Filomat 30:14 (2016), 3795–3806
DOI 10.2298/FIL1614795L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Value Distribution of L-Functions Concerning Sharing Sets

Peiqiang Lin[a], Weichuan Lin[a]

[a]*Department of Mathematics, Fujian Normal University, Fuzhou 350007, China*

Abstract. We use the concept of set centroid to study the value distribution of L-functions in the (extended) Selberg class, which shows how an L-function and a meromorphic function are uniquely determined by their two sharing sets. The results in this paper extend Theorem 1 in Li [A result on value distribution of L-functions, Proc. Amer. Math. Soc., 138(2010):2071–2077]. In addition, we show the accuracy of the results by giving some examples.

1. Introduction and main result

The Riemann hypothesis, proposed by Bernhard Riemann [8] in 1859, is a conjecture that all the non-trivial zeros of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ lie on the critical line consisting of the complex numbers $\frac{1}{2} + ti$, where t is a real number and i is the imaginary unit. It is a clue worth exploring that the Riemann hypothesis implies results about the distribution of prime numbers. Along with suitable generalizations, some mathematicians consider it the most important unresolved problem in pure mathematics (see [2]). Furthermore, Riemann hypothesis can be generalized by replacing the Riemann zeta function by the formally similar, but much more general, global L-functions. In this broader setting, L-functions, with the Riemann zeta function as a prototype, are more important in mathematics, specially, in number theory. Recently, the value distribution of L-functions has been studied extensively, such as Garunkstis [3], Li [5], and Steuding [16].

Two meromorphic functions f and g are said to share a value $a \in \hat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ CM (counting multiplicities) if $E(a, f) = E(a, g)$. Here $E(a, f) := \{s \in \mathbb{C} : f(s) - a = 0\}$ denotes the preimage of a under f, where a zero of $f - a$ with multiplicity m counts m times in $E(a, f)$. Moreover, f and g are said to share a value a IM (ignoring multiplicities) if $\overline{E}(a, f) = \overline{E}(a, g)$. Here $\overline{E}(a, f)$ denotes the set of the distinct elements in $E(a, f)$, which is called the simplified preimage of a under f. In terms of sharing values, two nonconstant meromorphic functions in $\mathbb{C}$ must be identically equal if they share five values IM, and one must be a Möbius transform of the other if they share four values CM, the numbers five and four are the best possible, as shown by Nevanlinna (see [4, 19]). In recent thirty years, there have been many generalizations based on the Nevanlinna's results (see [7, 9, 17, 18, 20]). Similar to above definitions, $E(S, f) := \bigcup_{a \in S} \{s \in \mathbb{C} : f(s) - a = 0\}$ denotes

2010 *Mathematics Subject Classification*. Primary 30D35; Secondary 30D30, 11M06, 11M36
Keywords. Riemann zeta function, L-function, sharing set, uniqueness
Received: 28 October 2014; Accepted: 10 March 2015
Communicated by Hari M. Srivastava
Research supported by the National Natural Science Foundation of China(11371225) and the Natural Science Foundation of Fujian Province in China (2011J01006).
Email address: sxlwc936@fjnu.edu.cn (Weichuan Lin)

the preimage of S under f, where $S \subset \hat{\mathbb{C}}$, a zero of $f - a$ with multiplicity m counts m times in $E(S, f)$. And $\overline{E}(S, f)$ denotes the set of the distinct elements in $E(S, f)$, which is called the simplified preimage of S under f. If $E(S, f) = E(S, g)$, then it is said that f and g share the set S CM. If $\overline{E}(S, f) = \overline{E}(S, g)$, then it is said that f and g share the set S IM. In this note, we present some new properties of L-functions with two shared-sets.

Throughout this paper, an L-function always means an L-function in the (extended) Selberg class, which includes the Riemann zeta function and essentially those Dirichlet series where one might expect a Riemann hypothesis. Introduced by Selberg [10], the Selberg class $\mathcal{S}$ is the set of all Dirichlet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

absolutely convergent for $\mathrm{Re}(s) > 1$ that satisfy four axioms:

(1) Ramanujan hypothesis: for any $\epsilon > 0$, $a(n) \ll n^{\epsilon}$;
(2) Analytic continuation: the function $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order for some non-negative integer k;
(3) Functional equation: $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega \overline{\Lambda_{\mathcal{L}}(1 - \bar{s})}, \quad \text{where} \quad \Lambda_{\mathcal{L}} := \mathcal{L}(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j, and complex numbers ν_j, ω with $\mathrm{Re}(\nu_j) \geq 0$ and $|\omega| = 1$.
(4) Euler product hypothesis: $\mathcal{L}$ can be written as a product over primes:

$$\mathcal{L}(s) = \prod_{p} \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$.

The extended Selberg class $\mathcal{S}^{\sharp}$ is defined as the set of all functions $\mathcal{L}(s)$ satisfying axioms (1)(3). Usually, a function is said to be an L-function if it possesses an Euler product hypothesis. However, it appears that $\mathcal{S}^{\sharp}$ contains interesting examples of functions which do not have an Euler product, and in some aspects it is worthwhile to study the extended Selberg class.

We first recall the following result, which actually holds without the Euler product hypothesis.

Proposition 1.1 ([15]). *For any nonconstant L-function $\mathcal{L}$, as $r \to \infty$, $T(r, \mathcal{L}) = \frac{d_{\mathcal{L}}}{\pi} r \log r + O(r)$.*

Here, $d_{\mathcal{L}}$ is called the degree of $\mathcal{L}$, which is given by $d_{\mathcal{L}} = 2 \sum_{j=1}^{K} \lambda_j$, where K and λ_j are the numbers in the axiom (3). Although the data of the functional equation are not unique, this quantity $d_{\mathcal{L}}$ is well-defined.

It is known that the number of sharing values can be substantially reduced for the uniqueness of two L-functions, as seen from the following result due to Steuding [16], which actually holds without the Euler product hypothesis.

