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Sufficient Conditions for a Certain General Class of Carathéodory Functions

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Abstract. In the present paper, the authors derive sufficient conditions for functions to be in a certain general class of Carathéodory functions in the open unit disk by using the Miller-Mocanu lemma. As an application of our main result, we deduce sufficient conditions for functions belonging to the class $\mathcal{ST}\hat{\mathcal{S}}_{\beta}^{\mu}$ which is introduced here. The various results presented here would generalize and extend many known results.

1. Introduction and Definitions

Let $\mathcal{H}[a_0, n]$ denote the class of functions $p(z)$ of the form:

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k \qquad (n \in \mathbb{N} = \{1, 2, 3, \ldots\};\ a_0 \in \mathbb{C}),$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

$\mathbb{C}$ being, as usual, the set of complex numbers.

Definition 1. If the function $p(z) \in \mathcal{H}[a_0, n]$ satisfies the following argument inequality:

$$|\arg[\lambda p(z) + 1 - \lambda]| < \frac{\pi}{2}\mu \qquad (z \in \mathbb{U};\ 0 < \mu \leqq 1;\ \lambda \in \mathbb{C} \setminus \{0\}),$$

then we say that $p(z)$ is a strongly Carathéodory function of type λ and order μ in $\mathbb{U}$ and we write $p(z) \in \mathcal{P}(\lambda, \mu)$.

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We note that, in the special case when $\lambda = 1$, the class $\mathcal{P}(\lambda, \mu)$ reduces to the class $\mathcal{STP}(\mu)$ which was studied by Shiraishi *et al.* [5] and Kim *et al.* [2] (see also [1], [4] and [9]).

Definition 2. Let $\mathcal{A}_n$ denote the class of functions of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \qquad (n \in \mathbb{N}),$$

which are analytic in $\mathbb{U}$. Also let

$$\mathcal{A} = \mathcal{A}_1.$$

A function $f \in \mathcal{A}_n$ is called strongly starlike of order μ $(0 < \mu \leqq 1)$ if and only if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\mu \ \ (z \in \mathbb{U};\ \ 0 < \mu \leqq 1).$$

We denote by $\mathcal{STS}(\mu)$ the class of all strongly starlike functions of order μ in $\mathbb{U}$. We also write

$$\mathcal{S}^* := \mathcal{STS}(1)$$

for the familiar class of starlike functions in $\mathbb{U}$.

The above-defined function class $\mathcal{STS}(\mu)$ was investigated by Shiraishi *et al.* [5] and Kim *et al.* [2]

Spaček [6] extended the class of starlike functions by introducing the class of spirallike functions of type β in $\mathbb{U}$ and gave the following analytical characterization of spirallikeness functions of type β in $\mathbb{U}$ (see also the recent work [8]).

Theorem 1 (see Spaček [6]). *Let the function $f \in \mathcal{A}$ and suppose that the parameter β is constrained by $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. Then $f(z)$ is a spirallike function of type β in $\mathbb{U}$ if and only if*

$$\Re\left(e^{i\beta}\ \frac{zf'(z)}{f(z)}\right) > 0 \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right).$$

We denote the class of spiralike functions of type β in $\mathbb{U}$ by $\hat{\mathcal{S}}_\beta$.

From Definition 1 and Theorem 1, it is easy to see that strongly starlike functions of order μ and spirallike functions of type β have some geometric relationships. Spirallike functions of type β map $\mathbb{U}$ into the right-half complex plane by the mapping $e^{i\beta}\ \frac{zf'(z)}{f(z)}$, while strongly starlike functions of order μ map $\mathbb{U}$ into the angular region of the right-half complex plane by the mapping $\frac{zf'(z)}{f(z)}$. Since

$$\lim_{z\to 0} e^{i\beta}\ \frac{zf'(z)}{f(z)} = e^{i\beta},$$

we can deduce that, if we restrict the image of the mapping $e^{i\beta}\ \frac{zf'(z)}{f(z)}$ in the angular region of right-half complex plane, then the vertex of the angular region lies on imaginary axis, and the angular region must be symmetric with respect to the straight line parallel to the real axis and passing through the point $e^{i\beta}$. This means that

$$|\arg(\zeta - i\sin\beta)| < \frac{\pi}{2}\mu,$$

where $\zeta \in \mathbb{C}$ and $\mu \in (0, 1]$. In view of this observation, we extend the classes $\mathcal{STS}(\mu)$ and $\hat{\mathcal{S}}_\beta$ by introducing the analytic function class $\mathcal{ST}\hat{\mathcal{S}}_\beta^\mu$ in $\mathbb{U}$ as follows.

