathcal{L}_{n-k+2}^- \cdots\ {}_x\mathcal{L}_{n-1}^-\ {}_x\mathcal{L}_n^-\right)\{{}_HA_n(x,y,z)\} \\ &= \frac{(n-k)!}{n!}D_x^k\{{}_HA_n(x,y,z)\}, \end{aligned}$$

$$\begin{aligned} {}_HA_{n-1}(x,y,z) &= {}_x\mathcal{L}_n^-\{{}_HA_n(x,y,z)\} \\ &= \frac{1}{n}D_x\{{}_HA_n(x,y,z)\} \end{aligned}$$

and

$$
\begin{aligned}
{}_HA_{n-2}(x,y,z) &= \left({}_x\mathcal{L}_{n-1}^{-}\ {}_x\mathcal{L}_n^{-}\right)\{{}_HA_n(x,y,z)\}\\
&= \frac{1}{n(n-1)}D_x^2\{{}_HA_n(x,y,z)\}.
\end{aligned}
$$

By substituting from these relations into the recurrence relation (21), we have

$$
{}_HA_{n+1}(x,y,z) = \left(x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k+2yD_x+3zD_x^2\right)\{{}_HA_n(x,y,z)\}.
$$

which yields the raising operator (27).

To obtain the raising operator (28), we use the following relations:

$$
\begin{aligned}
{}_HA_{n-k}(x,y,z) &= \left({}_y\mathcal{L}_{n-k+1}^{-}\ {}_y\mathcal{L}_{n-k+2}^{-}\cdots\ {}_y\mathcal{L}_{n-1}^{-}\ {}_y\mathcal{L}_n^{-}\right)\{{}_HA_n(x,y,z)\}\\
&= \frac{(n-k)!}{n!}D_x^{-k}D_y^k\{{}_HA_n(x,y,z)\},
\end{aligned}
$$

$$
\begin{aligned}
{}_HA_{n-1}(x,y,z) &= {}_y\mathcal{L}_n^{-}\{{}_HA_n(x,y,z)\}\\
&= \frac{1}{n}D_x^{-1}D_y\{{}_HA_n(x,y,z)\}
\end{aligned}
$$

and

$$
\begin{aligned}
{}_HA_{n-2}(x,y,z) &= \left({}_y\mathcal{L}_{n-1}^{-}\ {}_y\mathcal{L}_n^{-}\right)\{{}_HA_n(x,y,z)\}\\
&= \frac{1}{n(n-1)}D_x^{-2}D_y^2\{{}_HA_n(x,y,z)\}.
\end{aligned}
$$

Upon substituting from these relations into the recurrence relation (21), we get

$$
\begin{aligned}
&{}_HA_{n+1}(x,y,z)\\
&\quad= \left(x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-k}D_y^k+2yD_x^{-1}D_y+3zD_x^{-2}D_y^2\right)\{{}_HA_n(x,y,z)\},
\end{aligned}
$$

which leads us to the raising operator (28).

The derivation of the raising operator (29) would similarly make use of the following relations:

$$
\begin{aligned}
{}_HA_{n-k}(x,y,z) &= \left({}_z\mathcal{L}_{n-k+1}^{-}\ {}_z\mathcal{L}_{n-k+2}^{-}\cdots\ {}_z\mathcal{L}_{n-1}^{-}\ {}_z\mathcal{L}_n^{-}\right)\{{}_HA_n(x,y,z)\}\\
&= \frac{(n-k)!}{n!}D_x^{-2k}D_z^k\{{}_HA_n(x,y,z)\},
\end{aligned}
$$

$$
\begin{aligned}
{}_HA_{n-1}(x,y,z) &= {}_z\mathcal{L}_n^{-}\{{}_HA_n(x,y,z)\}\\
&= \frac{1}{n}D_x^{-2}D_z\{{}_HA_n(x,y,z)\}
\end{aligned}
$$

and

$$
\begin{aligned}
{}_HA_{n-2}(x,y,z) &= \left({}_z\mathcal{L}_{n-1}^{-}\ {}_z\mathcal{L}_n^{-}\right)\{{}_HA_n(x,y,z)\}\\
&= \frac{1}{n(n-1)}\ D_x^{-4}D_z^2\{{}_HA_n(x,y,z)\},
\end{aligned}
$$