Theorem A ([16], page 152). *Assume that $\mathcal{L}_1$, $\mathcal{L}_2$ satisfy the axioms (1)(3) with $a(1) = 1$. If $\mathcal{L}_1$, $\mathcal{L}_2$ share a value $c \neq \infty$ CM, then $\mathcal{L}_1 \equiv \mathcal{L}_2$.*

This implies that two L-functions with $a(1) = 1$ must be identically equal if they have the same zeros with counting multiplicities, and two L-functions with "enough" common zeros (without counting multiplicities) are expected to be dependent in a certain sense, see Bombieri and Perelli [1]. In addition, L-functions can be analytically continued as meromorphic functions in the complex plane,so the study on how an L-function and a meromorphic function are uniquely determined seems to be quite valuable. In order to study how an L-function is uniquely determined by preimages of complex values, one should examine the situation involving an arbitrary L-function and an arbitrary meromorphic function. Considering the meromorphic functions with finitely many poles, Li [5] established the following uniqueness theorem in Proc. Amer. Math. Soc..

Theorem B ([5]). *Let $a, b \in \mathbb{C}$ be two distinct values and let f be a meromorphic function in $\mathbb{C}$ with finitely many poles. If f and a nonconstant L-function $\mathcal{L}$ share a CM and b IM, then $\mathcal{L} \equiv f$.*

Remark 1.2. *The following example given by Li [5] shows that Theorem B no longer holds without "finitely many poles".*

Example 1.3. *Consider the function $\mathcal{L} = \zeta$ and $f = \dfrac{2\zeta}{\zeta+1}$, where ζ is the Riemann zeta function. It is clear that f has infinitely many poles, and f and $\mathcal{L}$ share $0, 1$ CM, but they are not identically equal.*

In particular, note that $s = 1$ is the only possible pole for an L-function, Theorem B implies that f is meromorphic in $\mathbb{C}$ and shares one value CM, a second value IM and the value ∞ IM with a non-constant L-function $\mathcal{L}$, then $\mathcal{L} \equiv f$. Furthermore, under the condition that f and $\mathcal{L}$ have three shared-values, Garunkštis, Grahl and Steuding [3] obtained the following result, which was established in Comment. Math. Univ. St. Pauli.

Theorem C ([3]). *Let $a, b \in \hat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ and $c \in \mathbb{C}$ be three distinct values and let f be a meromorphic function in $\mathbb{C}$. If f and a nonconstant L-function $\mathcal{L}$ share a, b CM and c IM, then $\mathcal{L} \equiv f$.*

Recently, investigating the case that f and $\mathcal{L}$ share three distinct finite complex values IM (without counting multiplicities), Li and Yi [6] got the following result in Mathematische Nachrichten.

Theorem D ([6]). *Let f be a transcendental meromorphic function in the complex plane such that f has finitely many poles in the complex plane, and let b_1, b_2, b_3 be three distinct finite complex values. If f shares b_1, b_2, b_3 IM with a nonconstant L-function $\mathcal{L}$, then $f \equiv \mathcal{L}$.*

Remark 1.4. *The condition that f is a meromorphic function with "finitely many poles" in Theorem D is necessary, as shown by the following example. Moreover, Example 1.2 also shows that the condition that f and $\mathcal{L}$ share two distinct complex values CM in Theorem C is necessary.*

Example 1.5. *Let $\mathcal{L}$ be a nonconstant L-function such that $\mathcal{L}$ has no poles, and $f = \dfrac{2\mathcal{L}}{\mathcal{L}^2+1}$. We find that f has infinitely many poles, f and $\mathcal{L}$ share 0 CM, $-1, 1$ IM. However, $f \not\equiv \mathcal{L}$.*

It is desirable to explore a problem about the shared-sets between a meromorphic function and an L-function. Furthermore, it would be valuable to know what characteristics these shared-sets would have to be. In order to make our result concise, we need the following definition:

Definition 1.6. *Suppose S is a subset of $\mathbb{C}$, the centroid of S is defined by*

$$C(S) = \frac{\int_{\mathbb{C}} z\chi_S(z)dz}{\int_{\mathbb{C}} \chi_S(z)dz},$$

where χ_S is the indicator function of the subset S of $\mathbb{C}$, which is defined as

$$\chi_S(z) := \begin{cases} 1 & \text{if } z \in S, \\ 0 & \text{if } z \notin S. \end{cases}$$

Especially, if $S = \{a_i | i = 1, 2, \cdots, n\}$ is a finite set in $\mathbb{C}$, the centroid of S can also be computed by

$$C(S) = \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right).$$

Using the notion of the centroid , we get the following main theorem.

Theorem 1.7. *Suppose that f is a meromorphic function in $\mathbb{C}$ with finitely many poles, and $S_1, S_2 \subset \mathbb{C}$ are two different sets such that $S_1 \bigcap S_2 = \varnothing$ and $\#(S_i) \le 2$, $i = 1, 2$, where $\#(S)$ denotes the cardinality of the set S. Let f and a nonconstant L-function $\mathcal{L}$ share S_1 CM and S_2 IM, then the following statements hold:*

(i) $\mathcal{L} \equiv f$, *if* $C(S_1) \neq C(S_2)$,
(ii) $\mathcal{L} \equiv f$ *or* $\mathcal{L} + f \equiv 2C(S_1)$, *if* $C(S_1) = C(S_2)$.

As an immediate consequence of Theorem 1.7, we obtain the result as follows.

Theorem 1.8. *Suppose that f is a meromorphic function in $\mathbb{C}$ with finitely many poles, and $S_1, S_2 \subset \mathbb{C}$ are two different sets such that $S_1 \bigcap S_2 = \varnothing$, $C(S_1) \neq C(S_2)$ and $\#(S_i) \le 2$, $i = 1, 2$, where $\#(S)$ denotes the cardinality of the set S. Let f and a nonconstant L-function $\mathcal{L}$ share S_1 CM and S_2 IM, then $\mathcal{L} \equiv f$.*

Remark 1.9. *When $\#(S_1) = \#(S_2) = 1$, Theorem 1.7 and Theorem 1.8 yield that Theorem B is valid.*

Remark 1.10. *The condition that f is a meromorphic function with "finitely many poles" in Theorem 1.7 and Theorem 1.2 is necessary, as shown by Example 1.5 and the following examples.*

Example 1.5 implies that f has infinitely many poles, f and $\mathcal{L}$ share $S_1 = \{0\}$ CM, $S_2 = \{-1, 1\}$ IM, and $C(\{0\}) = C(\{-1, 1\}) = 0$, but neither $\mathcal{L} \equiv f$ nor $\mathcal{L} + f \equiv 0$.

