Filomat 30:13 (2016), 3587–3599
DOI 10.2298/FIL1613587H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Spectra of 2×2 Upper Triangular Operator Matrices

Junjie Huang[a], Aichun Liu[a,b], Alatancang Chen[a,b]

[a] *School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China*
[b] *Department of Mathematics, Hohhot University for Nationalities, Hohhot 010051, China*

Abstract. The spectra of the 2×2 upper triangular operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ acting on a Hilbert space $H_1 \oplus H_2$ are investigated. We obtain a necessary and sufficient condition of $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$, in terms of the spectral properties of two diagonal elements A and B of M_C. Also, the analogues for the point spectrum, residual spectrum and continuous spectrum are further presented. Moveover, we construct some examples illustrating our main results. In particular, it is shown that the inclusion $\sigma_r(M_C) \subseteq \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$ is not correct in general. Note that $\sigma(T)$ (resp. $\sigma_r(T)$) denotes the spectrum (resp. residual spectrum) of an operator T, and $\mathcal{B}(H_2, H_1)$ is the set of all bounded linear operators from H_2 to H_1.

1. Introduction

Let H_1 and H_2 be separable, infinite dimensional, complex Hilbert spaces. $\mathcal{B}(H_1)$, $\mathcal{B}(H_2)$ and $\mathcal{B}(H_1 \oplus H_2)$ denote the set of all bounded linear operators on H_1, H_2 and $H_1 \oplus H_2$, respectively. When the operators $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ are given, the operator

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{B}(H_1 \oplus H_2)$$

with $C \in \mathcal{B}(H_2, H_1)$ has been extensively studied. The spectrum and related problems of M_C are considered, for example, in [1–7, 9–11, 14] and the references therein. It is well known that the spectrum of upper triangular block matrices is described as that of their diagonal elements, but this in general is not true for upper triangular operator matrices in infinite dimensional spaces. In [5], the perturbation of spectra of 2×2 operator matrices is characterized, and it is further shown that if $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ are normal, then

$$\sigma(A) \cup \sigma(B) = \sigma(M_C) \quad \text{for every } C \in \mathcal{B}(H_2, H_1). \tag{1.1}$$

The relationship between $\sigma(A) \cup \sigma(B)$ and $\sigma(M_C)$ is considered in [9], and the authors prove that if $\sigma(A) \cap \sigma(B)$ has no interior points, then (1.1) is valid. In [1], the set of operators $C \in \mathcal{B}(H_2, H_1)$ for which (1.1) holds is investigated. For more relevant researches, we refer the reader to [2, 3, 6, 7, 11].

2010 *Mathematics Subject Classification*. Primary 47A10; Secondary 47A55
Keywords. Spectrum, perturbation theory, 2×2 upper triangular operator matrix
Received: 27 October 2014; Accepted: 13 December 2014
Communicated by Dragan S. Djordjević
Corresponding author: Junjie Huang
Research supported by the National Natural Science Foundation of China (No. 11461049), and the Natural Science Foundation of Inner Mongolia (No. 2013JQ01)
Email addresses: huangjunjie@imu.edu.cn (Junjie Huang), lac41980881@163.com (Aichun Liu), alatanca@imu.edu.cn (Alatancang Chen)

The purpose of this paper is to study when the spectrum and its certain parts of upper triangular operator matrices can be formulated as the union of the corresponding spectrum of their diagonal elements. To be precise, we obtain some necessary and sufficient conditions which ensure that (1.1) and the similar results for the point spectrum, residual spectrum and continuous spectrum hold. See Theorems 2.1, 2.3–2.6 and Corollary 2.2 in Section 2.

For the operators $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$ (most of them appear in the literature) having certain properties, we verify that these operators exactly satisfy our results. See Corollaries 3.4–3.8 in Section 3. We also give examples showing that if all conditions in our theorems are not fulfilled, then, in general, (1.1) and the similar results are invalid. See Examples 4.1–4.6 in Section 4. In addition, Example 4.3 shows that the inclusion $\sigma_r(M_C) \subseteq \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$ is not necessarily correct.

Our main tools are various parts of the point spectrum and residual spectrum, which are closely related to the space decomposition technique.

In the following, we fix some notation and terminology. Let $T \in \mathcal{B}(H_1, H_2)$. Then, we use $\mathcal{N}(T)$ and $\mathcal{R}(T)$ to denote the null space and range of T, respectively. The symbol $n(T)$ represents the nullity of T which is equal to $\dim \mathcal{N}(T)$, and $d(T)$ stands for the deficiency of T which is equal to $\dim \mathcal{N}(T^*)$. As usual, we say T is right (resp. left) invertible if there exists an operator $S \in \mathcal{B}(H_2, H_1)$ such that $TS = I_{H_2}$ (resp. $ST = I_{H_1}$); and if T is both left invertible and right invertible, we call it invertible. It is well known that T is left invertible if and only if T is bounded below, and if and only if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed; and T is right invertible if and only if T is surjective, i.e., $\mathcal{R}(T) = H_2$. Now, let $H_1 = H_2$, i.e., $T \in \mathcal{B}(H_1)$. Then, by the Closed Graph Theorem, the resolvent set $\rho(T)$ of T consists of the complex numbers λ such that $T - \lambda I$ is a bijection on H_1; the spectrum $\sigma(T)$ of T is the complement of $\rho(T)$ in $\mathbb{C}$; the approximate point spectrum or left spectrum $\sigma_{ap}(T)$ of T is the set

$$\sigma_{ap}(T) = \left\{\lambda \in \mathbb{C};\ T - \lambda I \text{ is not left invertible}\right\}:$$

the defect spectrum or right spectrum $\sigma_\delta(T)$ of T is defined by

$$\sigma_\delta(T) = \{\lambda \in \mathbb{C}; T - \lambda I \text{ is nonsurjective}\}:$$

and the sets

$$\begin{aligned}
\sigma_p(T) &= \{\lambda \in \mathbb{C}:\ T - \lambda I \text{ is noninjective}\},\\
\sigma_r(T) &= \left\{\lambda \in \mathbb{C}:\ T - \lambda I \text{ is injective and } \overline{\mathcal{R}(T - \lambda I)} \neq H_1\right\},\\
\sigma_c(T) &= \left\{\lambda \in \mathbb{C}:\ T - \lambda I \text{ is injective}, \overline{\mathcal{R}(T - \lambda I)} = H_1 \text{ and } \mathcal{R}(T - \lambda I) \neq H_1\right\}
\end{aligned}$$

are called the point spectrum, residual spectrum and continuous spectrum, respectively. The point spectrum and residual spectrum are divided into the following disjoint union: $\sigma_p(T) = \cup_{i=1}^4 \sigma_{p,i}(T)$ and $\sigma_r(T) = \sigma_{r,1}(T) \cup \sigma_{r,2}(T)$, in terms of the density and closedness of the range $\mathcal{R}(T - \lambda I)$ of $T - \lambda I$, where

$$\begin{aligned}
\sigma_{p,1}(T) &= \left\{\lambda \in \mathbb{C}:\ \lambda \in \sigma_p(T) \text{ and } \mathcal{R}(T - \lambda I) = H_1\right\},\\
\sigma_{p,2}(T) &= \left\{\lambda \in \mathbb{C}:\ \lambda \in \sigma_p(T), \overline{\mathcal{R}(T - \lambda I)} = H_1 \text{ and } \mathcal{R}(T - \lambda I) \neq H_1\right\},\\
\sigma_{p,3}(T) &= \left\{\lambda \in \mathbb{C}:\ \lambda \in \sigma_p(T), \overline{\mathcal{R}(T - \lambda I)} \neq H_1 \text{ and } \mathcal{R}(T - \lambda I) \text{ is closed}\right\},\\
\sigma_{p,4}(T) &= \left\{\lambda \in \mathbb{C}:\ \lambda \in \sigma_p(T), \overline{\mathcal{R}(T - \lambda I)} \neq H_1 \text{ and } \mathcal{R}(T - \lambda I) \text{ is nonclosed}\right\},\\
\sigma_{r,1}(T) &= \left\{\lambda \in \mathbb{C}:\ \lambda \in \sigma_r(T) \text{ and } \mathcal{R}(T - \lambda I) \text{ is closed}\right\},\\
\sigma_{r,2}(T) &= \left\{\lambda \in \mathbb{C}:\ \lambda \in \sigma_r(T) \text{ and } \mathcal{R}(T - \lambda I) \text{ is nonclosed}\right\}.
\end{aligned}$$

Note that throughout this paper, for convenience, we write $\sigma_{p,ij}(T) = \sigma_{p,i}(T) \cup \sigma_{p,j}(T)$, $\sigma_{pr}(T) = \sigma_p(T) \cup \sigma_r(T)$ and $\sigma_{cr}(T) = \sigma_c(T) \cup \sigma_r(T)$, and use M_0 to denote the operator M_C with $C = 0$.