which, in conjunction with the recurrence relation (21), yields

$$
\begin{aligned}
&{}_HA_{n+1}(x,y,z)\\
&\quad=\left(x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-2k}D_z^k+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2\right)\{{}_HA_n(x,y,z)\}
\end{aligned}
$$

and, consequently, we have the raising operator (29). □

Remark 1. By appropriately choosing $A(t)$ in Theorem 1, we can deduce the following corollaries for the Hermite-based Bernoulli polynomials ${}_HB_n(x,y,z)$ and the Hermite-based Euler polynomials ${}_HE_n(x,y,z)$ defined by the generating functions (15) and (16), respectively.

Corollary 1. *The recurrence relations of the Hermite-based Bernoulli polynomials ${}_HB_n(x,y,z)$ are given by*

$$
\begin{aligned}
{}_HB_{n+1}(x,y,z)&=\left(x-\frac{1}{2}\right)\,{}_HB_n(x,y,z)+2ny\,{}_HB_{n-1}(x,y,z)\\
&\quad+3n(n-1)z\,{}_HB_{n-2}(x,y,z)-\frac{1}{n+1}\sum_{k=2}^{n+1}\binom{n+1}{k}\,{}_HB_{n-k+1}(x,y,z)B_k,
\end{aligned}
$$

where B_k denotes the Bernoulli number of order k and

$$
{}_HB_{-n}(x,y,z):=0 \qquad (n\in\mathbb{N}).
$$

The shift operators are given by

$$
\begin{aligned}
{}_x\mathcal{L}_n^-&=\frac{1}{n}D_x,\\
{}_y\mathcal{L}_n^-&=\frac{1}{n}D_x^{-1}D_y,\\
{}_z\mathcal{L}_n^-&=\frac{1}{n}D_x^{-2}D_z,\\
{}_x\mathcal{L}_n^+&=x-\frac{1}{2}+2yD_x+3zD_x^2-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{k-1},\\
{}_y\mathcal{L}_n^+&=x-\frac{1}{2}+2yD_x^{-1}D_y+3zD_x^{-2}D_y^2-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{1-k}D_y^{k-1}
\end{aligned}
$$

and

$$
{}_z\mathcal{L}_n^+=x-\frac{1}{2}+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2-2k}D_z^{k-1}.
$$

Corollary 2. *The recurrence relations of the Hermite-based Euler polynomials ${}_HE_n(x,y,z)$ are given by*

$$
\begin{aligned}
{}_HE_{n+1}(x,y,z)&=\left(x-\frac{1}{2}\right)\,{}_HE_n(x,y,z)+\frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}e_k\,{}_HE_{n-k}(x,y,z)\\
&\quad+2ny\,{}_HE_{n-1}(x,y,z)+3zn(n-1)\,{}_HE_{n-2}(x,y,z).
\end{aligned}
$$

The shift operators are given by

$$
\begin{aligned}
{}_x\mathcal{L}_n^-&=\frac{1}{n}D_x,\\
{}_y\mathcal{L}_n^-&=\frac{1}{n}D_x^{-1}D_y,
\end{aligned}
$$

$$ {}_{z}\mathcal{L}_{n}^{-} = \frac{1}{n} D_x^{-2} D_z, $$

$$ {}_{x}\mathcal{L}_{n}^{+} = x - \frac{1}{2} + 2yD_x + 3zD_x^2 + \frac{1}{2}\sum_{k=1}^{n} \frac{e_k}{k!} D_x^k, $$

$$ {}_{y}\mathcal{L}_{n}^{+} = x - \frac{1}{2} + 2yD_x^{-1}D_y + 3zD_x^{-2}D_y^2 + \frac{1}{2}\sum_{k=1}^{n} \frac{e_k}{k!} D_x^{-k} D_y^k $$

and

$$ {}_{z}\mathcal{L}_{n}^{+} = x - \frac{1}{2} + 2yD_x^{-2}D_z + 3zD_x^{-4}D_z^2 + \frac{1}{2}\sum_{k=0}^{n} \frac{e_k}{k!} D_x^{-2k} D_z^k, $$

where e_k *are the coefficients that are linked with the Euler numbers* E_k by

$$ e_k = -\frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} E_{k-j}. $$

Remark 2. The results asserted by Corollary 2 can easily be restated in terms of the Hermite-based Genocchi polynomials ${}_HG_n(x,y,z)$ defined by the generating function (17) by simply making use of the relationships (19) and (20). The details involved are being omitted here.