Example 1.11. *Let $\mathcal{L} = \zeta$ and $f = \frac{1}{\zeta}$, where ζ is the Riemann zeta function. We find that f has infinitely many poles, f and $\mathcal{L}$ share $S_1 = \{i, -i\}$ CM, $S_2 = \{1, -1\}$ CM, and $C(S_1) = C(S_2)$, but neither $\mathcal{L} \equiv f$ nor $\mathcal{L} + f \equiv 0$.*

Therefore, Example 1.5 and Example 1.11 show that Theorem 1.7 fails in the case that $C(S_1) = C(S_2)$ if f has infinitely many poles.

Example 1.12. *Let $\mathcal{L} = \zeta$ and $f = \frac{4\zeta}{3\zeta - 4}$, where ζ is the Riemann zeta function. We find that f has infinitely many poles, f and $\mathcal{L}$ share $S_1 = \{0\}$ CM, $S_2 = \{2, 4\}$ CM, and $C(S_1) \neq C(S_2)$, but $\mathcal{L} \not\equiv f$.*

Example 1.13. *Let $\mathcal{L} = \zeta$ and $f = \frac{1}{\zeta}$, where ζ is the Riemann zeta function. We find that f has infinitely many poles, f and $\mathcal{L}$ share $S_1 = \{i, -i\}$ CM, $S_2 = \{2, \frac{1}{2}\}$ CM, and $C(S_1) \neq C(S_2)$, but $\mathcal{L} \not\equiv f$.*

Therefore, Example 1.12 and Example 1.13 show that Theorem 1.7 fails in the case that $C(S_1) \neq C(S_2)$ if f has infinitely many poles.

Remark 1.14. *It is shown by the following example that $\mathcal{L} + f \equiv 2C(S_1)$ is possible in Theorem 1.7 when $C(S_1) = C(S_2)$.*

Example 1.15. *Let $\mathcal{L} = \zeta$ and $f = -\zeta$, where ζ is the Riemann zeta function. We find that f has finitely many poles, and f and $\mathcal{L}$ share $\{0\}$ CM, $\{-1, 1\}$ CM and $\{-2, 2\}$ CM. Obviously, $C(\{0\}) = C(\{-1, 1\}) = C(\{-2, 2\}) = 0$ and $\mathcal{L} + f \equiv 0$.*

Above all examples, we can draw a conclusion that there exist a meromorphic function f and a nonconstant L-function $\mathcal{L}$ and two sets S_1, S_2 ($\#(S_i) \le 2$, $i = 1, 2$) such that f and $\mathcal{L}$ share S_1, S_2 CM, but $f \not\equiv \mathcal{L}$. However, it is natural to pose the following two open questions:

Question 1.16. *Can CM shared-set S_1 be replaced by an IM shared-set in Theorem 1.7 and Theorem 1.8?*

Question 1.17. *What happens to Theorem 1.7 and Theorem 1.8 if* $\max\{\#(S_1), \#(S_2)\} \geq 3$*?*

To prove our results, we will employ Nevanlinna theory. For the convenience of the readers,next we list the following standard notations and results from Nevanlinna theory(see the references [4, 18, 19]).

Let f be a nonconstant meromorphic function in the complex plane. Then the definitions of the proximity function $m(r, f)$, the counting function $N(r, f)$, the reduced counting function $\overline{N}(r, f)$, the Nevanlinna characteristic function $T(r, f)$ of a nonconstant meromorphic function f and the order $\lambda(f)$ are defined as

$$\begin{aligned}
m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,\\
N(r,f) &= \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r,\\
\overline{N}(r,f) &= \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt + \overline{n}(0,f)\log r,\\
T(r,f) &= m(r,f) + N(r,f),\\
\lambda(f) &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},
\end{aligned}$$

respectively, where $\log^+ x = max\{\log x, 0\}$ for all $x \geq 0$, $n(t, f)$ denotes the number of poles of f in the disc $|z| < t$, counting multiplicities and $\overline{n}(t, f)$ denotes the number of poles of f in the disc $|z| < t$, ignoring multiplicities.

We recall the following results:

(i) The arithmetic properties of $T(r, f)$ and $m(r, f)$:

$$T(r, fg) \leq T(r, f) + T(r, g), T(r, f + g) \leq T(r, f) + T(r, g) + O(1).$$

The same inequalities holds for $m(r, f)$.

(ii) $T(r, f)$ is an increasing function of r. Moreover, f is a rational function if and only if

$$T(r, f) = O(\log r).$$

(iii) The Nevanlinna first fundamental theorem: $T(r, f) = T(r, \frac{1}{f}) + O(1)$.

(iv) The logarithmic derivative lemma: $m(r, \frac{f'}{f}) = O(\log r)$ if the order $\lambda(f)$ of f is finite.

(v) The Nevanlinna second fundamental theorem:

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) + S(r,f),$$

where $a_1, a_2, \cdots, a_q$ are distinct complex values in $\hat{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ and $S(r, f)$ denotes a quantity satisfying $S(r, f) = O(\log(rT(r, f)))$ for all r outside possibly a set of finite Lebesgue measure. If f is of finite order, then $S(r, f) = O(\log r)$ for all r.

In addition, we need the following definition:

Definition 1.18. *Let f, g be nonconstant meromorphic functions in the complex plane, and a, b be two distinct complex numbers.*

$$\overline{N}(r, f = a|g = b) = \int_0^r \frac{\overline{n}(t, f = a|g = b) - \overline{n}(0, f = a|g = b)}{t}dt + \overline{n}(0, f = a|g = b)\log r,$$

where $\overline{n}(t, f = a|g = b)$ denotes the number of common zeros of $f - a$ and $g - b$ in the disc $|z| \leq t$, and any of them is counted only once.