The organization of this paper is as follows. In Section 2, the main results of this paper are given; while proofs and some corollaries of the main results are presented in Section 3. Section 4 is devoted to examples to illustrate the previous results.

2. Main Results

First, we obtain a necessary and sufficient condition of $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$ by using the spectral properties of the operators A and B.

Theorem 2.1. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,* (1.1) *is valid if and only if the following statements are fulfilled:*
(i) $\lambda \in \sigma_{r,1}(A) \setminus \sigma_\delta(B)$ implies $n(B-\lambda I) = 0$ or $n(B-\lambda I) \neq d(A-\lambda I)$;
(ii) $\lambda \in \sigma_{p,1}(B) \setminus \sigma_{ap}(A)$ implies $d(A-\lambda I) = 0$ or $n(B-\lambda I) \neq d(A-\lambda I)$.

Proof. See Section 3. □

In this theorem, we actually emphasis the conditions $n(B-\lambda I) = 0$ and $d(A-\lambda I) = 0$ in the corresponding statements. Indeed, they are contained by $n(B-\lambda I) \neq d(A-\lambda I)$, since $\lambda \in \sigma_{r,1}(A)$ (resp. $\lambda \in \sigma_{p,1}(B)$) implies $d(A-\lambda I) \neq 0$ (resp. $n(B-\lambda I) \neq 0$). Note that $\sigma(A)$ is the disjoint union of $\sigma_{ap}(A)$ and $\sigma_{r,1}(A)$, and $\sigma(B)$ is the disjoint union of $\sigma_{p,1}(B)$ and $\sigma_\delta(B)$. Then, we immediately have the following result.

Corollary 2.2. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,* (1.1) *is valid if and only if $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B)$ implies $n(B-\lambda I) \neq d(A-\lambda I)$.*

Next, a necessary and sufficient condition of $\sigma_p(M_C) = \sigma_p(A) \cup \sigma_p(B)$ for every $C \in \mathcal{B}(H_2, H_1)$ is given.

Theorem 2.3. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_p(M_C) = \sigma_p(A) \cup \sigma_p(B) \quad \text{for every } C \in \mathcal{B}(H_2, H_1) \tag{2.1}$$

if and only if for each $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$, one of the following statements is fulfilled:
(i) $\lambda \in \rho(A)$;
(ii) $\lambda \in \sigma_{r,1}(A)$ and $n(B-\lambda I) > d(A-\lambda I)$.

Proof. See Section 3. □

Similarly, we may extend the previous theorems to some other parts of the spectrum, such as the residual spectrum, **point residual spectrum** (the union of the point spectrum and residual spectrum) and continuous spectrum.

Theorem 2.4. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_r(M_C) = \sigma_r(A) \cup \sigma_r(B) \quad \text{for every } C \in \mathcal{B}(H_2, H_1) \tag{2.2}$$

if and only if the following statements are fulfilled:
(i) if $\lambda \in \sigma_c(A) \cap \sigma_p(B)$, then $\lambda \in \sigma_{p,12}(B)$;
(ii) if $\lambda \in \sigma_r(B)$, then $\lambda \notin \sigma_p(A)$;
(iii) if $\lambda \in \sigma_r(A) \setminus \sigma_r(B)$, then $\lambda \in \rho(B)$.

Proof. See Section 3. □

Theorem 2.5. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_{pr}(M_C) = \sigma_{pr}(A) \cup \sigma_{pr}(B) \quad \text{for every } C \in \mathcal{B}(H_2, H_1) \tag{2.3}$$

if and only if the statements (i) and (ii) are fulfilled.
(i) If $\lambda \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$, then one of the statements (a) *and* (b) *holds:*
(a) *$A-\lambda I$ is left invertible and $n(B-\lambda I) > d(A-\lambda I)$;*
(b) *$\lambda \in \sigma_{p,13}(B)$ and $n(B-\lambda I) < d(A-\lambda I)$.*
(ii) If $\lambda \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$, then one of the statements (c), (d) *and* (e) *holds:*
(c) *$\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,12}(B)$ and $n(B-\lambda I) > d(A-\lambda I)$;*
(d) *$\lambda \in \sigma_{p,1}(B)$ and $n(B-\lambda I) < d(A-\lambda I)$;*
(e) *$\lambda \in \rho(B)$ and $d(A-\lambda I) > 0$.*

Proof. See Section 3. □

Theorem 2.6. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_c(M_C) = \sigma_c(A) \cup \sigma_c(B) \quad \text{for every } C \in \mathcal{B}(H_2, H_1) \tag{2.4}$$

if and only if the statements (i), (ii) and (iii) are fulfilled.

(i) If $\lambda \in \sigma_r(A) \cap \sigma_p(B)$, then one of the statements (a)–(f) holds:

(a) $\lambda \in \sigma_{r,1}(A)$ and $n(B - \lambda I) > d(A - \lambda I)$;

(b) $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,34}(B)$ and $n(B - \lambda I) \le d(A - \lambda I)$;

(c) $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B)$ and $n(B - \lambda I) < d(A - \lambda I)$;

(d) $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B)$ and $n(B - \lambda I) = d(A - \lambda I) < \infty$;

(e) $\lambda \in \sigma_{r,2}(A) \cap \sigma_{p,34}(B)$;

(f) $\lambda \in \sigma_{r,2}(A) \cap \sigma_{p,1}(B)$ and $n(B - \lambda I) \ne d(A - \lambda I)$.

(ii) If $\lambda \in \sigma_c(A)$, then $\lambda \notin \sigma_{pr}(B)$.

(iii) If $\lambda \in \sigma_c(B)$, then $\lambda \notin \sigma_{pr}(A)$.

Proof. See Section 3. □

3. Proofs

In this section, we first review some basic results, and then present proofs of the main results of this paper, i.e., Theorems 2.1, and 2.3–2.6.

Lemma 3.1. *Let X and Y be Hilbert spaces, and let $T \in \mathcal{B}(X, Y)$ with $\mathcal{R}(T)$ being nonclosed. Then, there exists a closed subspace $\Omega \subsetneq \overline{\mathcal{R}(T)}$ of Y, such that $\mathcal{R}(T) \cap \Omega = \{0\}$ and $\dim \Omega = \infty$. Moreover, we may further require $\mathcal{R}(T) \dot{+} \Omega \subsetneq \overline{\mathcal{R}(T)}$.*

Proof. The lemma is a direct consequence of Lemma 16.2 in [12]. □

The following lemmas are obvious, for details, see [14].

Lemma 3.2. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then, $M_C - \lambda I$ is injective for every $C \in \mathcal{B}(H_2, H_1)$ if and only if $A - \lambda I$ and $B - \lambda I$ are both injective.*

Lemma 3.3. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then, $\overline{\mathcal{R}(M_C - \lambda I)} = H_1 \oplus H_2$ for every $C \in \mathcal{B}(H_2, H_1)$ if and only if $\overline{\mathcal{R}(A - \lambda I)} = H_1$ and $\overline{\mathcal{R}(B - \lambda I)} = H_2$.*

Proof of Theorem 2.1. Sufficiency. Obviously, $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$. Now, we prove the opposite inclusion. Let $\lambda \in \sigma_{ap}(A) \cup \sigma_\delta(B)$. Then, $M_C - \lambda I$ is not left invertible or is nonsurjective, which indicates that $\lambda \in \sigma(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Let $\lambda \in \sigma_{r,1}(A) \setminus \sigma_\delta(B)$. Then, we know that $\lambda \in \sigma_r(A)$, $\mathcal{R}(B - \lambda I) = H_2$ and $\mathcal{R}(A - \lambda I)$ is closed. If $n(B - \lambda I) = 0$, then $\lambda \in \rho(B)$, and hence

$$\begin{pmatrix} I & -C(B-\lambda I)^{-1} \\ 0 & I \end{pmatrix} (M_C - \lambda I) = \begin{pmatrix} A - \lambda I & 0 \\ 0 & B - \lambda I \end{pmatrix}, \tag{3.1}$$

which deduces that $\lambda \in \sigma_r(M_C) \subseteq \sigma(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. If $n(B - \lambda I) \ne 0$, then $\mathcal{N}(B - \lambda I)$ and $\mathcal{R}(A - \lambda I)^\perp$ are nontrivial subspaces in H_2 and H_1, respectively. Thus, for every $C \in \mathcal{B}(H_2, H_1)$, $M_C - \lambda I$ admits the following block representation

$$M_C - \lambda I = \begin{pmatrix} (A-\lambda I)_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & (B-\lambda I)_1 \end{pmatrix} : \begin{pmatrix} H_1 \\ \mathcal{N}(B-\lambda I) \\ \mathcal{N}(B-\lambda I)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A-\lambda I) \\ \mathcal{R}(A-\lambda I)^\perp \\ H_2 \end{pmatrix}, \tag{3.2}$$

where both $(A-\lambda I)_1 : H_1 \to \mathcal{R}(A-\lambda I)$ and $(B-\lambda I)_1 : \mathcal{N}(B-\lambda I)^\perp \to H_2$ are invertible. So, we obtain

$$F_C(M_C-\lambda I)E_C = \begin{pmatrix} (A-\lambda I)_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & (B-\lambda I)_1 \end{pmatrix}$$

with

$$E_C = \begin{pmatrix} I & -(A-\lambda I)_1^{-1}C_1 & -(A-\lambda I)_1^{-1}C_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},\ F_C = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -C_4(B-\lambda I)_1^{-1} \\ 0 & 0 & I \end{pmatrix}. \tag{3.3}$$