3. Differential, Integro-Differential and Partial Differential Equations for the Hermite-Based Appell Polynomials

In this section, we obtain differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials via the factorization method. Moreover, we list the corresponding equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials.

Theorem 2. *The Hermite-based Appell polynomials satisfy the following differential equation*:

$$ \left((x+\alpha_0)D_x + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{k+1} + 2yD_x^2 + 3zD_x^3 - n \right) \{ {}_HA_n(x,y,z) \} = 0. \tag{32} $$

Proof. Using the following factorization relation:

$$ {}_{x}\mathcal{L}_{n+1}^{-}\, {}_{x}\mathcal{L}_{n}^{+} \{ {}_HA_n(x,y,z) \} = {}_HA_n(x,y,z) $$

and the shift operators (24) and (27), we get the desired result (32). □

Theorem 3. *The Hermite-based Appell polynomials satisfy the following integro-differential equations*:

$$ \left((x+\alpha_0)D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-k} D_y^{k+1} + 2D_x^{-1}D_y + 2yD_x^{-1}D_y^2 + 3zD_x^{-2}D_y^3 - (n+1)D_x \right) \{ {}_HA_n(x,y,z) \} = 0, \tag{33} $$

$$ \left((x+\alpha_0)D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-2k} D_z^{k+1} + 2yD_x^{-2}D_z^2 + 3D_x^{-4}D_z^2 + 3zD_x^{-4}D_z^3 - (n+1)D_x^2 \right) \{ {}_HA_n(x,y,z) \} = 0, \tag{34} $$

$$\Bigg((x+\alpha_0)D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-2k} D_z^k D_y + 2D_x^{-2} D_z + 2yD_x^{-2} D_z D_y + 3zD_x^{-4} D_z^2 D_y - (n+1)D_x\Bigg)\{{}_HA_n(x,y,z)\} = 0 \tag{35}$$

and

$$\Bigg((x+\alpha_0)D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_z D_x^{-k} D_y^k + 2yD_x^{-1} D_y D_z + 3D_x^{-2} D_y^2 + 3zD_x^{-2} D_y^2 D_z - (n+1)D_x^2\Bigg)\{{}_HA_n(x,y,z)\} = 0. \tag{36}$$

Proof. Using the following factorization relation:

$$\mathcal{L}_{n+1}^{-}\mathcal{L}_n^{+}\ {}_HA_n(x,y,z) = {}_HA_n(x,y,z)$$

and the shift operators (25), (26), (28) and (29), we get the integro-differential equations (33) and (34), respectively. Again, by using the above factorization relation together with the shift operators (25) and (29), we get the integro-differential equation (35). To obtain the integro-differential equation (36), we use the shift operators (26) and (28) in the above factorization relation. □

Theorem 4. *The Hermite-based Appell polynomials satisfy the following partial differential equations*:

$$\Bigg((x+\alpha_0)D_x^{2n} D_z + 2nD_x^{2n-1} D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{2n-2k} D_z^{k+1} + 2yD_x^{2n-2} D_z^2 + 3D_x^{2n-4} D_z^2 + 3zD_x^{2n-4} D_z^3 - (n+1)D_x^{2n+2}\Bigg)\{{}_HA_n(x,y,z)\} = 0, \tag{37}$$

$$\Bigg((x+\alpha_0)D_x^{n} D_y + nD_x^{n-1} D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{n-k} D_y^{k+1} + 2D_x^{n-1} D_y + 2yD_x^{n-1} D_y^2 + 3zD_x^{n-2} D_y^3 - (n+1)D_x^{n+1}\Bigg)\{{}_HA_n(x,y,z)\} = 0, \tag{38}$$

$$\Bigg((x+\alpha_0)D_x^{2n} D_y + 2nD_x^{2n-1} D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_y D_x^{2n-2k} D_z^{k} + 2D_x^{2n-2} D_z + 2yD_x^{2n-2} D_z D_y + 3zD_x^{2n-4} D_z^2 D_y - (n+1)D_x^{2n+1}\Bigg)\{{}_HA_n(x,y,z)\} = 0 \tag{39}$$

and

$$\Bigg((x+\alpha_0)D_x^{n} D_z + nD_x^{n-1} D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_z D_x^{n-k} D_y^{k} + 2yD_x^{n-1} D_y D_z + 3D_x^{n-2} D_y^2 + 3zD_x^{n-2} D_y^2 D_z - (n+1)D_x^{n+2}\Bigg)\{{}_HA_n(x,y,z)\} = 0. \tag{40}$$