2. Lemmas and Propositions

Next the following lemmas are given to prove the main results of this paper:

Lemma 2.1. *Suppose that f is a meromorphic function in $\mathbb{C}$ with finitely many poles, and $a, b, c \in \mathbb{C}$ are three distinct values. If f and a nonconstant L-function $\mathcal{L}$ share $\{a\}$ CM and $\{b, c\}$ IM, then $2T(r, f) \leq 3T(r, \mathcal{L}) + S(r, f)$, $2T(r, \mathcal{L}) \leq 3T(r, f) + S(r, \mathcal{L})$.*

Proof. Since f only has finitely many poles, then $\overline{N}(r, f) = S(r, f)$. Moreover, according to axioms (2) of the Selberg class, the poles of $\mathcal{L}$ only occur at $s = 1$, then $\overline{N}(r, \mathcal{L}) = S(r, \mathcal{L})$. Therefore, using the Nevanlinna second fundamental theorem, we have

$$\begin{aligned} 2T(r, f) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + \overline{N}\left(r, \frac{1}{f-c}\right) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + \overline{N}\left(r, \frac{1}{f-c}\right) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{\mathcal{L}-a}\right) + \overline{N}\left(r, \frac{1}{\mathcal{L}-b}\right) + \overline{N}\left(r, \frac{1}{\mathcal{L}-c}\right) + S(r, f) \\ &\leq 3T(r, \mathcal{L}) + S(r, f). \end{aligned}$$

Similarly, we have $2T(r, \mathcal{L}) \leq 3T(r, f) + S(r, \mathcal{L})$. □

Remark 2.2. *If f satisfies the conditions in Lemma 2.1, then f is a transcendental meromorphic function of finite order, and $\lambda(f) = \lambda(\mathcal{L}) \leq 1$, $S(r, f) = S(r, \mathcal{L}) = O(\log r)$.*

Lemma 2.3. *Suppose $\mathcal{L}$ is a nonconstant L-function, there is no generalized Picard exceptional value of $\mathcal{L}$ in the complex plane.*

Proof. Note that ∞ is a generalized Picard exceptional value of $\mathcal{L}$, so there is at most one generalized Picard exceptional value of $\mathcal{L}$ in the complex plane. According to axioms (2) of the Selberg class, $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order for some non-negative integer k. Suppose $a \in \mathbb{C}$ is a generalized Picard exceptional value of $\mathcal{L}$, let $\alpha_1, \alpha_2, \cdots, \alpha_t$ be all the zeros with multiplicity $k_1, k_2, \cdots, k_t$ of $\mathcal{L} - a$ respectively, then

$$\frac{(s-1)^k(\mathcal{L}(s) - a)}{(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_t)^{k_t}} = e^{p(s)},$$

where $p(s)$ is a polynomial. According to Proposition 1.1, $\lambda(\mathcal{L}) \leq 1$. Obviously, $deg(p) \geq 1$, or $\mathcal{L}$ will be a rational function. Therefore, $deg(p) = \lambda(\mathcal{L}) = 1$. Let $p(s) = a_0 s + a_1$. By calculation, we have

$$T(r, \mathcal{L}) = \frac{|a_0|}{\pi} r + O(\log r) = O(r),$$

which is in contradiction with Proposition 1.1. Therefore, there is no generalized Picard exceptional value of $\mathcal{L}$ in the complex plane. □

We also need the following main results, which play an important role in the proof of Theorem 1.7.

Proposition 2.4. *Suppose f is a meromorphic function in $\mathbb{C}$ with finitely many poles, and $a, b, c \in \mathbb{C}$ are three distinct values. Let f and a nonconstant L-function $\mathcal{L}$ share $\{a\}$ CM and $\{b, c\}$ IM, then the following statements hold:*

(i) $\mathcal{L} \equiv f$, *if* $2a \neq b + c$,
(ii) $\mathcal{L} \equiv f$ *or* $\mathcal{L} + f \equiv 2a$, *if* $2a = b + c$.

Proof. Note that f only has finitely many poles, we let $\alpha_1, \alpha_2, \cdots, \alpha_t$ be all the poles of order $k_1, k_2, \cdots, k_t$ of f respectively, then $(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_t)^{k_t} f(s)$ is an entire function of finite order. Moreover, according to axioms (2) of the Selberg class, $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order for some non-negative integer k. Let

$$F_1 = \frac{\mathcal{L}-a}{Q(f-a)},$$

where

$$Q = \frac{(s-\alpha_1)^{k_1}(s-\alpha_2)^{k_2}\cdots(s-\alpha_t)^{k_t}}{(s-1)^k} \tag{1}$$

is a rational function. Since f and $\mathcal{L}$ share $\{a\}$ CM, F_1 has neither a pole nor a zero in complex plane. Note that f, $\mathcal{L}$, Q are the function of finite order, and hence there is a polynomial p such that

$$F_1 = \frac{\mathcal{L}-a}{Q(f-a)} = e^p.$$

According to Proposition 1.1 and Lemma 2.1, we have $\lambda(f) = \lambda(\mathcal{L}) \le 1$. Obviously, $deg(p) \le 1$. Assume that $p(s) = a_0 s + a_1$. by calculation, we have

$$T(r, e^p Q) = \frac{|a_0|}{\pi} r + O(\log r).$$

Furthermore, we conclude that any zero of $(\mathcal{L}-b)(\mathcal{L}-c)$ is a zero of $(e^pQ-1)(e^pQ-\frac{c-a}{b-a})(e^pQ-\frac{b-a}{c-a})$ since f and $\mathcal{L}$ share $\{b,c\}$ IM.

Suppose that $\mathcal{L} \not\equiv f$, $(b-a)(\mathcal{L}-a) \not\equiv (c-a)(f-a)$ and $(c-a)(\mathcal{L}-a) \not\equiv (b-a)(f-a)$. Note that $\overline{N}(r,\mathcal{L}) = S(r,\mathcal{L}) = O(\log r)$, by the Nevanlinna second fundamental theorem, we have

$$\begin{aligned}
T(r,\mathcal{L}) &\le \overline{N}\left(r, \frac{1}{\mathcal{L}-b}\right) + \overline{N}\left(r, \frac{1}{\mathcal{L}-c}\right) + \overline{N}(r,\mathcal{L}) + O(\log r) \\
&\le \overline{N}\left(r, \frac{1}{e^pQ-1}\right) + \overline{N}\left(r, \frac{1}{e^pQ-\frac{c-a}{b-a}}\right) + \overline{N}\left(r, \frac{1}{e^pQ-\frac{b-a}{c-a}}\right) + O(\log r) \\
&\le 3T(r, e^pQ) + O(\log r) \\
&= \frac{3|a_0|}{\pi} r + O(\log r) = O(r),
\end{aligned}$$

which is in contradiction with Proposition 1.1. It follows that one of the following holds:

(i) $\mathcal{L} \equiv f$,
(ii) $(b-a)(\mathcal{L}-a) \equiv (c-a)(f-a)$,
(iii) $(c-a)(\mathcal{L}-a) \equiv (b-a)(f-a)$.

