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Characterizations for Certain Subclasses of Starlike and Convex Functions Associated with Bessel Functions

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Abstract. In the present paper, we obtain some characterizations for a certain generalized Bessel function of the first kind to be in the subclasses $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\alpha,\beta)$, $\mathcal{UCT}(\alpha,\beta)$, $\mathcal{PT}(\alpha)$ and $\mathcal{CPT}(\alpha)$ of normalized analytic functions in the open unit disk $\mathbb{U}$. Furthermore, we consider an integral operator related to the generalized Bessel Function which we have characterized here.

1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0). \tag{1.2}$$

Let $\mathcal{T}^*(\alpha)$ and $C(\alpha)$ denote the subclasses of $\mathcal{T}$ consisting of starlike and convex functions of order α $(0 \leqq \alpha < 1)$ (see [16]), respectively. In 1997, Bharati *et al.* [5] introduced the following subclasses of starlike and convex functions.

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Definition 1.1. A function f of the form (1.1) is said to be in the class $\mathcal{S_P T}(\alpha,\beta)$ if it satisfies the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geqq \alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \qquad (\alpha \geqq 0;\ 0 \leqq \beta < 1)$$

and $f \in \mathcal{UCV}(\alpha,\beta)$ if and only if $zf' \in \mathcal{S_P}(\alpha,\beta)$.

Definition 1.2. A function f of the form (1.1) is said to be in the class $\mathcal{P}(\alpha)$ if it satisfies the condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)+\alpha \geqq \left|\frac{zf'(z)}{f(z)}-\alpha\right| \qquad (0<\alpha<\infty)$$

and $f \in \mathcal{CP}(\alpha)$ if and only if $zf' \in \mathcal{P}(\alpha)$. We write

$$\mathcal{PT}(\alpha)=\mathcal{P}(\alpha)\cap\mathcal{T} \qquad \textit{and} \qquad \mathcal{CPT}(\alpha)=\mathcal{CP}(\alpha)\cap\mathcal{T}.$$

Bharati *el al.* [5] showed that

$$\mathcal{S_P T}(\alpha,\beta)=\mathcal{T}^*\left(\frac{\alpha+\beta}{1+\alpha}\right),$$

$$\mathcal{UCT}(\alpha,\beta)=C\left(\frac{\alpha+\beta}{1+\alpha}\right),$$

$$\mathcal{PT}(\alpha)=\mathcal{T}^*(1-\alpha) \qquad \left(\frac{1}{2}<\alpha<1\right)$$

and

$$\mathcal{CPT}(\alpha)=C(1-\alpha) \qquad \left(\frac{1}{2}<\alpha<1\right).$$

In particular, we note that $\mathcal{UCV}(1,0)$ is the class of uniformly convex functions given by Goodman [10]. For more interesting developments of some related subclasses of $\mathcal{UCV}(\alpha,\beta)$, the readers may be referred to the works of Goodman [11], Ma and Minda [12] and Rønning (see [14] and [15]).

Recently, Baricz [1] defined a generalized Bessel function $\omega_{p,b,c} \equiv \omega$ as follows:

$$\omega(z)=\omega_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\,c^n}{n!\,\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p}, \tag{1.3}$$

which is the particular solution of the following second-order linear homogeneous differential equation:

$$z^2\omega''(z)+bz\omega'(z)+[cz^2-p^2+(1-b)]\omega(z)=0 \qquad (b,p,c\in\mathbb{C}), \tag{1.4}$$

which, in turn, is a natural generalization of the classical Bessel's equation.

Solutions of (1.4) are regarded as the generalized Bessel function of order p. The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order p. We also note that the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$, although the series defined above is convergent everywhere.

Now, we consider the function $u_{p,b,c}(z)$ defined by the following transformation:

$$u_{p,b,c}(z)=2^p\Gamma\left(p+\frac{b+1}{2}\right)z^{-\frac{p}{2}}\,\omega_{p,b,c}(\sqrt{z}) \qquad (\sqrt{1}:=1).$$

By using the well-known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0) \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1,2,3,\ldots\}), \end{cases}$$

we can express $u_{p,b,c}(z) \equiv u$ as follows:

$$u_p(z) = u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p+\frac{b+1}{2}\right)_n} \frac{z^n}{n!} \tag{1.5}$$

$$\left(p + \frac{b+1}{2} \notin \mathbb{N}^- \cup \{0\};\ \mathbb{N}^- = \{-1,-2,\cdots\}\right)$$