Proof. If we differentiate the integro-differential equation (34) $2n$ times with respect to x, we get the partial differential equation (37). Similarly, by taking the derivatives of the integro-differential equation (33) n times with respect to x, we get the partial differential equation (38). In order to derive the partial differential equation (39), we take the derivatives of the integro-differential equation (35) $2n$ times with respect to x. Similarly, in order to obtain the partial differential equation (40), we take the derivatives of the integro-differential equation (36) n times with respect to x. □

Remark 3. Just as we indicated in Remark 1, by suitably specializing the function $A(t)$ in Theorems 2, 3 and 4, we can deduce the following corollaries which provide the differential, integro-differential and partial differential equations for the Hermite-based Bernoulli polynomials ${}_HB_n(x,y,z)$ and the Hermite-based Euler polynomials ${}_HE_n(x,y,z)$ defined by the generating functions (15) and (16), respectively.

Corollary 3. *The Hermite-based Bernoulli polynomials satisfy the following differential equation*:

$$\left(\left(x-\frac{1}{2}\right)D_x + 2yD_x^2 + 3zD_x^3 - \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^k - n\right)\{{}_HB_n(x,y,z)\} = 0, \tag{41}$$

where B_k denotes the Bernoulli number of order k.

Corollary 4. *The Hermite-based Bernoulli polynomials satisfy the following integro-differential equations*:

$$\begin{aligned}\Bigg(\left(x-\frac{1}{2}\right)D_y &+ 2D_x^{-1}D_y + 2yD_x^{-1}D_y^2 \\ &+ 3zD_x^{-2}D_y^3 - \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{1-k}D_y^k - (n+1)D_x\Bigg)\{{}_HB_n(x,y,z)\} = 0,\end{aligned} \tag{42}$$

$$\begin{aligned}\Bigg(\left(x-\frac{1}{2}\right)D_z &+ 2yD_x^{-2}D_z^2 + 3D_x^{-4}D_z^2 + 3zD_x^{-4}D_z^3 \\ &- \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2-2k}D_z^k - (n+1)D_x^2\Bigg)\{{}_HB_n(x,y,z)\} = 0,\end{aligned} \tag{43}$$

$$\begin{aligned}\Bigg(\left(x-\frac{1}{2}\right)D_y &+ 2D_x^{-2}D_z + 2yD_x^{-2}D_zD_y + 3zD_x^{-4}D_z^2D_y \\ &- \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2-2k}D_z^{k-1}D_y - (n+1)D_x\Bigg)\{{}_HB_n(x,y,z)\} = 0\end{aligned} \tag{44}$$

and

$$\begin{aligned}\Bigg(\left(x-\frac{1}{2}\right)D_z &+ 2yD_x^{-1}D_yD_z + 3D_x^{-2}D_y^2 + 3zD_x^{-2}D_y^2D_z \\ &- \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{1-k}D_y^{k-1}D_z - (n+1)D_x^2\Bigg)\{{}_HB_n(x,y,z)\} = 0,\end{aligned} \tag{45}$$

where B_k denotes the Bernoulli number of order k.