We distinguish the following two cases to discuss.

Case 1 When $2a \ne b+c$, we claim that $\mathcal{L} \equiv f$.

Indeed, if $(b-a)(\mathcal{L}-a) \equiv (c-a)(f-a)$, then the set sharing properties of f and $\mathcal{L}$ yield $\overline{E}(c,\mathcal{L}) = \overline{E}(b,f)$, and hence, $\overline{E}(b,\mathcal{L}) = \overline{E}(c,f)$. Note that b is not the Picard exceptional value of $\mathcal{L}$, this implies that $(b-a)^2 = (c-a)^2$, i.e. $b = c$ or $2a = b+c$. Thus we get a contradiction.

On the other hand, if $(c-a)(\mathcal{L}-a) \equiv (b-a)(f-a)$, we also can get a contradiction in the same way as above. Therefore, we have $\mathcal{L} \equiv f$.

Case 2 When $2a = b+c$, we see that $b-a = a-c$. Therefore, we have $\mathcal{L} \equiv f$ or $\mathcal{L}+f \equiv 2a$. □

Proposition 2.5. *Suppose f is a meromorphic function in $\mathbb{C}$ with finitely many poles, and $a, b, c \in \mathbb{C}$ are three distinct values. Let f and a nonconstant L-function $\mathcal{L}$ share $\{a\}$ IM and $\{b,c\}$ CM, then the following statements hold:*

(i) $\mathcal{L} \equiv f$, *if* $2a \neq b + c$,
(ii) $\mathcal{L} \equiv f$ *or* $\mathcal{L} + f \equiv 2a$, *if* $2a = b + c$.

Proof. We discuss the following two cases.
Case 1 When $2a \neq b + c$, we are going to show that $\mathcal{L} \equiv f$.
Let

$$F_2 = \frac{(\mathcal{L} - b)(\mathcal{L} - c)}{Q^2(f - b)(f - c)},$$

where Q is defined by (1). In the same way in the proof of Proposition 2.4, we can prove that there exists a polynomial q such that

$$F_2 = \frac{(\mathcal{L} - b)(\mathcal{L} - c)}{Q^2(f - b)(f - c)} = e^q,$$

and $deg(q) \leq 1$, $T(r, e^q Q^2) = O(r)$. Note that a is not the Picard exceptional value of $\mathcal{L}$, and $\mathcal{L}$ and f share a IM, but $2a - b - c \neq 0$. Therefore, we have

$$f + \mathcal{L} - b - c \not\equiv 0. \tag{2}$$

Suppose $\mathcal{L} \not\equiv f$, note that

$$e^q Q^2 - 1 = \frac{(f - \mathcal{L})(f + \mathcal{L} - b - c)}{(f - b)(f - c)}, \tag{3}$$

thus $e^q Q^2 \not\equiv 1$. We see that any zero of $\mathcal{L} - a$ is a zero of $e^q Q^2 - 1$, we have

$$\overline{N}\left(r, \frac{1}{\mathcal{L} - a}\right) \leq \overline{N}\left(r, \frac{1}{e^q Q^2 - 1}\right) \leq T(r, e^q Q^2) + O(1) = O(r). \tag{4}$$

Let's consider the following two functions:

$$G_0 = \left(\frac{\mathcal{L}'}{(\mathcal{L} - a)(\mathcal{L} - b)(\mathcal{L} - c)} - \frac{f'}{(f - a)(f - b)(f - c)}\right)(f - \mathcal{L})(f + \mathcal{L} - b - c),$$

$$G_1 = \left(\frac{(b - a)(b - c)\mathcal{L}'}{(\mathcal{L} - a)(\mathcal{L} - b)(\mathcal{L} - c)} - \frac{(c - a)(c - b)f'}{(f - a)(f - b)(f - c)}\right)(f - \mathcal{L})(f + \mathcal{L} - b - c).$$

First, we claim that $G_1 \not\equiv 0$. Otherwise, suppose that $G_1 \equiv 0$. By (2) and $\mathcal{L} \not\equiv f$, we have

$$\frac{(b - a)(b - c)\mathcal{L}'}{(\mathcal{L} - a)(\mathcal{L} - b)(\mathcal{L} - c)} \equiv \frac{(c - a)(c - b)f'}{(f - a)(f - b)(f - c)}. \tag{5}$$

Note that $\mathcal{L}$ has no Picard exceptional value in the complex plane, we may assume that s_0 is a zero of $\mathcal{L} - b$ of multiplicity k, the set sharing properties of f and $\mathcal{L}$ yield that s_0 is the zero of $f - b$ of multiplicity k or the zero of $f - c$ of multiplicity k.

If s_0 is the zero of $f - b$ of multiplicity k, then we can deduce that the principal part of the Laurent expansion of $\frac{(b - a)(b - c)\mathcal{L}'}{(\mathcal{L} - a)(\mathcal{L} - b)(\mathcal{L} - c)}$ at $s = s_0$ is $\frac{k}{s - s_0}$ and the one for the function $\frac{(c - a)(c - b)f'}{(f - a)(f - b)(f - c)}$ is $\frac{(a - c)k}{(b - a)(s - s_0)}$. It shows from (5) that $\frac{a - c}{b - a} = 1$, i. e , $2a = b + c$. Thus we get a contradiction. Therefore, s_0 should be the zero of $f - c$, and hence, $E(b, \mathcal{L}) \subset E(c, f)$.

Similarly, we can also get $E(c, f) \subset E(b, \mathcal{L})$. Therefore, we obtain that $E(b, \mathcal{L}) = E(c, f)$. Furthermore, we have $E(c, \mathcal{L}) = E(b, f)$.