From $n(B-\lambda I) \neq d(A-\lambda I)$, it follows that $C_3 : \mathcal{N}(B-\lambda I) \to \mathcal{R}(A-\lambda I)^\perp$ is either noninjective or nonsurjective, which together with the invertibilities of E_C and F_C demonstrates that $\lambda \in \sigma(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Let $\lambda \in \sigma_{p,1}(B) \backslash \sigma_{ap}(A)$. Then, we see that $\lambda \in \sigma_p(B), \mathcal{R}(B-\lambda I) = H_2$ and $A-\lambda I$ is left invertible. If $d(A-\lambda I) = 0$, then $\lambda \in \rho(A)$, and hence

$$(M_C-\lambda I)\begin{pmatrix} I & -(A-\lambda I)^{-1}C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A-\lambda I & 0 \\ 0 & B-\lambda I \end{pmatrix}, \tag{3.4}$$

which derives that $\lambda \in \sigma_p(M_C) \subseteq \sigma(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. If $d(A-\lambda I) \neq 0$, then the subspaces $\mathcal{N}(B-\lambda I)$ and $\mathcal{R}(A-\lambda I)^\perp$ are nontrivial as above. Thus, for every $C \in \mathcal{B}(H_2, H_1)$, $M_C - \lambda I$ has the matrix form (3.2) with $(A-\lambda I)_1 : H_1 \to \mathcal{R}(A-\lambda I)$ and $(B-\lambda I)_1 : \mathcal{N}(B-\lambda I)^\perp \to H_2$ being invertible. From the above arguments, it follows that $\lambda \in \sigma(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Therefore, $\sigma(M_C) \supseteq \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Necessity. By Theorem 2 in [5] and the assumption $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$, we immediately have that the assertions (i) and (ii) hold. □

Corollary 3.4. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad \textit{for every } C \in \mathcal{B}(H_2, H_1)$$

if one of the following assumptions is fulfilled:

(i) $\sigma(A) \cap \sigma(B)$ has no interior point (see [9]);

(ii) A^* or B has single valued extension property (SVEP) (see [7]);

(iii) A is cohypernormal, or B is hypernormal (see [9]).

Proof. Let the assumption (i) hold. It is clear that $\sigma(A) \cap \sigma(B) \subseteq \sigma_{ap}(A) \cup \sigma_\delta(B)$, since $\sigma(A) \cap \sigma(B)$ has no interior point. If $\lambda \in \sigma_{r,1}(A) \setminus \sigma_\delta(B)$, then $\lambda \notin \sigma_{ap}(A) \cup \sigma_\delta(B)$, and hence $\lambda \notin \sigma(A) \cap \sigma(B)$. Thus, $\lambda \in \rho(B)$, which deduces $n(B-\lambda I) = 0$. Similarly, $\lambda \in \sigma_{p,1}(B) \setminus \sigma_{ap}(A)$ implies $d(A-\lambda I) = 0$. By Theorem 2.1, it follows that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Let A^* or B have SVEP. By Corollary 7 and Theorem 2 in [8], we see that $\sigma(A) = \sigma_{ap}(A)$ or $\sigma(B) = \sigma_\delta(B)$, and hence $\sigma_{r,1}(A) = \emptyset$ or $\sigma_{p,1}(B) = \emptyset$. Thus, $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$ by Theorem 2.1.

To prove the case when the assumption (iii) is satisfied, it suffices to note that $\sigma_r(A) = \emptyset$ for cohypernormal operator A, and $\sigma_{p,12}(B) = \emptyset$ for hypernormal operator B. □

Proof of Theorem 2.3. Sufficiency. Obviously, $\sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$ for every $C \in \mathcal{B}(H_2, H_1)$. We prove the opposite inclusion as follows. Let $\lambda \in \sigma_p(A)$. Set $z_0 = (x_0, 0)$ with $x_0 \in \mathcal{N}(A-\lambda I) \setminus \{0\}$, then $(M_C - \lambda I)z_0 = 0$, i.e., $\lambda \in \sigma_p(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Now, let $\lambda \in \sigma_p(B) \backslash \sigma_p(A)$. If $\lambda \in \rho(A)$, then for every $C \in \mathcal{B}(H_2, H_1)$, we have the factorization (3.4), and hence $\lambda \in \sigma_p(M_C)$. If $\lambda \in \sigma_{r,1}(A)$ and $n(B-\lambda I) > d(A-\lambda I)$, then $\mathcal{N}(B-\lambda I) \neq \{0\}$, $d(A-\lambda I) > 0$ and $\mathcal{R}(A-\lambda I)$ $(\neq \{0\})$ is closed. Suppose, without loss of generality, that $\mathcal{N}(B-\lambda I) \neq H_2$. Then, we have the decompositions $H_1 = \mathcal{R}(A-\lambda I) \oplus \mathcal{R}(A-\lambda I)^\perp$ and $H_2 = \mathcal{N}(B-\lambda I) \oplus \mathcal{N}(B-\lambda I)^\perp$. Thus, for every $C \in \mathcal{B}(H_2, H_1)$, $M_C - \lambda I$ has the matrix form (3.2), in which $(A-\lambda I)_1 : H_1 \to \mathcal{R}(A-\lambda I)$ is invertible, and $(B-\lambda I)_1 : \mathcal{N}(B-\lambda I)^\perp \to H_2$ is injective. So, we obtain

$$(M_C - \lambda I)E_C = \Delta_C$$

with E_C being defined as in (3.3), and

$$\Delta_C = \begin{pmatrix} (A-\lambda I)_1 & 0 & 0 \\ 0 & C_3 & C_4 \\ 0 & 0 & (B-\lambda I)_1 \end{pmatrix}.$$

To prove $\lambda \in \sigma_p(M_C)$, it suffices to show that Δ_C is noninjective, since E_C is invertible. In view of $n(B-\lambda I) > d(A-\lambda I)$, it follows that $C_3 : \mathcal{N}(B-\lambda I) \to \mathcal{R}(A-\lambda I)^\perp$ is noninjective. Set $x_0^2 \in \mathcal{N}(C_3) \setminus \{0\}$ and write $z_0 = (0, x_0^2, 0)$, then $\Delta_C z_0 = 0$, which shows that Δ_C is noninjective. Therefore, $\sigma_p(M_C) \supseteq \sigma_p(A) \cup \sigma_p(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Necessity. Assume not, and let $\lambda_0 \in \sigma_p(B) \backslash \sigma_p(A)$, but neither of the assertions (i) and (ii) holds. There are two possible cases.