Definition 3. Let the function $f \in \mathcal{A}$. Suppose also that the parameters μ and β are constrained by

$$0 < \mu \leqq 1 \qquad \text{and} \qquad -\frac{\pi}{2} < \beta < \frac{\pi}{2}.$$

We then say that $f \in \mathcal{ST}\hat{\mathcal{S}}_{\beta}^{\mu}$ if and only if

$$\left|\arg\left(e^{i\beta}\,\frac{zf'(z)}{f(z)} - i\sin\beta\right)\right| < \frac{\pi}{2}\mu \qquad \left(z \in \mathbb{U};\ 0 < \mu \leqq 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right).$$

Obviously, we have

$$\mathcal{ST}\hat{\mathcal{S}}_{\beta}^{1} = \hat{\mathcal{S}}_{\beta} \qquad \text{and} \qquad \mathcal{ST}\hat{\mathcal{S}}_{0}^{\mu} = \mathcal{STS}(\mu).$$

Definition 4 (see, for example, [3]; see also [7]). For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0 \qquad \text{and} \qquad |\mathfrak{w}(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\mathfrak{w}(z)\big) \qquad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We denote by $\mathcal{Q}$ the class of functions $q(z)$ which are analytic and injective on $\overline{\mathbb{U}} \setminus \mathbb{E}(q)$, where

$$\mathbb{E}(q) = \left\{\zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that

$$q'(\zeta) \neq 0 \qquad \big(\zeta \in \partial\mathbb{U} \setminus \mathbb{E}(q)\big).$$

Here, as usual, we write

$$\overline{\mathbb{U}} := \mathbb{U} \cup \partial\mathbb{U} \quad \text{and} \quad \partial\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| = 1\}.$$

Finally, let the subclass of $\mathcal{Q}$ for which $q(0) = a_0$ be denoted by $\mathcal{Q}(a_0)$.

The main object of this investigation is to derive sufficient conditions for functions to be in the general class $\mathcal{H}[a_0, n]$ of Carathéodory functions in the open unit disk $\mathbb{U}$. We also apply our main result (Theorem 2 below) in order to deduce the corresponding sufficient conditions for functions belonging to the class $\mathcal{ST}\hat{\mathcal{S}}_{\beta}^{\mu}$ which we have introduced here. The various results presented here would generalize and extend many known results.

2. Main Results

To prove our main results, we need the following lemma due to Miller and Mocanu [3].

Lemma. *Let $q(z) \in \mathcal{Q}(a_0)$ and let $h(z) \in \mathcal{H}[a_0, n]$ with $h(z) \neq a_0$. If $h(z) \nprec q(z)$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus \mathbb{E}(q)$ for which*

$$h(z_0) = q(\zeta_0)$$

and

$$z_0h'(z_0) = m\zeta_0 q'(\zeta_0) \qquad (m \geqq n \geqq 1).$$

By applying the above Lemma, we derive the following theorem.

Theorem 2. *Let the parameters α and λ be constrained by*

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \qquad \textit{and} \qquad \lambda \in \mathbb{C} \setminus \{0\},$$

respectively. Suppose also that the function $g(z)$ is analytic in $\mathbb{U}$ with

$$A := \inf_{z\in\mathbb{U}} \left\{ \left[\Re\big(g(z)\big)\Re(\lambda) + \Im(g(z))\Im(\lambda)\right]\cos\alpha - \Big|[\Re(g(z))\Im(\lambda) + \Im(g(z))\Re(\lambda)]\sin\alpha\Big| \right\} > 0. \tag{1}$$

If $p(z) \in \mathcal{H}[1, n]$ satisfies the following conditions:

$$p(z) \neq 1 \qquad (z \in \mathbb{U})$$

and

$$\Re\{p(z) + g(z)zp'(z)\} > \max\{A_1, A_2\},$$

where

$$A_1 = \frac{\cos\alpha}{2nA}\left(\frac{nA\tan\alpha + \Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha}{|\lambda|}\right)^2 + \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \cos\alpha} \qquad (z \in \mathbb{U})$$

and

$$A_2 = \frac{\cos\alpha}{2nA}\left(\frac{nA\tan\alpha + \Re(\lambda)\sin\alpha - \Im(\lambda)\cos\alpha}{|\lambda|}\right)^2 + \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \cos\alpha} \qquad (z \in \mathbb{U}),$$

then

$$|\arg[\lambda p(z) + 1 - \lambda]| < \frac{\pi}{2} - |\alpha| \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2};\ \lambda \in \mathbb{C}\setminus\{0\}\right).$$