Then the function $u_{p,b,c}$ is analytic on $\mathbb{C}$ and satisfies the following second-order linear differential equation:

$$4z^2u''(z) + 2(2p+b+1)zu'(z) + cu(z) = 0.$$

The study of the generalized Bessel function is a recent interesting topic in geometric function theory. We refer, in this connection, to the works of Baricz (see [1], [2], [3] and [4]) and Cho *et al.* [6] and Mondal and Swaminathan [13] and Deniz *et al.* ([8] and [9]) and others (see [7] and [18]). The corresponding developments involving the Struve functions can be found in the recent investigation by Srivastava *et al.* [17].

In the present paper, we determine sufficient conditions for zu_p to be in $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\alpha,\beta)$ and $\mathcal{UCV}(\alpha,\beta)$ and also give necessary and sufficient conditions for $z(2-u_p)$ to be in the function classes $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\alpha,\beta)$, $\mathcal{UCT}(\alpha,\beta)$, $\mathcal{PT}(\alpha)$ and $\mathcal{CPT}(\alpha)$. Futhermore, we consider an integral operator related to the function u_p. Throughout this paper, we will use the following notation for convenience in (1.5):

$$m = p + \frac{b+1}{2}.$$

2. Main Results

To establish our main results, we need the following Lemmas due to Bharati *et al.* [5].

Lemma 2.1. (see [5]) (i) *A sufficient condition for a function f of the form (1.1) to be in the class $\mathcal{S}_{\mathcal{P}}(\alpha,\beta)$ is that*

$$\sum_{n=2}^{\infty}(n(1+\alpha)-(\alpha+\beta))|a_n| \leqq 1-\beta \qquad (\alpha \geqq 0;\ 0 \leqq \beta < 1) \tag{2.1}$$

and a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\alpha,\beta)$ is that the condition (2.1) is satisfied.

(ii) *A sufficient condition for a function f of the form (1.1) to be in the class $\mathcal{UCV}(\alpha,\beta)$ is that*

$$\sum_{n=2}^{\infty} n(n(1+\alpha)-(\alpha+\beta))|a_n| \leqq 1-\beta \qquad (\alpha \geqq 0;\ 0 \leqq \beta < 1) \tag{2.2}$$

and a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{UCT}(\alpha,\beta)$ is that the condition (2.2) is satisfied.

Lemma 2.2. (see [5]) (i) *A a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{PT}(\alpha)$ is that*

$$\sum_{n=2}^{\infty}(n-1-\alpha)a_n \leqq \alpha \qquad \left(\frac{1}{2} < \alpha \leqq 1\right). \tag{2.3}$$

(ii) *A necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{CPT}(\alpha)$ is that*

$$\sum_{n=2}^{\infty} n(n-1-\alpha)a_n \leqq \alpha \qquad \left(\frac{1}{2} < \alpha \leqq 1\right). \tag{2.4}$$

Lemma 2.3. (see [4]) *If*

$$b, p, c \in \mathbb{C} \qquad and \qquad m \notin \mathbb{N}^{-} \cup \{0\},$$

then the function u_p defined by (1.5) satisfies the following recursive relation:

$$4mu_p'(z) = -cu_{p+1}(z) \qquad (z \in \mathbb{C}). \tag{2.5}$$

Theorem 2.1. *If $c < 0$ and $m > 0$, then $zu_p \in \mathcal{S}_{\mathcal{P}}(\alpha, \beta)$ if*

$$(1+\alpha)u_p'(1) + (1-\beta)[u_p(1)-1] \leqq 1-\beta \qquad (\alpha \geqq 0;\ 0 \leqq \beta < 1). \tag{2.6}$$

Proof. Since

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} z^n,$$

by virtue of (i) in Lemma 2.1, it suffices to show that

$$\mathcal{L}(c, m, \alpha, \beta) := \sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)] \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} \leqq 1-\beta.$$

by simple computation, we get

$$\begin{aligned}
\mathcal{L}(c, m, \alpha, \beta) &= \sum_{n=2}^{\infty} [(n-1)(1+\alpha) + (1-\beta)] \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} \\
&= (1+\alpha) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-2)!} + (1-\beta) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} \\
&= (1+\alpha) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\,n!} + (1-\beta) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\,(n+1)!} \\
&= (1+\alpha) \frac{(-c/4)}{m} \sum_{n=0}^{\infty} \frac{(-c/4)^{n}}{(m+1)_{n}\,n!} + (1-\beta) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\,(n+1)!} \\
&= (1+\alpha) \frac{(-c/4)}{m} u_{p+1}(1) + (1-\beta)[u_p(1)-1] \\
&= (1+\alpha)u_p'(1) + (1-\beta)[u_p(1)-1].
\end{aligned} \tag{2.7}$$