Case 1: $\lambda_0 \in \sigma_p(B) \backslash \sigma_p(A)$, but $\lambda_0 \notin \rho(A) \cup \sigma_{r,1}(A)$. In this case, we see that $\overline{\mathcal{R}(A-\lambda_0 I)} = H_1$ and $\mathcal{R}(A-\lambda_0 I) \neq H_1$, or $\overline{\mathcal{R}(A-\lambda_0 I)} \neq H_1$ and $\mathcal{R}(A-\lambda_0 I)$ is nonclosed, which both imply that $\mathcal{R}(A-\lambda_0 I)$ is nonclosed. By Lemma 3.1, there exists a closed subspace

$$\Omega_1 \subsetneqq \overline{\mathcal{R}(A-\lambda_0 I)}, \tag{3.5}$$

such that $\mathcal{R}(A-\lambda_0 I) \cap \Omega_1 = \{0\}$ and $\dim \Omega_1 = \infty$. Then, there exists an isometry

$$C_1 : \mathcal{N}(B-\lambda_0 I) \to \Omega_1, \tag{3.6}$$

since $n(B-\lambda_0 I) \leq \dim \Omega_1$. Without loss of generality, we may suppose that $\mathcal{N}(B-\lambda_0 I) \neq H_2$. Define an operator C_0 from H_2 to H_1 by

$$C_0 = (C_1 \ \ 0) : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to H_1. \tag{3.7}$$

Thus, $M_{C_0} - \lambda_0 I$ admits the following block representation

$$M_{C_0} - \lambda_0 I = \begin{pmatrix} A-\lambda_0 I & C_1 & 0 \\ 0 & 0 & (B-\lambda_0 I)_1 \end{pmatrix} : \begin{pmatrix} H_1 \\ \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}. \tag{3.8}$$

Clearly, $(B-\lambda_0 I)_1 : \mathcal{N}(B-\lambda_0 I)^\perp \to H_2$ is injective, which together with the facts $\lambda_0 \notin \sigma_p(A)$ and $\mathcal{R}(A-\lambda_0 I) \cap \mathcal{R}(C_1) = \{0\}$ demonstrates that $M_{C_0} - \lambda_0 I$ is injective. Therefore, $\lambda_0 \notin \sigma_p(M_{C_0})$. This contradicts the assumption $\sigma_p(M_C) = \sigma_p(A) \cup \sigma_p(B)$ for every $C \in \mathcal{B}(H_2, H_1)$, since $\lambda_0 \in \sigma_p(A) \cup \sigma_p(B)$.

Case 2: $\lambda_0 \in \sigma_p(B) \backslash \sigma_p(A)$, but $\lambda_0 \notin \rho(A)$ and $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$. In this case, we know that $\mathcal{N}(B-\lambda_0 I) \neq \{0\}$, and $\mathcal{R}(A-\lambda_0 I)^\perp$ is a nontrivial subspace in H_1. Without loss of generality, suppose that $\mathcal{N}(B-\lambda_0 I)^\perp \neq \{0\}$. Since $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$, we can define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} 0 & 0 \\ C_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}$$

with $C_3 : \mathcal{N}(B-\lambda_0 I) \to \mathcal{R}(A-\lambda_0 I)^\perp$ being an isometry. Then, $M_{C_0} - \lambda_0 I$ can be written as

$$M_{C_0} - \lambda_0 I = \begin{pmatrix} (A-\lambda_0 I)_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & (B-\lambda_0 I)_1 \end{pmatrix} : \begin{pmatrix} H_1 \\ \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \\ H_2 \end{pmatrix}.$$

where $(A-\lambda_0 I)_1 : H_1 \to \overline{\mathcal{R}(A-\lambda_0 I)}$ and $(B-\lambda_0 I)_1 : \mathcal{N}(B-\lambda_0 I)^\perp \to H_2$ are both injective. It is readily seen that $M_{C_0} - \lambda_0 I$ is injective, i.e., $\lambda_0 \notin \sigma_p(M_{C_0})$. Thus, the same contradiction as *Case 1* appears. $\square$

Corollary 3.5. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_p(M_C) = \sigma_p(A) \cup \sigma_p(B) \quad \text{for every } C \in \mathcal{B}(H_2, H_1)$$

if one of the following assumptions is fulfilled:
(i) A is an operator with pure point spectrum;
(ii) A is cohypernormal and $\sigma_c(A) = \emptyset$.

Proof. Let the assumption (i) be satisfied. Since the operator A only has pure point spectrum, $\sigma_{cr}(A) = \emptyset$. If $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$, then $\lambda \in \rho(A)$. By Theorem 2.3, This proves $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{B}(H_2, H_1)$. The case, when the assumption (ii) holds, can be further verified, since $\sigma_r(A) = \emptyset$ for cohypernormal operator A. □

Proof of Theorem 2.4. Sufficiency. Let $\lambda \in \sigma_r(M_C)$ for some $C \in \mathcal{B}(H_2, H_1)$. Obviously, $\lambda \notin \sigma_p(A)$. In the sequel, we claim $\lambda \notin \sigma_c(A) \cap \sigma_p(B)$. To see this, suppose to the contrary that $\lambda \in \sigma_c(A) \cap \sigma_p(B)$, which implies $\lambda \in \sigma_{p,12}(B)$ by the assertion (i). Then, $\overline{\mathcal{R}(A - \lambda I)} = H_1$ and $\overline{\mathcal{R}(B - \lambda I)} = H_2$. By Lemma 3.3, it follows that $\overline{\mathcal{R}(M_C - \lambda I)} = H_1 \oplus H_2$ for every $C \in \mathcal{B}(H_2, H_1)$. Thus, $\lambda \notin \sigma_r(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$, giving a contradiction. This proves $\lambda \notin \sigma_c(A) \cap \sigma_p(B)$. From the above discussions, we have $\lambda \in \rho(A) \cup \sigma_r(A) \cup \rho(B) \cup \sigma_{rc}(B) \cup (\sigma_c(A) \setminus \sigma_p(B)) \cup (\sigma_p(B) \setminus \sigma_c(A))$. Since $\lambda \in \sigma_r(M_C)$ and $\lambda \notin \sigma_p(A)$, it is not hard to show that if $\lambda \in \rho(A) \cup (\sigma_c(A) \setminus \sigma_p(B))$, we clearly have $\lambda \in \sigma_r(B)$, and if $\lambda \in \rho(B) \cup \sigma_c(B) \cup (\sigma_p(B) \setminus \sigma_c(A))$, we must have $\lambda \in \sigma_r(A)$. Therefore, $\sigma_r(M_C) \subseteq \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Now, we prove the opposite inclusion. If $\lambda \in \sigma_r(B)$, then $\lambda \notin \sigma_p(A)$ by the assertion (ii). Thus, for every $C \in \mathcal{B}(H_2, H_1)$, we know by Lemma 3.2 that $\lambda \notin \sigma_p(M_C)$. Note that the fact $\lambda \in \sigma_r(B)$ implies $\overline{\mathcal{R}(B - \lambda I)} \neq H_2$, which immediately deduces $\overline{\mathcal{R}(M_C - \lambda I)} \neq H_1 \oplus H_2$. Hence, $\lambda \in \sigma_r(M_C)$. If $\lambda \in \sigma_r(A) \setminus \sigma_r(B)$, then $\lambda \in \rho(B)$ by the assertion (iii). Thus, for every $C \in \mathcal{B}(H_2, H_1)$, we have $\lambda \notin \sigma_p(M_C)$ and the factorization (3.1), which together with $\lambda \in \sigma_r(A)$ imply that $\lambda \in \sigma_r(M_C)$. Therefore, $\sigma_r(M_C) \supseteq \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Necessity. Assume to the contrary that there exists $\lambda_0 \in \mathbb{C}$, such that one of the assertions (i), (ii) and (iii) fails to hold. There are three possible cases.

Case 1: $\lambda_0 \in \sigma_c(A) \cap \sigma_p(B)$, but $\lambda_0 \notin \sigma_{p,12}(B)$, i.e., $\lambda_0 \in \sigma_c(A) \cap \sigma_{p,34}(B)$. In this case, we see that $\mathcal{N}(B - \lambda_0 I) \neq \{0\}$, $\mathcal{R}(A - \lambda_0 I)^\perp = \{0\}$ and $\mathcal{R}(A - \lambda_0 I)$ is nonclosed. Without loss of generality, suppose that $\mathcal{N}(B - \lambda_0 I)^\perp \neq \{0\}$. Use the operator C_0 defined as in (3.7), then $M_{C_0} - \lambda_0 I$ possesses the matrix form (3.8). Thus, $M_{C_0} - \lambda_0 I$ is injective. Note that $\lambda_0 \in \sigma_{p,34}(B)$ implies $\overline{\mathcal{R}(B - \lambda_0 I)} \neq H_2$. Therefore, $\lambda_0 \in \sigma_r(M_{C_0})$. This contradicts the assumption $\sigma_r(M_C) = \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$, since $\lambda_0 \in \sigma_c(A) \cap \sigma_p(B)$.

Case 2: $\lambda_0 \in \sigma_r(B)$, but $\lambda_0 \in \sigma_p(A)$, i.e., $\lambda_0 \in \sigma_r(B) \cap \sigma_p(A)$. In this case, by $\lambda_0 \in \sigma_p(A)$, it can be readily seen that $\lambda_0 \in \sigma_p(M_C)$, and clearly $\lambda_0 \notin \sigma_r(M_C)$, for every $C \in \mathcal{B}(H_2, H_1)$. This contradicts the assumption $\sigma_r(M_C) = \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$, since $\lambda_0 \in \sigma_r(A) \cup \sigma_r(B)$.