Corollary 5. *The Hermite-based Bernoulli polynomials satisfy the following partial differential equations*:

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_x^nD_y+nD_x^{n-1}D_y+2D_x^{n-1}D_y+2yD_x^{n-1}D_y^2+3zD_x^{n-2}D_y^3 \\ -\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{n-k+1}D_y^k-(n+1)D_x^{n+1}\Bigg)\{_HB_n(x,y,z)\}=0, \tag{46}$$

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_x^{2n}D_z+2nD_x^{2n-1}D_z+2yD_x^{2n-2}D_z^2+3D_x^{2n-4}D_z^2+3zD_x^{2n-4}D_z^3 \\ -\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2n-2k+2}D_z^k-(n+1)D_x^{2n+2}\Bigg)\{_HB_n(x,y,z)\}=0, \tag{47}$$

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_x^{2n}D_y+2nD_x^{2n-1}D_y+2D_x^{2n-2}D_z+2yD_x^{2n-2}D_zD_y+3zD_x^{2n-4}D_z^2D_y \\ -\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2n-2k+2}D_z^{k-1}D_y-(n+1)D_x^{2n+1}\Bigg)\{_HB_n(x,y,z)\}=0 \tag{48}$$

and

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_x^nD_z+nD_x^{n-1}D_z+2yD_x^{n-1}D_yD_z+3D_x^{n-2}D_y^2+3zD_x^{n-2}D_y^2D_z \\ -\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{n-k+1}D_y^{k-1}D_z-(n+1)D_x^{n+2}\Bigg)\{_HB_n(x,y,z)\}=0, \tag{49}$$

where B_k denotes the Bernoulli number of order k.

Corollary 6. *The differential equation satisfied by the Hermite-based Euler polynomials is given by*

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_x+2yD_x^2+3zD_x^3+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{k+1}-n\Bigg)\{_HE_n(x,y,z)\}=0, \tag{50}$$

where e_k is given, in terms of the Euler number E_k, as in Corollary 2.

Corollary 7. *The Hermite-based Euler polynomials satisfy the following integro-differential equations*:

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_y+2D_x^{-1}D_y+2yD_x^{-1}D_y^2+3zD_x^{-2}D_y^3 \\ +\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-k}D_y^{k+1}-(n+1)D_x\Bigg)\{_HE_n(x,y,z)\}=0, \tag{51}$$

$$\Bigg(\Big(x-\frac{1}{2}\Big)D_z+2yD_x^{-2}D_z^2+3D_x^{-4}D_z^2+3zD_x^{-4}D_z^3 \\ +\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-2k}D_z^{k+1}-(n+1)D_x^2\Bigg)\{_HE_n(x,y,z)\}=0, \tag{52}$$

$$\left(\left(x-\frac{1}{2}\right)D_y+2D_x^{-2}D_z+2yD_x^{-2}D_zD_y+3zD_x^{-4}D_z^2D_y\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-2k}D_z^kD_y-(n+1)D_x\right)\{_HE_n(x,y,z)\}=0 \tag{53}$$

and

$$\left(\left(x-\frac{1}{2}\right)D_z+2yD_x^{-1}D_yD_z+3D_x^{-2}D_y^2+3zD_x^{-2}D_y^2D_z\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-k}D_y^kD_z-(n+1)D_x^2\right)\{_HE_n(x,y,z)\}=0. \tag{54}$$

Corollary 8. *The Hermite-based Euler polynomials satisfy the following partial differential equations*:

$$\left(\left(x-\frac{1}{2}\right)D_x^nD_y+nD_x^{n-1}D_y+2D_x^{n-1}D_y+2yD_x^{n-1}D_y^2+3zD_x^{n-2}D_y^3\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{n-k}D_y^{k+1}-(n+1)D_x^{n+1}\right)\{_HE_n(x,y,z)\}=0, \tag{55}$$

$$\left(\left(x-\frac{1}{2}\right)D_x^{2n}D_z+2nD_x^{2n-1}D_z+2yD_x^{2n-2}D_z^2+3D_x^{2n-4}D_z^2+3zD_x^{2n-4}D_z^3\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{2n-2k}D_z^{k+1}-(n+1)D_x^{2n+2}\right)\{_HE_n(x,y,z)\}=0, \tag{56}$$

$$\left(\left(x-\frac{1}{2}\right)D_x^{2n}D_y+2nD_x^{2n-1}D_y+2D_x^{2n-2}D_z+2yD_x^{2n-2}D_zD_y+3zD_x^{2n-4}D_z^2D_y\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{2n-2k}D_z^kD_y-(n+1)D_x^{2n+1}\right)\{_HE_n(x,y,z)\}=0 \tag{57}$$

and

$$\left(\left(x-\frac{1}{2}\right)D_x^nD_z+nD_x^{n-1}D_z+2yD_x^{n-1}D_yD_z+3D_x^{n-2}D_y^2+3zD_x^{n-2}D_y^2D_z\right.$$
$$\left.+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{n-k}D_y^kD_z-(n+1)D_x^{n+2}\right)\{_HE_n(x,y,z)\}=0, \tag{58}$$

e_k *being given*, *in terms of the Euler number* E_k, *as in Corollary* 2.