Therefore, we see that the last expression (2.7) is bounded above by $1-\beta$ if (2.6) is satisfied. □

Corollary 2.1. *If $c < 0$ and $m > 0$, then*

$$z(2-u_p(z)) \in \mathcal{S}_{\mathcal{P}}\mathcal{T}(\alpha, \beta)$$

if and only if the condition (2.6) *is satisfied.*

Proof. Since

$$z(2-u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} z^n,$$

by using the same techniques given in the proof of Theorem 2.1, we have immediately Corollary 2.1. □

Theorem 2.2. *If* $c < 0$ *and* $m > 0$, *then*

$$zu_p \in \mathcal{UCV}(\alpha, \beta)$$

if

$$(1+\alpha)u_p''(1) + (3+2\alpha-\beta)u_p'(1) + (1-\beta)[u_p(1)-1] \leqq 1-\beta \quad (\alpha \geqq 0;\ 0 \leqq \beta < 1). \tag{2.8}$$

Proof. Since

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} z^n,$$

by virtue of (ii) in Lemma 2.1, it suffices to show that

$$\mathcal{P}(c,m,\alpha,\beta) := \sum_{n=2}^{\infty} n[n(1+\alpha)-(\alpha+\beta)] \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} \leqq 1-\beta.$$

Writing

$$n^2 = (n-1)(n-2) + 3(n-1) + 1 \qquad \text{and} \qquad n = (n-1)+1,$$

we can rewrite the above terms as follows:

$$\begin{aligned}
&\mathcal{P}(c,m,\alpha,\beta)\\
&\quad = (1+\alpha)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}\\
&\qquad\qquad + (3+2\alpha-\beta)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} + (1-\beta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}\\
&\quad = (1+\alpha)\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-3)!}\\
&\qquad\qquad + (3+2\alpha-\beta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-2)!} + (1-\beta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}\\
&\quad = (1+\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\,(n-2)!}\\
&\qquad\qquad + (3+2\alpha-\beta)\sum_{n=1}^{\infty}\frac{(-c/4)^{n}}{(m)_{n}\,(n-1)!} + (1-\beta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\,(n+1)!}\\
&\quad = (1+\alpha)\frac{(-c/4)^2}{m(m+1)}\sum_{n=1}^{\infty}\frac{(-c/4)^{n-1}}{(m+2)_{n-1}\,(n-1)!}\\
&\qquad\qquad + (3+2\alpha-\beta)\frac{(-c/4)}{m}\sum_{n=1}^{\infty}\frac{(-c/4)^{n-1}}{(m+1)_{n-1}\,(n-1)!} + (1-\beta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\,(n+1)!}\\
&\quad = (1+\alpha)\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1) + (3+2\alpha-\beta)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\beta)[u_p(1)-1]\\
&\quad = (1+\alpha)u_p''(1) + (3+2\alpha-\beta)u_p'(1) + (1-\beta)[u_p(1)-1].
\end{aligned}$$

Therefore, we see that the last expression is bounded above by $1-\beta$ if (2.8) is satisfied. □

By using a similar method as in the proof of Corollary 2.1, we have the following result.

Corollary 2.2. *If $c<0$ and $m>0$, then*

$$z(2-u_p) \in \mathcal{UCT}(\alpha, \beta)$$

if and only if the condition (2.8) is satisfied.

The proofs of Theorem 2.3 and Theorem 2.4 are much akin to that of Theorem 2.1 or Theorem 2.2, and so the details may be omitted.