Case 3: $\lambda_0 \in \sigma_r(A) \setminus \sigma_r(B)$, but $\lambda_0 \notin \rho(B)$. In this case, it follows that $\lambda_0 \in \sigma_p(B)$ or $\mathcal{R}(B - \lambda_0 I)$ is nonclosed. If $\lambda_0 \in \sigma_p(B)$, then $\lambda_0 \in \sigma_p(M_0)$, and clearly $\lambda_0 \notin \sigma_r(M_0)$, but $\lambda_0 \in \sigma_r(A) \cup \sigma_r(B)$. The same contradiction as *Case 2* occurs. If $\lambda_0 \notin \sigma_p(B)$ and $\mathcal{R}(B - \lambda_0 I)$ is nonclosed, we combine $\lambda_0 \in \sigma_r(A) \setminus \sigma_r(B)$ then have $\lambda_0 \in \sigma_r(A) \cap \sigma_c(B)$. To complete the proof of *Case 3*, it suffices to find some $C_0 \in \mathcal{B}(H_2, H_1)$ satisfying $\lambda_0 \notin \sigma_r(M_{C_0})$ (actually, λ_0 must belong to $\sigma_c(M_{C_0})$).

Indeed, by Closed Range Theorem (see [13]), $\mathcal{R}(B - \lambda_0 I)$ is nonclosed implies that $\mathcal{R}(B^* - \overline{\lambda_0} I)$ is nonclosed. Then, by Lemma 3.1, there exists a closed subspace

$$\Omega_2 \subsetneqq \overline{\mathcal{R}(B^* - \overline{\lambda_0} I)}, \tag{3.9}$$

such that $\mathcal{R}(B^* - \overline{\lambda_0} I) \cap \Omega_2 = \{0\}$, $\dim \Omega_2 = \infty$, and hence $\dim \Omega_2 = \dim H_1$. Picking a unitary operator U_1 from H_1 onto Ω_2, and writing $C_0 = U_1^*$, we have the operator matrix

$$\begin{pmatrix} A^* - \overline{\lambda_0} & 0 \\ U_1 & B^* - \overline{\lambda_0} \end{pmatrix}. \tag{3.10}$$

Since $\lambda_0 \in \sigma_c(B)$ and $\mathcal{R}(B^* - \overline{\lambda_0}I) \cap \mathcal{R}(U_1) = \{0\}$, we deduce that the operator matrix (3.10) is injective. Thus, the range of its adjoint

$$M_{C_0} - \lambda_0 I = \begin{pmatrix} A - \lambda_0 I & C_0 \\ 0 & B - \lambda_0 I \end{pmatrix}$$

is dense in $H_1 \oplus H_2$. Therefore, $\lambda_0 \notin \sigma_r(M_{C_0})$. □

Corollary 3.6. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_r(M_C) = \sigma_r(A) \cup \sigma_r(B) \quad \textit{for every } C \in \mathcal{B}(H_2, H_1)$$

if one of the following assumptions is fulfilled:
(i) A and B are normal, and $\sigma_{p,34}(B) = \emptyset$;
(ii) A and B are hypernormal, and $\sigma_{p,34}(A) = \sigma_{p,34}(B) = \sigma_c(B) = \emptyset$;
(iii) A and B are cohypernormal, and $\sigma_c(A) = \emptyset$.

Proof. Note that the normal operators are hypernormal and cohypernormal. The proof follows from Theorem 2.4 and the properties of point spectrum and residual spectrum of these three kinds of operators. □

Proof of Theorem 2.5. Sufficiency. Obviously, $\sigma_{pr}(M_C) \subseteq \sigma_{pr}(A) \cup \sigma_{pr}(B)$ for every $C \in \mathcal{B}(H_2, H_1)$. To complete the proof, it suffices to prove the opposite inclusion.

Let $\lambda \in \sigma_p(A) \cup \overline{\sigma_p(B^*)}$. Then, for every $C \in \mathcal{B}(H_2, H_1)$, we know that either $M_C - \lambda I$ is noninjective or $\overline{\mathcal{R}(M_C - \lambda I)} \neq H_1 \oplus H_2$, i.e., $\lambda \in \sigma_{pr}(M_C)$.

Let $\lambda \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$. If the assertion ($a$) holds, i.e., $A - \lambda I$ is left invertible and $n(B - \lambda I) > d(A - \lambda I)$, then $\lambda \notin \sigma_p(A)$, $\mathcal{R}(A - \lambda I)$ is closed and $n(B - \lambda I) > d(A - \lambda I)$. In this case, we see that $\mathcal{N}(B - \lambda I) \neq \{0\}$ and $\mathcal{R}(A - \lambda I) \neq \{0\}$. When $d(A - \lambda I) = 0$, we have $\lambda \in \rho(A)$, which together with $\lambda \in \sigma_p(B)$ demonstrates that $\lambda \in \sigma_p(M_C) \subseteq \sigma_{pr}(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$; when $d(A - \lambda I) > 0$, similar to the proof of Theorem 2.3, we obtain that $\lambda \in \sigma_p(M_C) \subseteq \sigma_{pr}(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. If the assertion ($b$) is fulfilled, i.e., $\lambda \in \sigma_{p,13}(B)$ and $n(B - \lambda I) < d(A - \lambda I)$, then $\mathcal{R}(B - \lambda I)$ is closed, which together with $\lambda \notin \overline{\sigma_p(B^*)}$ deduces $\mathcal{R}(B - \lambda I) = H_2$. In this case, we see that $\mathcal{N}(B - \lambda I)$ and $\mathcal{R}(A - \lambda I)^\perp$ are nontrivial subspaces in H_2 and H_1, respectively. Thus, for every $C \in \mathcal{B}(H_2, H_1)$, the operator matrix $M_C - \lambda I$ has the matrix form

$$M_C - \lambda I = \begin{pmatrix} (A - \lambda I)_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & (B - \lambda I)_1 \end{pmatrix} : \begin{pmatrix} H_1 \\ \mathcal{N}(B - \lambda I) \\ \mathcal{N}(B - \lambda I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A - \lambda I)} \\ \mathcal{R}(A - \lambda I)^\perp \\ H_2 \end{pmatrix}.$$

It can be readily seen that $(B - \lambda I)_1 : \mathcal{N}(B - \lambda I)^\perp \to H_2$ is invertible, and

$$\begin{pmatrix} I & 0 & -C_2(B - \lambda I)_1^{-1} \\ 0 & I & -C_4(B - \lambda I)_1^{-1} \\ 0 & 0 & I \end{pmatrix} (M_C - \lambda I) = \begin{pmatrix} (A - \lambda I)_1 & C_1 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & (B - \lambda I)_1 \end{pmatrix},$$

which implies that $\overline{\mathcal{R}(C_3)} \neq \mathcal{R}(A - \lambda I)^\perp$, since $n(B - \lambda I) < d(A - \lambda I)$. Thus, for every $C \in \mathcal{B}(H_2, H_1)$, we always have $\overline{\mathcal{R}(M_C - \lambda I)} \neq H_1 \oplus H_2$, and hence $\lambda \in \sigma_{pr}(M_C)$.

Let $\lambda \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$. If the assertion ($c$) is true, i.e., $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,12}(B)$ and $n(B - \lambda I) > d(A - \lambda I)$, then $\lambda \notin \sigma_p(A)$ and $\mathcal{R}(A - \lambda I)$ is closed. The rest of the proof is the same as in the case $\lambda \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$ with the assertion (a). If the assertion (d) holds, i.e., $\lambda \in \sigma_{p,1}(B)$ and $n(B - \lambda I) < d(A - \lambda I)$, then $\mathcal{R}(B - \lambda I) = H_2$. The rest of the proof is the same as in the case $\lambda \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$ with the assertion (b). If the assertion (e) is fulfilled, i.e., $\lambda \in \rho(B)$ and $d(A - \lambda I) > 0$, then we derive that $\lambda \in \sigma_r(M_C) \subseteq \sigma_{pr}(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Now, let $\lambda \in \sigma_r(B) \setminus \sigma_p(A)$. Then, we clearly have $\lambda \in \sigma_r(M_C) \subseteq \sigma_{pr}(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Therefore, $\sigma_{pr}(M_C) \supseteq \sigma_{pr}(A) \cup \sigma_{pr}(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Necessity. Assume not, and let $\lambda_0 \in \mathbb{C}$, but one of the assertions (i) and (ii) fails to hold. There are nine possible cases.