Proof. Consider the functions $h_1(z)$ and $q_1(z)$ given by

$$h_1(z) = e^{i\alpha}[\lambda p(z) + 1 - \lambda] \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2};\ \lambda \in \mathbb{C}\setminus\{0\}\right) \tag{2}$$

and

$$q_1(z) = \frac{e^{i\alpha} + e^{-i\alpha}z}{1 - z} \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{3}$$

Then, clearly, the functions $h_1(z)$ and $q_1(z)$ are analytic in $\mathbb{U}$ with

$$h_1(0) = q_1(0) = e^{i\alpha} \in \mathbb{C}$$

and

$$q_1(\mathbb{U}) = \{w:\ w \in \mathbb{C} \quad \text{and} \quad \Re(w) > 0\}.$$

We now suppose that $h_1(z) \nprec q_1(z)$. Then, by using the above Lemma, we can deduce that there exist points

$$z_1 \in \mathbb{U} \quad \text{and} \quad \zeta_1 \in \partial\mathbb{U}\setminus\{1\}$$

such that

$$h_1(z_1) = q_1(\zeta_1) = i\rho_1 \qquad (\rho_1 \in \mathbb{R}) \tag{4}$$

and

$$z_1 h'(z_1) = m\zeta_1 q'(\zeta_1) \qquad (m \geqq n \geqq 1). \tag{5}$$

We note that

$$\zeta_1 = q_1^{-1}\big(h_1(z_1)\big) = \frac{h_1(z_1) - e^{i\alpha}}{h_1(z_1) + e^{-i\alpha}} \tag{6}$$

and

$$\zeta_1 q_1'(\zeta_1) = -\frac{\rho_1^2 - 2\rho_1 \sin\alpha + 1}{2\cos\alpha} =: \sigma_1(\rho_1) < 0. \tag{7}$$

For such points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$, we obtain

$$\begin{aligned}
&\Re\{p(z_1) + g(z_1)z_1 p'(z_1)\}\\
&\quad = \Re\left(\frac{\lambda - 1 + e^{-i\alpha}h_1(z_1)}{\lambda} + g(z_1)\frac{e^{-i\alpha}z_1 h_1'(z_1)}{\lambda}\right)\\
&\quad = \Re\left(\frac{\lambda - 1 + e^{-i\alpha}q_1(\zeta_1)}{\lambda} + g(z_1)\frac{e^{-i\alpha}m\zeta_1 q_1'(\zeta_1)}{\lambda}\right)\\
&\quad = \Re\left(\frac{\lambda - 1 + e^{-i\alpha}i\rho_1}{\lambda} + g(z_1)\frac{e^{-i\alpha}m\sigma_1(\rho_1)}{\lambda}\right)\\
&\quad = \frac{|\lambda|^2 - \Re(\lambda) + [\Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha]\rho_1}{|\lambda|^2}\\
&\qquad + \frac{m\{\Re\big(g(z_1)\big)[\Re(\lambda)\cos\alpha - \Im(\lambda)\sin\alpha] + \Im\big(g(z_1)\big)[\Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha]\}\sigma_1(\rho_1)}{|\lambda|^2}\\
&\quad \leqq \frac{|\lambda|^2 - \Re(\lambda) + [\Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha])\rho_1}{|\lambda|^2} + \frac{nA\sigma_1(\rho_1)}{|\lambda|^2}\\
&\quad = \frac{|\lambda|^2 - \Re(\lambda) + [\Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha]\rho_1}{|\lambda|^2} - \frac{nA}{|\lambda|^2}\left(\frac{\rho_1^2 - 2\rho_1\sin\alpha + 1}{2\cos\alpha}\right)\\
&\quad = B\rho_1^2 + C\rho_1 + D\\
&\quad =: \kappa(\rho_1),
\end{aligned} \tag{8}$$

where

$$B = -\frac{nA}{2|\lambda|^2\ \cos\alpha},$$

$$C = \frac{nA\tan\alpha + \Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha}{|\lambda|^2}$$

and

$$D = \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \cos\alpha}.$$

We can thus see that the function $\kappa(\rho_1)$ in (8) takes on the maximum value at ρ_1^* given by

$$\rho_1^* = \frac{nA\sin\alpha + [\Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha]\cos\alpha}{nA}.$$