Theorem 2.3. *If $c<0$ and $m>0$, then*

$$z(2-u_p) \in \mathcal{PT}(\alpha)$$

if and only if

$$u_p'(1) + \alpha u_p(1) \leqq 2\alpha \qquad \left(\frac{1}{2} < \alpha \leqq 1\right). \tag{2.9}$$

Theorem 2.4. *If $c<0$ and $m>0$, then $z(2-u_p) \in \mathcal{CPT}(\alpha)$ if and only if*

$$u_p''(1) + (2+\alpha)u_p'(1) + \alpha u_p(1) \leqq 2\alpha \qquad \left(\frac{1}{2} < \alpha \leqq 1\right). \tag{2.10}$$

In the next theorems stated below, we obtain results of similar types in connection with a particular integral operator $\mathcal{I}(c, m; z)$ given by

$$\mathcal{I}(c, m; z) = \int_0^z [2 - u_p(t)]dt \tag{2.11}$$

Theorem 2.5. *If $c<0$ and $m>0$, then $\mathcal{I}(c, m; z) \in \mathcal{UCT}(\alpha, \beta)$ if and only if the the condition (2.6) is satisfied.*

Proof. Since

$$\mathcal{I}(c, m; z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\, n!} z^n,$$

by virtue of (ii) in Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta)) \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} \leqq 1-\beta. \tag{2.12}$$

The remaining part of the proof of Theorem 2.5 is similar to that of Theorem 2.1, and so we omit the details. □

Similarly, by virtue of (ii) in Lemma 2.2 and Theorem 2.3, we have the following theorem.

Theorem 2.6. *If $c<0$ and $m>0$, then $\mathcal{I}(c, m; z) \in \mathcal{CPT}(\alpha)$ if and only if the the condition (2.9) is satisfied.*

3. Concluding Remarks and Observations

In our present investigation, we have successfully derived several characterizations for a generalized Bessel function of the first kind, which is defined by (1.5), to be in the subclasses $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\alpha, \beta)$, $\mathcal{UCT}(\alpha, \beta)$, $\mathcal{PT}(\alpha)$ and $\mathcal{CPT}(\alpha)$ of normalized analytic functions in the open unit disk $\mathbb{U}$. We have also considered the integral operator (2.11) which is related to the generalized Bessel Function in (1.5) characterized here.

References

[1] Á. Baricz, Geometric properties of generalized Bessel functions, *Publ. Math. Debrecen* **73** (2008), 155–178.
[2] Á. Baricz, Geometric properties of generalized Bessel functions of complex order, *Mathematica* (*Cluj*) **48** (**71**) (2006), 13–18.
[3] Á. Baricz, *Generalized Bessel Functions of the First Kind*, Ph.D. Thesis, Babeş-Bolyai University, Cluj-Napoca, 2008.
[4] Á. Baricz, *Generalized Bessel Functions of the First Kind*, Lecture Notes in Mathematics, Vol. **1994**, Springer-Verlag, Berlin, Heidelberg and New York, 2010.
[5] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and a corresponding class of starlike functions, *Tamkang J. Math.* **28** (1997), 17–32.
[6] N. E. Cho, G. Murugusundaramoorthy and T. Janani, Some inclusion results of certain subclass of analytic functions associated with Bessel functions, Preprint 2014.
[7] J. Choi and P. Agarwal, Certain unified integrals involving a product of Bessel functions of the first kind, *Honam Math. J.* **35** (2013), 667–677.
[8] E. Deniz, Convexity of integral operators involving generalized Bessel functions, *Integral Transforms Spec. Funct.* **24** (2013), 201–216.
[9] E. Deniz, H. Orhan and H. M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, *Taiwanese J. Math.* **15** (2011), 883–917.
[10] A. W. Goodman, On uniformly convex functions, *Ann Polon. Math.* **56** (1991), 87–92.
[11] A. W. Goodman, On uniformly convex functions, *J. Math. Anal. Appl.* **155** (1991), 364–370.
[12] W. Ma and D. Minda, Uniformly convex funtions, *Ann. Polon. Math.* **57** (1992), 166–175.
[13] S. R. Mondal and A. Swaminathan, Geometric properties of generalized Bessel functions, *Bull. Malaysian Math. Sci. Soc.* **35** (2012), 179–194.
[14] F. Rønning, On starlike functions associated with parabolic regions, *Ann Univ. Marie-Curie Skłodowska Sect. A* **45** (1991), 117–122.
[15] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.* **118** (1993), 190–196.
[16] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.
[17] H. M. Srivastava, G. Murugusundaramoorthy and T. Janani, Uniformly starlike functions and uniformly convex functions associated with the Struve function, *Internat. J. Appl. Math. Engrg. Sci.* **8** (2) (2014) (in press).
[18] N. Yagmur and H. Orhan, Starlikeness and convexity of generalized Struve functions, *Abstr. Appl. Anal.* **2013** (2013), Article ID 954513, 1–6.