Case 1: $\lambda_0 \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$, but $\lambda_0 \notin \sigma_{p,13}(B)$ and $\mathcal{R}(A-\lambda_0 I)$ is nonclosed. In this case, $\mathcal{R}(B-\lambda_0 I)$ is nonclosed, then there exists a closed subspace $\Omega_2 \subsetneqq \overline{\mathcal{R}(B^* - \overline{\lambda_0} I)}$ as in (3.9) with $\mathcal{R}(B^* - \overline{\lambda_0} I) \cap \Omega_2 = \{0\}$ and $\dim \Omega_2 = \infty$. Note that $\mathcal{R}(A-\lambda_0 I) \neq \{0\}$ and $\mathcal{N}(B-\lambda_0 I)$ is a nontrivial subspace. Without loss of generality, suppose that $\mathcal{R}(A-\lambda_0 I)^\perp \neq \{0\}$. Thus, there exists an isometry U_2 from $\mathcal{R}(A-\lambda_0 I)^\perp$ to $\Omega_2 \subsetneqq \overline{\mathcal{R}(B^* - \overline{\lambda_0} I)}$. Also, since $\mathcal{R}(A-\lambda_0 I)$ is nonclosed, we may have the isometry C_1 as in (3.6). Define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} C_1 & 0 \\ 0 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}, \tag{3.11}$$

where $C_4 = U_2^*$. Clearly, $M_{C_0} - \lambda I$ is injective. Note that

$$C_0^* = \begin{pmatrix} C_1^* & 0 \\ 0 & U_0 \end{pmatrix} : \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix}$$

and $\lambda_0 \notin \sigma_p(A) \cup \overline{\sigma_p(B^*)}$. It is readily seen that $M_{C_0}^* - \overline{\lambda_0} I$ is injective, i.e., $\overline{\mathcal{R}(M_{C_0} - \lambda_0 I)} = H_1 \oplus H_2$. Therefore, $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 2: $\lambda_0 \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$, but $\mathcal{R}(A-\lambda_0 I)$ is nonclosed and $n(B-\lambda_0 I) \geq d(A-\lambda_0 I)$. In this case, we know that $\mathcal{R}(A-\lambda_0 I) \neq \{0\}$, and $\mathcal{N}(B-\lambda_0 I)$ is a nontrivial subspace in H_2. Without loss of generality, we may assume that $\mathcal{R}(A-\lambda_0 I)^\perp \neq \{0\}$. Since $\mathcal{R}(A-\lambda_0 I)$ is nonclosed, we have the operator C_1 as in *Case 1*. By $n(B-\lambda_0 I) \geq d(A-\lambda_0 I)$, there exists a bounded operator $C_3 : \mathcal{N}(B-\lambda_0 I) \to \mathcal{R}(A-\lambda_0 I)^\perp$ with densely range. Define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}.$$

Then, $M_{C_0} - \lambda_0 I$ is injective and $\overline{\mathcal{R}(M_{C_0} - \lambda_0 I)} = H_1 \oplus H_2$, i.e., $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 3: $\lambda_0 \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$, but $\lambda_0 \notin \sigma_{p,13}(B)$ and $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$. In this case, we see that $\mathcal{N}(B-\lambda_0 I)$ and $\mathcal{R}(A-\lambda_0 I)^\perp$ are nontrivial subspaces in H_2 and H_1, respectively. Since $\lambda_0 \notin \sigma_{p,13}(B)$ implies that $\mathcal{R}(B-\lambda_0 I)$ is nonclosed, we have the operator C_4 as in *Case 1*. By $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$, there exists an isometry $C_3 : \mathcal{N}(B-\lambda_0 I) \to \mathcal{R}(A-\lambda_0 I)^\perp$. Define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} 0 & 0 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}. \tag{3.12}$$

Then, $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 4: $\lambda_0 \in \sigma_p(B) \setminus (\sigma_p(A) \cup \overline{\sigma_p(B^*)})$, but $n(B-\lambda_0 I) = d(A-\lambda_0 I)$. In this case, we still have that $\mathcal{N}(B-\lambda_0 I)$ and $\mathcal{R}(A-\lambda_0 I)^\perp$ are nontrivial subspaces in H_2 and H_1, respectively. By $n(B-\lambda_0 I) = d(A-\lambda_0 I)$, there exists a unitary operator $C_3 : \mathcal{N}(B-\lambda_0 I) \to \mathcal{R}(A-\lambda_0 I)^\perp$. Define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} 0 & 0 \\ C_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}.$$

Then, $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 5: $\lambda_0 \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$, but $\lambda_0 \notin \sigma_{r,1}(A)$, $\lambda_0 \notin \sigma_{p,1}(B)$ and $\lambda_0 \notin \rho(B)$, i.e., $\lambda_0 \in \sigma_{r,2}(A)$ and $\lambda_0 \in \sigma_{p,2}(B) \cup \sigma_c(B)$. In this case, we see that $\lambda_0 \notin \sigma_p(A)$, and both $\mathcal{R}(A-\lambda_0 I)$ and $\mathcal{R}(B-\lambda_0 I)$ are nonclosed. Define the operator C_0 as in *Case 1*, then $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 6: $\lambda_0 \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$, but $\lambda_0 \in \sigma_{r,2}(A)$, $\lambda_0 \notin \rho(B)$ and $n(B-\lambda_0 I) \geq d(A-\lambda_0 I)$, i.e., $\lambda_0 \in \sigma_{r,2}(A)$, $\lambda_0 \in \sigma_{p,12}(B)$ and $n(B-\lambda_0 I) \geq d(A-\lambda_0 I) > 0$. In this case, if $\lambda_0 \in \sigma_{p,2}(B)$, then $\lambda_0 \notin \sigma_p(A)$, and both $\mathcal{R}(A-\lambda_0 I)$ and $\mathcal{R}(B-\lambda_0 I)$ are nonclosed. Define the operator C_0 as in *Case 1*, then $\lambda_0 \notin \sigma_{pr}(M_{C_0})$. However, if $\lambda_0 \in \sigma_{p,1}(B)$, then $\lambda_0 \notin \sigma_p(A)$, $n(B-\lambda_0 I) \geq d(A-\lambda_0 I)$ and $\mathcal{R}(A-\lambda_0 I)$ is nonclosed. Define the operator C_0 as in *Case 2*, then $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 7: $\lambda_0 \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$, but $\lambda_0 \notin \sigma_{p,12}(B)$, $\lambda_0 \notin \sigma_{p,1}(B)$ and $\lambda_0 \notin \rho(B)$, i.e., $\lambda_0 \in \sigma_r(A) \cap \sigma_c(B)$. In this case, we see that $\lambda_0 \notin \sigma_p(A)$, $\mathcal{N}(B-\lambda_0 I) = \{0\}$ and $\mathcal{R}(B-\lambda_0 I)$ is nonclosed. Clearly, $\mathcal{R}(A-\lambda_0 I)^\perp$ is a nontrivial subspace in H_1. Since $\mathcal{R}(B-\lambda_0 I)$ is nonclosed, we may define the operator C_4 as in Case 1. Define

$$C_0 = \begin{pmatrix} 0 \\ C_4 \end{pmatrix} : \mathcal{N}(B-\lambda_0 I)^\perp = H_2 \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}.$$

Then, $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 8: $\lambda_0 \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$, but $\lambda_0 \notin \sigma_{p,1}(B)$, $\lambda_0 \notin \rho(B)$ and $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$, i.e., $\lambda_0 \in \sigma_r(A)$, $\lambda_0 \in \sigma_{p,2}(B) \cup \sigma_c(B)$ and $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$. In this case, we see that $\lambda_0 \notin \sigma_p(A)$, $\mathcal{R}(B-\lambda_0 I)$ is nonclosed, and $\mathcal{R}(A-\lambda_0 I)^\perp$ is clearly a nontrivial subspace in H_1. If $\lambda_0 \in \sigma_c(B)$, then $\mathcal{N}(B-\lambda_0 I) = \{0\}$. Define the operator C_0 as in *Case 7*, then $\lambda_0 \notin \sigma_{pr}(M_{C_0})$. If $\lambda_0 \in \sigma_{p,2}(B)$, then $\mathcal{N}(B-\lambda_0 I)$ is a nontrivial subspace in H_2. By $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$, defining the operator C_0 as in *Case 3*, we have $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Case 9: $\lambda_0 \in \sigma_r(A) \setminus \overline{\sigma_p(B^*)}$, but $\lambda_0 \notin \rho(B)$ and $n(B-\lambda_0 I) = d(A-\lambda_0 I)$, i.e., $\lambda_0 \in \sigma_r(A) \cap \sigma_{p,12}(B)$ and $n(B-\lambda_0 I) = d(A-\lambda_0 I)$. In this case, we know that $\mathcal{N}(B-\lambda_0 I)$ and $\mathcal{R}(A-\lambda_0 I)^\perp$ are nontrivial subspaces in H_2 and H_1, respectively. Define the operator C_0 as in *Case 4*, then $\lambda_0 \notin \sigma_{pr}(M_{C_0})$.