Assume that s_1 is a zero of $\mathcal{L}-c$ and $f-b$ of multiplicity l, we know that the principal part of the Laurent expansion of $\frac{(b-a)(b-c)\mathcal{L}'}{(\mathcal{L}-a)(\mathcal{L}-b)(\mathcal{L}-c)}$ at $s=s_1$ is $\frac{(b-a)l}{(a-c)(s-s_1)}$ and the one for the function $\frac{(c-a)(c-b)f'}{(f-a)(f-b)(f-c)}$ is $\frac{(a-c)l}{(b-a)(s-s_1)}$. It implies that $\frac{b-a}{a-c}=\frac{a-c}{b-a}$, i.e. $(b+c-2a)(b-c)=0$, so $b=c$ or $2a=b+c$. Thus we get a contradiction. Therefore, $G_1\not\equiv 0$.

Next, we claim that $G_0\equiv 0$. Otherwise, suppose that $G_0\not\equiv 0$.

Note that any zeros of $\mathcal{L}-a$, $\mathcal{L}-b$ or $\mathcal{L}-c$ are not the poles of G_0, we get to know that the possible poles of G_0 only come from the poles of f and $\mathcal{L}$, which are finitely many. Therefore, G_0 has at most finitely many poles, which implies that

$$N(r,G_0)=O(\log r). \tag{6}$$

According to (3), we can rewrite G_0 as follows:

$$G_0=\frac{\mathcal{L}'}{\mathcal{L}-a}(1-\frac{1}{e^qQ^2})-\frac{f'}{f-a}(e^qQ^2-1).$$

Thus

$$\begin{aligned} m(r,G_0) &\leq m\left(r,\frac{\mathcal{L}'}{\mathcal{L}-a}\right)+m\left(r,1-\frac{1}{e^qQ^2}\right)+m\left(r,\frac{f'}{f-a}\right)+m(r,e^qQ^2-1)+\log 2\\ &\leq T\left(r,1-\frac{1}{e^qQ^2}\right)+T(r,e^qQ^2-1)+O(\log r)\\ &\leq 2T(r,e^qQ^2)+O(\log r)=O(r), \end{aligned}$$

which together with (6) obtains that $T(r,G_0)=O(r)$.

Note that G_1 can be rewritten as

$$G_1=\frac{(b-a)(b-c)\mathcal{L}'}{\mathcal{L}-a}(1-\frac{1}{e^qQ^2})-\frac{(c-a)(c-b)f'}{f-a}(e^qQ^2-1).$$

From similar estimate, we can also deduce that $T(r,G_1)=O(r)$.

Suppose that s_2 is a zero of $\mathcal{L}-b$ of multiplicity k, then s_2 is the zero of $f-b$ of multiplicity k or the zero of $f-c$ of multiplicity k since f and $\mathcal{L}$ share $\{b,c\}$ CM. If s_2 is the zero of $f-b$ of multiplicity k, we conclude that s_2 is also the zero of G_0 of multiplicity at least k, then we have

$$\overline{N}(r,\mathcal{L}=b|f=b)\leq\overline{N}\left(r,\frac{1}{G_0}\right)\leq T(r,G_0)+O(1)=O(r). \tag{7}$$

If s_2 is the zero of $f-c$ of multiplicity k, we conclude that s_2 is also the zero of G_1 of multiplicity at least k, then we have

$$\overline{N}(r,\mathcal{L}=b|f=c)\leq\overline{N}\left(r,\frac{1}{G_1}\right)\leq T(r,G_1)+O(1)=O(r). \tag{8}$$

Combining (4), (7), (8), and using the Nevanlinna second fundamental theorem, we have

$$\begin{aligned} T(r,\mathcal{L}) &\leq \overline{N}\left(r,\frac{1}{\mathcal{L}-a}\right)+\overline{N}\left(r,\frac{1}{\mathcal{L}-b}\right)+\overline{N}(r,\mathcal{L})+O(\log r)\\ &= \overline{N}\left(r,\frac{1}{\mathcal{L}-a}\right)+\overline{N}(r,\mathcal{L}=b|f=b)+\overline{N}(r,\mathcal{L}=b|f=c)+O(\log r)\\ &= O(r), \end{aligned}$$

which is in contradiction with Proposition 1.1. Thus $G_0\equiv 0$, as claimed above.

Since $G_0 \equiv 0$, by (2) and the assumption that $\mathcal{L} \not\equiv f$, we have

$$\frac{\mathcal{L}'}{(\mathcal{L}-a)(\mathcal{L}-b)(\mathcal{L}-c)} \equiv \frac{f'}{(f-a)(f-b)(f-c)}. \tag{9}$$

Since f and $\mathcal{L}$ must share $\{a\}$ IM, by Lemma 2.2, we assume that s_3 is a zero of $\mathcal{L}-a$ of multiplicity m and a zero of $f-a$ of multiplicity n. By simple calculation, we get to know that the principal part of the Laurent expansion of $\frac{\mathcal{L}'}{(\mathcal{L}-a)(\mathcal{L}-b)(\mathcal{L}-c)}$ at $s = s_3$ is $\frac{m}{(a-b)(a-c)(s-s_3)}$ and the one for the function $\frac{f'}{(f-a)(f-b)(f-c)}$ is $\frac{n}{(a-b)(a-c)(s-s_3)}$. It follows from (9) that $m = n$. Thus f and $\mathcal{L}$ must share $\{a\}$ CM. According to Proposition 2.4, we will get a contradiction. Therefore, $\mathcal{L} \equiv f$.

Case 2 When $2a = b + c$, let

$$P(\omega) = \omega^2 - 2a\omega + bc.$$

It is easy to know that $P(b) = P(c) = 0$.