Hence we have

$$\begin{aligned}\Re\{p(z_1) + g(z_1)z_1p'(z_1)\} &\leqq \kappa(\rho_1^*)\\ &= \frac{\cos\alpha}{2nA}\left(\frac{nA\tan\alpha + \Re(\lambda)\sin\alpha + \Im(\lambda)\cos\alpha}{|\lambda|}\right)^2\\ &\qquad + \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2\cos\alpha|\lambda|^2}\\ &=: A_1,\end{aligned}$$

where A is given by (1). This evidently contradicts the assumption of Theorem 1. Therefore, we have

$$\Re\Big(h(z)\Big) = \Re\{e^{i\alpha}[\lambda p(z) + 1 - \lambda]\} > 0 \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{9}$$

We next put

$$h_2(z) = e^{-i\alpha}[\lambda p(z) + 1 - \lambda] \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right) \tag{10}$$

and

$$q_2(z) = \frac{e^{-i\alpha} + e^{i\alpha}z}{1 - z} \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{11}$$

We then see that the functions $h_2(z)$ and $q_2(z)$ are analytic in $\mathbb{U}$ with

$$h_2(0) = q_2(0) = e^{-i\alpha} \in \mathbb{C}$$

and

$$q_2(\mathbb{U}) = \{w :\ w \in \mathbb{C} \quad \text{and} \quad \Re(w) > 0\}.$$

We now suppose that

$$h_2(z) \not\prec q_2(z) \qquad (z \in \mathbb{U}).$$

Therefore, by using the above Lemma, we deduce that there exist points

$$z_2 \in \mathbb{U} \quad \text{and} \quad \zeta_2 \in \partial\mathbb{U} \setminus \{1\}$$

such that

$$h_2(z_2) = q_2(\zeta_2) = i\rho_2 \qquad (\rho_1 \in \mathbb{R}) \tag{12}$$

and

$$z_2h_2'(z_2) = m\zeta_2q_2'(\zeta_2) \qquad (m \geqq n \geqq 1). \tag{13}$$

Furthermore, we note that

$$\zeta_2 = q_2^{-1}\Big(h_2(z_2)\Big) = \frac{h_2(z_2) - e^{-i\alpha}}{h_2(z_2) + e^{i\alpha}} \tag{14}$$

and

$$\zeta_2 q_2'(\zeta_2) = -\frac{\rho_2^2 + 2\rho_2 \sin\alpha + 1}{2\cos\alpha} =: \sigma_2(\rho_2) < 0. \tag{15}$$

For such points $z_2 \in \mathbb{U}$ and $\zeta_2 \in \partial\mathbb{U} \setminus \{1\}$, we obtain

$$\begin{aligned}
&\Re\{p(z_2) + g(z_2)z_2 p'(z_2)\} \\
&= \Re\left(\frac{\lambda - 1 + e^{i\alpha}h_2(z_2)}{\lambda} + g(z_2)\frac{e^{i\alpha}z_2 h_2'(z_2)}{\lambda}\right) \\
&= \Re\left(\frac{\lambda - 1 + e^{i\alpha}q_2(\zeta_2)}{\lambda} + g(z_2)\frac{e^{i\alpha}m\zeta_2 q_2'(\zeta_2)}{\lambda}\right) \\
&= \Re\left(\frac{\lambda - 1 + e^{i\alpha}i\rho_2}{\lambda} + g(z_2)\frac{e^{i\alpha}m\sigma_2(\rho_2)}{\lambda}\right) \\
&= \frac{|\lambda|^2 - \Re(\lambda) + [\Im(\lambda)\cos\alpha - \Re(\lambda)\sin\alpha]\rho_2}{|\lambda|^2} \\
&\quad + \frac{m\{\Re\big(g(z_2)\big)[\Re(\lambda)\cos\alpha + \Im(\lambda)\sin\alpha] + \Im(g(z_2))[\Im(\lambda)\cos\alpha - \Re(\lambda)\sin\alpha]\}\sigma_2(\rho_2)}{|\lambda|^2} \\
&\leqq \frac{|\lambda|^2 - \Re(\lambda) + [\Im(\lambda)\cos\alpha - \Re(\lambda)\sin\alpha]\rho_2}{|\lambda|^2} + \frac{nA\sigma_2(\rho_2)}{|\lambda|^2} \\
&= \frac{|\lambda|^2 - \Re(\lambda) + [\Im(\lambda)\cos\alpha - \Re(\lambda)\sin\alpha]\rho_2}{|\lambda|^2} - \frac{nA}{|\lambda|^2}\left(\frac{\rho_2^2 + 2\rho_2\sin\alpha + 1}{2\cos\alpha}\right) \\
&= B\rho_2^2 + C\rho_2 + D \\
&=: \kappa(\rho_2),
\end{aligned} \tag{16}$$