Note that $\lambda_0 \in \sigma_{pr}(A) \cup \sigma_{pr}(B)$ in all above cases. Therefore, we always have the contradiction with the assumption $\sigma_{pr}(M_C) = \sigma_{pr}(A) \cup \sigma_{pr}(B)$ for every $C \in \mathcal{B}(H_2, H_1)$. This proves necessity. □

Corollary 3.7. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. If A is cohypernormal and B is hypernormal, then*

$$\sigma_{pr}(M_C) = \sigma_{pr}(A) \cup \sigma_{pr}(B) \quad \text{for every } C \in \mathcal{B}(H_2, H_1).$$

Proof. The proof is similar to that of the previous corollaries, and is omitted. □

Proof of Theorem 2.6. Sufficiency. Let $\lambda \in \sigma_c(M_C)$ for some $C \in \mathcal{B}(H_2, H_1)$. Obviously, $\lambda \notin \sigma_p(A) \cup \sigma_r(B)$. We further claim that $\lambda \notin \sigma_r(A) \cap \sigma_p(B)$, otherwise $\lambda \notin \sigma_c(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$ follows from the assertion (i), giving a contradiction. So, $\lambda \in \rho(A) \cup \sigma_c(A) \cup \rho(B) \cup \sigma_c(B) \cup (\sigma_r(A) \setminus \sigma_p(B)) \cup (\sigma_p(B) \setminus \sigma_r(A))$. Since $\lambda \in \sigma_c(M_C)$ and $\lambda \notin \sigma_p(A) \cup \sigma_r(B)$, it follows that if $\lambda \in \rho(A) \cup (\sigma_r(A) \setminus \sigma_p(B))$, we have $\lambda \in \sigma_c(B)$, and if $\lambda \in \rho(B) \cup (\sigma_p(B) \setminus \sigma_r(A))$, we obtain $\lambda \in \sigma_c(A)$. Thus, $\sigma_c(M_C) \subseteq \sigma_c(A) \cup \sigma_c(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Now, we prove the opposite inclusion. If $\lambda \in \sigma_c(A)$, then by the assertion (i) we have $\lambda \in \sigma_c(B) \cup \rho(B)$. When $\lambda \in \rho(B)$, the factorization (3.1) implies that $\lambda \in \sigma_c(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$; when $\lambda \in \sigma_c(B)$, we clearly have that $\lambda \in \sigma_c(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Thus, $\sigma_c(A) \subseteq \sigma_c(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. If $\lambda \in \sigma_c(B) \setminus \sigma_c(A)$, then by the assertion (ii) we have $\lambda \notin \sigma_{pr}(A)$. This immediately implies that $(\sigma_c(B) \setminus \sigma_c(A)) \subseteq \sigma_c(M_C)$ for every $C \in \mathcal{B}(H_2, H_1)$. Therefore, $\sigma_c(M_C) \supseteq \sigma_c(A) \cup \sigma_c(B)$ for every $C \in \mathcal{B}(H_2, H_1)$.

Necessity. Assume to the contrary that there exists $\lambda_0 \in \mathbb{C}$, such that one of the assertions (i), (ii) and (iii) fails to hold.

If the assertion (i) is invalid, then there are four possible cases.

Case 1: $\lambda_0 \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B)$, and $n(B-\lambda_0 I) = d(A-\lambda_0 I) = \infty$. In this case, we see that $\lambda_0 \notin \sigma_p(A)$, $\mathcal{R}(A-\lambda_0 I)$ is closed, and $\mathcal{R}(B-\lambda_0 I) = H_2$. Since $n(B-\lambda_0 I) = d(A-\lambda_0 I) = \infty$, there exists a bounded operator $C_3 : \mathcal{N}(B-\lambda_0 I) \to \mathcal{R}(A-\lambda_0 I)^\perp$, such that C_3 is injective, $\overline{\mathcal{R}(C_3)} = \mathcal{R}(A-\lambda_0 I)^\perp$ and $\mathcal{R}(C_3) \neq \mathcal{R}(A-\lambda_0 I)^\perp$. Define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} 0 & 0 \\ C_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A-\lambda_0 I) \\ \mathcal{R}(A-\lambda_0 I)^\perp \end{pmatrix}.$$

Then, $M_{C_0} - \lambda_0 I$ admits the following block representation

$$M_{C_0} - \lambda_0 I = \begin{pmatrix} (A-\lambda_0 I)_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & (B-\lambda_0 I)_1 \end{pmatrix} : \begin{pmatrix} H_1 \\ \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A-\lambda_0 I) \\ \mathcal{R}(A-\lambda_0 I)^{\perp} \\ H_2 \end{pmatrix}.$$

Thus, we immediately have $\lambda_0 \in \sigma_c(M_{C_0})$.

Case 2: $\lambda_0 \in \sigma_{r,1}(A) \cap \sigma_{p,2}(B)$, and $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$. In this case, we see that $\lambda_0 \notin \sigma_p(A)$, $\overline{\mathcal{R}(B-\lambda_0 I)} = H_2$, and $\mathcal{R}(B-\lambda_0 I)$ is nonclosed. Since $n(B-\lambda_0 I) \leq d(A-\lambda_0 I)$, we may define the operator C_0 as in (3.12). Then, we know that $\lambda_0 \notin \sigma_{pr}(M_{C_0})$. Clearly, $\mathcal{R}(M_{C_0} - \lambda_0 I)$ is nonclosed in $H_1 \oplus H_2$. Thus, $\lambda_0 \in \sigma_c(M_{C_0})$.

Case 3: $\lambda_0 \in \sigma_{r,2}(A) \cap \sigma_{p,2}(B)$. In this case, we see that $\overline{\mathcal{R}(B-\lambda_0 I)} = H_2$, and $\mathcal{R}(A-\lambda_0 I)$ and $\mathcal{R}(B-\lambda_0 I)$ are nonclosed. Choose the operator C_0 defined as in (3.11), then we know that $\lambda_0 \notin \sigma_{pr}(M_{C_0})$. Clearly, $\mathcal{R}(M_{C_0} - \lambda_0 I)$ is nonclosed in $H_1 \oplus H_2$. Thus, $\lambda_0 \in \sigma_c(M_{C_0})$.

Case 4: $\lambda_0 \in \sigma_{r,2}(A) \cap \sigma_{p,1}(B)$, and $n(B-\lambda_0 I) = d(A-\lambda_0 I)$. In this case, we see that $\mathcal{R}(A-\lambda_0 I)$ is nonclosed, and $\mathcal{R}(B-\lambda_0 I) = H_2$. Then, there exists a closed subspace $\Omega_1 \subsetneq \overline{\mathcal{R}(A-\lambda_0 I)}$ as in (3.5) with $\mathcal{R}(A-\lambda_0 I) \cap \Omega_1 = \{0\}$ and $\dim \Omega_1 = \infty$. Let $V \subsetneq \Omega_1$ be a closed subspace with $\dim V = \infty$. Thus, there exists an isometry $C_1 : \mathcal{N}(B-\lambda_0 I) \to V$, since $n(B-\lambda_0 I) \leq \dim V$. By $n(B-\lambda_0 I) = d(A-\lambda_0 I)$, there exists a unitary operator $C_3 : \mathcal{N}(B-\lambda_0 I) \to \mathcal{R}(A-\lambda_0 I)^{\perp}$. Define an operator C_0 from H_2 to H_1 by

$$C_0 = \begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0 I) \\ \mathcal{N}(B-\lambda_0 I)^{\perp} \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A-\lambda_0 I)} \\ \mathcal{R}(A-\lambda_0 I)^{\perp} \end{pmatrix}.$$

Then, it can be readily seen that $\lambda_0 \in \sigma_c(M_{C_0})$.

Note that $\lambda_0 \in \sigma_c(M_{C_0})$ in all previous cases. Therefore, we always have the contradiction with the assumption $\sigma_c(M_C) = \sigma_c(A) \cup \sigma_c(B)$ for every $C \in \mathcal{B}(H_2, H_1)$, since $\lambda_0 \in \sigma_r(A) \cap \sigma_p(B)$.