We claim that $P(f) \equiv P(\mathcal{L})$. Indeed, let's consider the following function:

$$U = \frac{P'(\mathcal{L})\mathcal{L}'}{P(\mathcal{L})} - \frac{P'(f)f'}{P(f)}.$$

Obviously, all zero of $P(f)$ and $P(L)$ cannot be the poles of U since f and $\mathcal{L}$ share $\{b, c\}$ CM. Thus, the poles of U only occur at the poles of $P'(f)f'$ and $P'(\mathcal{L})\mathcal{L}'$. Therefore,

$$N(r, U) \le \overline{N}(r, P'(f)f') + \overline{N}(r, P'(\mathcal{L})\mathcal{L}') \le \overline{N}(r, f) + \overline{N}(r, \mathcal{L}) = O(\log r). \tag{10}$$

Note that

$$m(r, U) \le m(r, \frac{P'(f)f'}{P(f)}) + m(r, \frac{P'(\mathcal{L})\mathcal{L}'}{P(\mathcal{L})}) + \log 2 = S(r, P(f)) + S(r, P(\mathcal{L})) = O(\log r). \tag{11}$$

Combining (10), (11), we have

$$T(r, U) = O(\log r).$$

This implies that U is a rational function. Moreover, it is obvious that any zero of $\mathcal{L} - a$ is a zero of $P'(\mathcal{L})$. Since $\mathcal{L}$ and f share a IM, any zero of $\mathcal{L} - a$ is a zero of $P'(f)$ as well. Hence any zero of $\mathcal{L} - a$ is a zero of U.

Suppose that $U \not\equiv 0$, we see that U has at most finitely many zeros. Therefore, $\mathcal{L} - a$ has at most finitely many zeros, that is, a is a generalized Picard exceptional value of $\mathcal{L}$. It is in contradiction with Lemma 2.3. Thus $U \equiv 0$, integrating this equation, we have $P(\mathcal{L}) \equiv AP(f)$, where A is a non-zero constant. Clearly, a is not the Picard exceptional value of f or $\mathcal{L}$. Therefore, $A = 1$, i.e. $P(\mathcal{L}) \equiv P(f)$.

Since $P(\mathcal{L}) \equiv P(f)$, we have

$$(\mathcal{L} + f - 2a)(\mathcal{L} - f) \equiv 0.$$

Therefore, we get $\mathcal{L} \equiv f$ or $\mathcal{L} + f \equiv 2a$. □

Proposition 2.6. *Suppose f is a meromorphic function in $\mathbb{C}$ with finitely many poles, and $a, b, c, d \in \mathbb{C}$ are four distinct values. Let f and a nonconstant L-function $\mathcal{L}$ share $\{a, b\}$ CM and $\{c, d\}$ IM, then the following statements hold:*

(i) $\mathcal{L} \equiv f$, *if* $a + b \neq c + d$,

(ii) $\mathcal{L} \equiv f$ *or* $\mathcal{L} + f \equiv a + b$, *if* $a + b = c + d$.

Proof. Let

$$F_3 = \frac{(\mathcal{L}-a)(\mathcal{L}-b)}{Q^2(f-a)(f-b)},$$

where Q is defined by (1). In the same way in the proof of Proposition 2.4, we can prove that there exists a polynomial u such that

$$F_3 = \frac{(\mathcal{L}-a)(\mathcal{L}-b)}{Q^2(f-a)(f-b)} = e^u,$$

where $deg(u) \leq 1$.

Since f and $\mathcal{L}$ share $\{c, d\}$ IM, it is easy to know that any zero of $(\mathcal{L}-c)(\mathcal{L}-d)$ is a zero of $(e^uQ^2 - 1)\left(e^uQ^2 - \frac{(c-a)(c-b)}{(d-a)(d-b)}\right)\left(e^uQ^2 - \frac{(d-a)(d-b)}{(c-a)(c-b)}\right)$.

Suppose that $\mathcal{L} \not\equiv f$, $\mathcal{L}+f \not\equiv a+b$, $(d-a)(d-b)(\mathcal{L}-a)(\mathcal{L}-b) \not\equiv (c-a)(c-b)(f-a)(f-b)$ and $(c-a)(c-b)(\mathcal{L}-a)(\mathcal{L}-b) \not\equiv (d-a)(d-b)(f-a)(f-b)$, note that $\overline{N}(r,\mathcal{L}) = S(r,\mathcal{L}) = O(\log r)$ and $T(r, e^uQ^2) = O(r)$, by the Nevanlinna second fundamental theorem, we have

$$\begin{aligned} T(r,\mathcal{L}) &\leq \overline{N}\left(r, \frac{1}{\mathcal{L}-c}\right) + \overline{N}\left(r, \frac{1}{\mathcal{L}-d}\right) + \overline{N}(r,\mathcal{L}) + O(\log r) \\ &\leq \overline{N}\left(r, \frac{1}{e^uQ^2-1}\right) + \overline{N}\left(r, \frac{1}{e^uQ^2 - \frac{(c-a)(c-b)}{(d-a)(d-b)}}\right) + \overline{N}\left(r, \frac{1}{e^uQ^2 - \frac{(d-a)(d-b)}{(c-a)(c-b)}}\right) + O(\log r) \\ &\leq 3T(r, e^uQ^2) + O(\log r) = O(r), \end{aligned}$$

which is in contradiction with Proposition 1.1. It follows that one of the following holds:

(i) $\mathcal{L} \equiv f$,
(ii) $\mathcal{L} + f \equiv a + b$,
(iii) $(c-a)(c-b)(\mathcal{L}-a)(\mathcal{L}-b) \equiv (d-a)(d-b)(f-a)(f-b)$,
(iv) $(d-a)(d-b)(\mathcal{L}-a)(\mathcal{L}-b) \equiv (c-a)(c-b)(f-a)(f-b)$.

We distinguish the following two cases to discuss.

Case 1 When $a+b \neq c+d$, we claim that $\mathcal{L} \equiv f$.

Indeed, if $\mathcal{L}+f \equiv a+b$, then f and $\mathcal{L}$ share c, d IM since $a+b \neq c+d$. This implies that $a+b = 2c = 2d$, thus we get a contradiction.

If $(c-a)(c-b)(\mathcal{L}-a)(\mathcal{L}-b) \equiv (d-a)(d-b)(f-a)(f-b)$, we obtain that $\overline{E}(d,\mathcal{L}) = \overline{E}(c, f)$ since f and $\mathcal{L}$ share $\{c, d\}$ IM, and hence, $\overline{E}(c,\mathcal{L}) = \overline{E}(d, f)$. Note that c is not the Picard exceptional value of $\mathcal{L}$, this implies that $(c-a)^2(c-b)^2 = (d-a)^2(d-b)^2$, so $c = d$ or $a+b = c+d$. Thus, we get a contradiction.