where

$$B = -\frac{nA}{2\cos\alpha|\lambda|^2},$$

$$C = -\frac{nA\tan\alpha + \Re(\lambda)\sin\alpha - \Im(\lambda)\cos\alpha}{|\lambda|^2}$$

and

$$D = \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \cos\alpha}.$$

We can see that the function $\kappa(\rho_2)$ in (16) takes on the maximum value at ρ_2^* given by

$$\rho_2^* = -\frac{nA\sin\alpha - [\Im(\lambda)\cos\alpha - \Re(\lambda)\sin\alpha]\cos\alpha}{nA}.$$

Hence we have

$$\begin{aligned}
\Re\{p(z_2) + g(z_2)z_2 p'(z_2)\} &\leqq \kappa(\rho_2^*) \\
&= \frac{\cos\alpha}{2nA}\left(\frac{nA\tan\alpha + \Re(\lambda)\sin\alpha - \Im(\lambda)\cos\alpha}{|\lambda|}\right)^2 \\
&\qquad + \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \cos\alpha} \\
&=: A_2,
\end{aligned}$$

where A is given by (1). This evidently contradicts the assumption of Theorem 1. Therefore, we have

$$\Re\{h(z)\} = \Re\{e^{-i\alpha}[\lambda p(z) + 1 - \lambda]\} > 0 \qquad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{17}$$

Consequently, by suitably combining the inequalities (9) and (17), we complete the proof of Theorem 1. □

3. Corollaries and Consequences

By setting $\lambda = 1$ in Theorem 1, we obtain the following corollary, which was obtained by Shiraishi *et al.* [5].

Corollary 1. *Let the function* $g(z)$ *be analytic in* $\mathbb{U}$ *with*

$$A := \inf_{z\in\mathbb{U}}\{\Re\big(g(z)\big)\cos\alpha - |\Im\big(g(z)\big)\sin\alpha|\} > 0. \tag{18}$$

If $p(z) \in \mathcal{H}[1, n]$ *satisfies the following conditions*:

$$p(z) \neq 1 \qquad (z \in \mathbb{U})$$

and

$$\Re\{p(z) + g(z)zp'(z)\} > \frac{1}{2nA}[(\cos\alpha + 2nA)\sin^2\alpha - n^2A^2\cos\alpha] \qquad (z \in \mathbb{U}),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - |\alpha| \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right).$$

If we let

$$\alpha = \frac{\pi}{2}(1 - \mu) \qquad (0 < \mu \leqq 1)$$

in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the parameters* μ *and* λ *be constrained by*

$$0 < \mu \leqq 1 \qquad \text{and} \qquad \lambda \in \mathbb{C} \setminus \{0\},$$

respectively. Suppose also that the function $g(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\begin{aligned} A := \inf_{z\in\mathbb{U}}\Big\{&\sin\left(\frac{\pi}{2}\mu\right)\left[\Re\big(g(z)\big)\Re(\lambda) + \Im\big(g(z)\big)\Im(\lambda)\right] \\ &- \left|\cos\left(\frac{\pi}{2}\mu\right)\left[\Re\big(g(z)\big)\Im(\lambda) + \Im\big(g(z)\big)\Re(\lambda)\right]\right|\Big\} > 0. \end{aligned} \tag{19}$$