4. Further Remarks and Observations

For the Hermite-based Appell polynomials defined by means of the generating function (9), many interesting properties and characteristics were investigated earlier by using operational techniques combined with the principle of monomiality. Here, in our present investigation, we have found the differential, integro-differential and partial differential equations for the Hermite-based Appell polynomials via the factorization method. We have also derived the corresponding equations for the Hermite-based Bernoulli polynomials and the Hermite-based Euler polynomials which are defined by the generating functions (15) and (16), respectively. Moreover, just as we indicated in Remark 2, we can easily deduce the corresponding results for the Hermite-based Genocchi polynomials defined by the generating function (17) from those involving the Hermite-based Euler polynomials by means of such simple relationships as (19) and (20).

References

[1] P. Appell, Sur une classes de polynômes, *Ann. Sci. École Norm.* (*Sér.* 2) **9** (1880), 119–144.

[2] G. Bretti and P. E. Ricci, Multidimensional extensions of the Bernoulli and Appell Polynomials, *Taiwanese J. Math.* **8** (2004), 415–428.

[3] C. Cassisa and P. E. Ricci, Multi-variable Gould-Hopper and Laguerre polynomials, *Matematiche* (*Catania*) **52** (2007), 107–125.

[4] Y. Ben Cheikh, On obtaining dual sequences via quasi-monomiality, *Georgian Math. J.* **9** (2002), 413–422.

[5] Y. Ben Cheikh, Some results on quasi-monomiality, *Appl. Math. Comput.* **141** (2003), 63–76.

[6] Y. Ben Cheikh and H. Chaggara, Connection problems via lowering operators, *J. Comput. Appl. Math.* **178** (2005), 45–61.

[7] G. Dattoli, Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle, in D. Cocolicchio, G. Dattoli and H. M. Srivastava (Eds.), *Advanced Topics in Mathematics and Physics* (Melfi, 9-12 May 1999), Aracne Editrice, Rome, 2000, pp. 147–164.

[8] G. Dattoli, A. Torre and G. Mazzacurati, Quasi-monomials and isospectral problems, *Nuovo Cimento B* **112** (1997), 133–138.

[9] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vols. **I** and **III**, McGraw-Hill Book Company, New York, Toronto and London, 1953 and 1955.

[10] M. X. He and P. E. Ricci, Differential equation of Appell polynomials via the factorization method, *J. Comput. Appl. Math.* **139** (2002), 231–237.

[11] S. Khan, G. Yasmin, R. Khan and N. A. M. Hassan, Hermite-based Appell polynomials: Properties and applications, *J. Math. Anal. Appl.* **351** (2009), 756–764.

[12] Y. L. Luke, *The Special Functions and Their Approximations*, Vol. **I**, Mathematics in Science and Engineering, Vol. **53-I**, A Series of Monographs and Textbooks, Academic Press, New York and London, 1969.

[13] M. A. Özarslan, Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials, *Adv. Difference Equations* **2013** (2013), Article ID 2013:116, 1–13.

[14] Á. Pintér and H. M. Srivastava, Addition theorems for the Appell polynomials and the associated classes of polynomial expansions, *Aequationes Math.* **85** (2013), 483–495.

[15] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* **5** (2011), 390–444.

[16] H. M. Srivastava and Y. Ben Cheikh, Orthogonality of some polynomial sets via quasi-monomiality, *Appl. Math. Comput.* **142** (2003), 415–425.

[17] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Acedemic Publishers, Dordrecht, Boston and London, 2001.

[18] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[19] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[20] H. M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **17** (2004), 375–380.

[21] B. Yılmaz and M. A. Özarslan, Differential equations for the extended 2D Bernoulli and Euler polynomials, *Adv. Difference Equations* **2013** (2013), Article ID 2013:103, 1–16.