If $(d-a)(d-b)(\mathcal{L}-a)(\mathcal{L}-b) \equiv (c-a)(c-b)(f-a)(f-b)$, we can also get a contradiction in the same way as above. Therefore, we have $\mathcal{L} \equiv f$.

Case 2 When $a+b = c+d$, we can deduce that $(c-a)(c-b) = (d-a)(d-b)$, thus (iii) and (iv) are both equivalent to $(\mathcal{L}+f-a-b)(\mathcal{L}-f) \equiv 0$. Therefore, we obtain that $\mathcal{L} \equiv f$ or $\mathcal{L}+f \equiv a+b$. □

3. Proof of Theorem

Proof. According to the cardinality of sharing sets, we distinguish the following four cases to discuss.

Case 1 When $\#(S_1) = \#(S_2) = 1$, the condition $C(S_1) \neq C(S_2)$ yields that there exist two distinct finite complex numbers $a, b \in \mathbb{C}$ such that f and $\mathcal{L}$ share a CM and b IM. Note that f has finitely many poles, by Theorem B, we obtain $\mathcal{L} = f$.

Case 2 When $\#(S_1) = 1$ and $\#(S_2) = 2$, we assume that $S_1 = \{a\}$, $S_2 = \{b, c\}$, where $a, b, c \in \mathbb{C}$ are three distinct finite complex numbers. This shows that f and $\mathcal{L}$ share $\{a\}$ CM and $\{b, c\}$ IM, where $C(S_1) = a$ and $C(S_2) = \frac{b+c}{2}$. Note that f has finitely many poles, by Proposition 2.4, we conclude that Theorem 1.7 holds.

Case 3 When $\#(S_1) = 2$ and $\#(S_2) = 1$, we assume that $S_1 = \{b, c\}$, $S_2 = \{a\}$, where $a, b, c \in \mathbb{C}$ are three distinct finite complex numbers. This shows that f and $\mathcal{L}$ share $\{a\}$ IM and $\{b, c\}$ CM, where $C(S_1) = \dfrac{b+c}{2}$ and $C(S_2) = \dfrac{a}{2}$. Note that f has finitely many poles, by Proposition2.5, we conclude that Theorem1.7 holds.

Case 4 When $\#(S_1) = \#(S_2) = 2$, we assume that $S_1 = \{a, b\}$, $S_2 = \{c, d\}$, where $a, b, c, d \in \mathbb{C}$ are four distinct finite complex numbers. This shows that f and $\mathcal{L}$ share $\{a, b\}$ CM and $\{c, d\}$ IM, where $C(S_1) = \dfrac{a+b}{2}$ and $C(S_2) = \dfrac{c+d}{2}$. Note that f has finitely many poles, by Proposition2.6, we conclude that Theorem1.7 holds. $\square$

4. Further Remarks

It is well known that the theory of the families of L-functions and partial zeta type functions, and also the family of zeta functions themselves, has became a very important part of Analytic Number Theory. In recent years, many authors introduced and investigated series associated with zeta functions and q-zeta functions (see [11, 12]). Furthermore, H. M. Srivastava, etc. constructed and investigated various properties of a unified presentation of certain meromorphic functions related to the families of the partial zeta type functions, q-zeta functions and (q-)L-functions(see [13, 14]).

In this paper, we mainly study the uniqueness of L-functions in the (extended) Selberg class, which include the Riemann zeta function and essentially those Dirichlet series where one might expect a Riemann hypothesis(For details, in Section 1). Naturally, we are interesting to know what happen on the subject of (q-)L-series and (q-)L-functions of the papers [13, 14] under the sharing-set conditions of Theorem1.7. Unfortunately, we do not find the effective method to resolve it.

Acknowledgements. The authors wish to express thanks to the referee for reading the manuscript very carefully and making a number of valuable suggestions and comments towards the improvement of the paper.

References

[1] E. Bombieri and A. Perelli, Distinct zeros of L-functions, Acta Arithmetica, 83(1998):271–281.
[2] E. Bombieri, The Riemann Hypothesis official problem description, Clay Mathematics Institute, 2000.
[3] R. Garunkštis, J. Grahl, and J. Steuding, Uniqueness Theorems for L-Functions, Comment. Math. Univ. St. Pauli, 60(2011):15–35.
[4] W. K. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.
[5] B. Q. Li, A result on value distribution of L-functions, Proc. Amer. Math. Soc., 138(2010):2071–2077.
[6] X. M. Li and H. X. Yi, Results on value distribution of L-functions, Mathematische Nachrichten, 286(2013):1326–1336.
[7] W. Lin, S. Mori and K. Tohge, Uniqueness theorems in an angular domain, Tohoku Math. J., 58(2006):509–527.
[8] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Größe, In Monatsberichte der Berliner Akademie, 1859.
[9] P. Sahooi, Unicity theorem for entire functions sharing one value, Filomat, 27(2013):797–809.
[10] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, In Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 367–385. 1992.
[11] H. M. Srivastava, J. Choi, Series Associated with the Zeta and related functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[12] H. M. Srivastava, J. Choi, Zeta and q-Zeta functions and associated series and integrals, Elsevier Science Publishers, Amsterdam,London and New York, 2012.
[13] H. M. Srivastava, H. Özden, I. N. Cangül and Y. Simsek, A unified presentation of certain meromorphic functions related to the families of the partial zeta type functions and the L-functions, Appl. Math. Comput.,219(2012): 3903–3913.
[14] H. M. Srivastava, T. Kim and Y. Simsek, q-Bernoulli numbers and polynomials associated with multiple q-Zeta functions and basic L-series, Russian J. Math. Phys., 12(2005): 241–268.
[15] J. Steuding, How many values can L-functions share ?, Fiz. Mat. Fak. Moksl. Semin. Darb, 7(2004):70–81.
[16] J. Steuding, Value-distribution of L-functions, Springer, Berlin, 2007.
[17] J. Xia and Y. Xu, Uniqueness and differential polynomials of meromorphic functions sharing one value, Filomat, 25(2011):185–194.
[18] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[19] L. Yang, Value Distribution Theory, Springer, Berlin, 1993.
[20] X. Zhang and W. Lin, Uniqueness and value-sharing of entire functions, J. Math. Anal. Appl., 343(2008):938–950.