If the assertion (ii) is not true, i.e., $\lambda_0 \in \sigma_c(A)$, but $\lambda_0 \in \sigma_{pr}(B)$. Then, it follows that $\lambda_0 \in \sigma_{pr}(M_0)$, and hence $\lambda_0 \notin \sigma_c(M_0)$. However, $\lambda_0 \in \sigma_c(A) \cup \sigma_c(B)$, which contradict the assumption $\sigma_c(M_C) = \sigma_c(A) \cup \sigma_c(B)$ for every $C \in \mathcal{B}(H_2, H_1)$. Similarly, we obtain the same contradiction, if the assertion (iii) is invalid. Therefore, the assertions (i), (ii) and (iii) are all fulfilled. □

Corollary 3.8. *Let $A \in \mathcal{B}(H_1)$ and $B \in \mathcal{B}(H_2)$. Then,*

$$\sigma_c(M_C) = \sigma_c(A) \cup \sigma_c(B) \quad \textit{for every } C \in \mathcal{B}(H_2, H_1)$$

if one of the following assumptions is fulfilled:

(i) A and B are hypernormal, and $\sigma_{p,34}(A) = \sigma_{p,34}(B) = \sigma_r(A) = \sigma_r(B) = \emptyset$;

(ii) A and B are cohypernormal, and $\sigma_p(A) = \sigma_p(B) = \emptyset$;

(iii) A and B are normal, and $\sigma_{p,34}(A) = \sigma_{p,34}(B) = \emptyset$.

Proof. The proof is trivial, and is omitted. □

4. Examples

In this section, some examples illustrating results of the previous sections are presented. To streamline the calculations, we work in the infinite-dimensional Hilbert space $\ell^2(1, +\infty)$ or simply ℓ^2, which consists of square summable complex-valued sequences.

Example 4.1. *Let $H_1 = H_2 = \ell^2$, and use the right shift operator as A and left shift operator as B, i.e., for $(x_1, x_2, \cdots) \in H_1 = H_2$,*

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, x_3, \cdots), \quad B(x_1, x_2, \cdots) = (x_2, x_3, \cdots).$$

Then, A and B are bounded operators on H_1 and H_2, respectively. We claim that $0 \in \sigma_{r,1}(A) \backslash \sigma_\delta(B)$ but $n(B) = d(A) \neq 0$; meanwhile $0 \in \sigma_{p,1}(B) \setminus \sigma_{ap}(A)$ but $n(B) = d(A) \neq 0$, and hence (1.1) *is invalid.*

It can be readily seen that $0 \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B)$, and $n(B) = d(A) = 1$. Then, we deduce that $0 \in \sigma_{r,1}(A) \setminus \sigma_{\delta}(B)$, $0 \in \sigma_{p,1}(B) \setminus \sigma_{ap}(A)$, but $n(B) = d(A) \neq 0$. By Theorem 2.1, we immediately know that (1.1) is not true. Indeed, define the operator C_0 by

$$C_0(x_1, x_2, \cdots) = (x_1, 0, 0, \cdots)$$

for $(x_1, x_2, \cdots) \in H_2$, then $0 \in \rho(M_{C_0})$, but $0 \in \sigma(A) \cup \sigma(B)$.

Example 4.2. *Let $H_1 = H_2 = \ell^2$. Consider the operators A and B defined by*

$$A(x_1, x_2, \cdots) = (x_1, x_2, x_1 + x_2, x_3, x_4, x_3 - x_4, x_5, \cdots), \quad B(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots),$$

for $(x_1, x_2, \cdots) \in H_1 = H_2$. Then, A and B are bounded operators on H_1 and H_2, respectively. We claim that $0 \in \sigma_{r,1}(A)$, but $n(B) < d(A)$, and hence (2.1) *is invalid.*

Direct calculations show that $\mathcal{R}(A)$ is closed, and for $(x_1, x_2, \cdots) \in H_1$,

$$A^*(x_1, x_2, \cdots) = (x_1 + x_3, x_2 + x_3, x_4 + x_6, x_5 - x_6, x_7, \cdots).$$

Since $0 \in \sigma_p(A^*)$, with an eigenspace spanned by $(1, 1, -1, 0, 0, \cdots)$ and $(0, 0, 0, 1, -1, -1, 0, 0, \cdots)$, we have $d(A) = \dim \mathcal{N}(A^*) = 2$. Again, it can be seen that $0 \in \sigma_p(B)$ with an eigenspace spanned by $(1, 0, 0, \cdots)$, and hence $n(B) = 1$. Thus, $0 \in \sigma_{r,1}(A)$ and $n(B) < d(A)$. By Theorem 2.3, we immediately know that (2.1) is invalid. Indeed, define the operator C_0 by

$$C_0(x_1, x_2, \cdots) = (x_1, 0, 0, ...)$$

for $(x_1, x_2, \cdots) \in H_2$. Then, we actually have $0 \notin \sigma_p(M_{C_0})$, but $0 \in \sigma_p(B) \subseteq \sigma_p(A) \cup \sigma_p(B)$.

Analogously, we may construct various types of examples (e.g., $0 \in \sigma_{r,1}(A)$, but $n(B) = d(A)$), such that neither of assertions (i) and (ii) holds. There are similar remarks for other examples below.

Example 4.3. *As in Example* 4.1*, let $H_1 = H_2 = \ell^2$, and use the right shift operator as A and the identity operator as B. Then, A and B are bounded operators on H_1 and H_2, respectively. We claim that $1 \in \sigma_c(A) \cap \sigma_p(B)$, but $1 \notin \sigma_{p,12}(B)$, and hence* (2.2) *is invalid.*

The claim $1 \in \sigma_c(A) \cap \sigma_p(B)$ is obvious. Since $B - I$ is the zero operator on H_2, i.e., $\mathcal{N}(B - I) = H_2$, it follows that $1 \notin \sigma_{p,12}(B)$. By Theorem 2.4, the assertion (2.2) is not true. Indeed, taking $C_0 = I$ yields $1 \in \sigma_r(M_{C_0})$, even though $(\sigma_c(A) \cap \sigma_p(B) \ni) 1 \notin \sigma_r(A) \cup \sigma_r(B)$.

Remark 4.4. *This example also shows that the inclusion $\sigma_r(M_C) \subseteq \sigma_r(A) \cup \sigma_r(B)$ for every $C \in \mathcal{B}(H_2, H_1)$ is not necessarily valid.*

Example 4.5. *As in Example* 4.1*, let $H_1 = H_2 = \ell^2$, and use the right shift operator as A. For $(x_1, x_2, \cdots) \in H_2$, the operator B is defined by*

$$B(x_1, x_2, \cdots) = (x_2, x_2 + x_3, x_3 + x_4, x_5, x_6, \cdots).$$

Then, A and B are bounded operators on H_1 and H_2, respectively. We claim that $0 \in (\sigma_p(B) \cap \sigma_r(A)) \setminus \sigma_p(B^*)$, but $n(B) = d(A)$, and hence (2.3) *is invalid.*

A straightforward calculation shows that $0 \in \sigma_p(B) \cap \sigma_r(A)$ and $n(B) = d(A) = 1$. Note that

$$B^*(x_1, x_2, \cdots) = (0, x_1 + x_2, x_2 + x_3, x_3, x_4, \cdots),$$

for $(x_1, x_2, \cdots) \in H_2$. Then, we have $0 \notin \sigma_p(B^*)$. By Theorem 2.5, it follows that (2.3) is invalid. Indeed, define the operator C_0 by

$$C_0(x_1, x_2, \cdots) = (x_1, 0, 0, \cdots)$$

for $(x_1, x_2, \cdots) \in H_2$, then $0 \in \rho(M_{C_0})$, but clearly $0 \in \sigma_{pr}(A) \cup \sigma_{pr}(B)$.

Example 4.6. *Let $H_1 = H_2 = \ell^2$. Consider the operators A and B defined by*

$$A(x_1, x_2, \cdots) = (x_1, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \cdots), \quad Bx = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

for $(x_1, x_2, \cdots) \in H_1 = H_2$. Then, A and B are bounded operators on H_1 and H_2, respectively. We claim that $0 \in \sigma_c(A)$, but $0 \in \sigma_r(B)$, and hence (2.4) *is invalid.*

Direct calculations show that $\overline{\mathcal{R}(A)} = H_1$ and $\mathcal{R}(A)$ is nonclosed. It is clear that A and B are injective, and $\overline{\mathcal{R}(B)} \neq H_2$. These prove $0 \in \sigma_c(A)$ and $0 \in \sigma_r(B)$. By Theorem 2.6, it follows that (2.4) is invalid. Indeed, define the operator C_0 by

$$C_0(x_1, x_2, \cdots) = (x_1, x_2, x_3, 0, 0, \cdots),$$

for $(x_1, x_2, \cdots) \in H_2$. Since A and B are injective, M_{C_0} is injective, which together with $\overline{\mathcal{R}(B)} \neq H_2$ implies $0 \in \sigma_r(M_{C_0})$. However, we clearly have $0 \in \sigma_c(A) \cup \sigma_c(B)$.

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