If $p(z) \in \mathcal{H}[1, n]$ *satisfies the following conditions*:

$$p(z) \neq 1 \qquad (z \in \mathbb{U})$$

and

$$\Re\{p(z) + g(z)zp'(z)\} > \max\{A_1, A_2\},$$

where

$$A_1 = \frac{\sin\left(\frac{\pi}{2}\mu\right)}{2nA}\left(\frac{nA\cot\left(\frac{\pi}{2}\mu\right) + \Re(\lambda)\cos\left(\frac{\pi}{2}\mu\right) + \Im(\lambda)\sin\left(\frac{\pi}{2}\mu\right)}{|\lambda|}\right)^2 + \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \sin\left(\frac{\pi}{2}\mu\right)} \qquad (z \in \mathbb{U}), \tag{20}$$

and

$$A_2 = \frac{\sin\left(\frac{\pi}{2}\mu\right)}{2nA}\left(\frac{nA\cot\left(\frac{\pi}{2}\mu\right) + \Re(\lambda)\cos\left(\frac{\pi}{2}\mu\right) - \Im(\lambda)\sin\left(\frac{\pi}{2}\mu\right)}{|\lambda|}\right)^2 + \frac{|\lambda|^2 - \Re(\lambda)}{|\lambda|^2} - \frac{nA}{2|\lambda|^2\ \sin\left(\frac{\pi}{2}\mu\right)} \qquad (z \in \mathbb{U}), \tag{21}$$

then

$$p(z) \in \mathcal{P}(\lambda, \mu).$$

If, in Corollary 2, we put

$$\lambda = 1 + i\tan\beta \qquad \left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$$

and

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + na_{n+1}z^n + \cdots \qquad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}_n$, we are led to the following result.

Corollary 3. *Let the parameters μ, β and λ be constrained by*

$$0 < \mu \leqq 1, \quad -\frac{\pi}{2} < \beta < \frac{\pi}{2} \qquad \text{and} \qquad \lambda \in \mathbb{C} \setminus \{0\},$$

respectively. Suppose also that the function $g(z)$ is analytic in $\mathbb{U}$ with

$$A := \inf_{z\in\mathbb{U}}\left\{\sin\left(\frac{\pi}{2}\mu\right)\left[\Re\big(g(z)\big) + \Im\big(g(z)\big)\tan\beta\right] - \left|\cos\left(\frac{\pi}{2}\mu\right)\left[\Re\big(g(z)\big)\tan\beta + \Im\big(g(z)\big)\right]\right|\right\} > 0. \tag{22}$$

If $f(z) \in \mathcal{A}_n$ satisfies the following conditions:

$$\frac{zf'(z)}{f(z)} \neq 1 \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{zf'(z)}{f(z)} + g(z)\,\frac{zf'(z)}{f(z)}\left[1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right]\right) > \max\{A_1, A_2\},$$

where

$$A_1 = \frac{\sin\left(\frac{\pi}{2}\mu\right)}{2nA}\left(\frac{nA\cot\left(\frac{\pi}{2}\mu\right)+\cos\left(\frac{\pi}{2}\mu\right)+\sin\left(\frac{\pi}{2}\mu\right)\tan\beta}{\sec\beta}\right)^2 + \sin^2\beta - \frac{nA\cos^2\beta}{2\sin\left(\frac{\pi}{2}\mu\right)} \qquad (z\in\mathbb{U}) \tag{23}$$

and

$$A_2 = \frac{\sin\frac{\pi}{2}\mu}{2nA}\left(\frac{nA\cot\frac{\pi}{2}\mu+\cos\frac{\pi}{2}\mu-\sin\frac{\pi}{2}\mu\tan\beta}{\sec\beta}\right)^2 + \sin^2\beta - \frac{nA\cos^2\beta}{2\sin\left(\frac{\pi}{2}\mu\right)} \qquad (z\in\mathbb{U}), \tag{24}$$

then

$$f(z)\in\mathcal{ST}\hat{\mathcal{S}}^{\mu}_{\beta}.$$

Proof. By applying Corollary 2, we deduce that

$$\left|\arg\left((1+i\tan\beta)\frac{zf'(z)}{f(z)}-i\tan\beta\right)\right| < \frac{\pi}{2}\mu$$

or, equivalently, that

$$\left|\arg\left(e^{i\beta}\,\frac{zf'(z)}{f(z)}-i\sin\beta\right)\right| < \frac{\pi}{2}\mu,$$

which implies that

$$f(z)\in\mathcal{ST}\hat{\mathcal{S}}^{\mu}_{\beta},$$

as claimed. This completes the proof of Corollary 3. □

If we set $\beta = 0$ in Corollary 3, we are led easily to Corollary 4 which was proven by Shiraishi *et al.* [5].

Corollary 4. *Let the function $g(z)$ be analytic in $\mathbb{U}$ with*

$$A := \inf_{z\in\mathbb{U}}\left\{\Re\big(g(z)\big)\sin\left(\frac{\pi}{2}\mu\right)-\left|\Im\big(g(z)\big)\cos\left(\frac{\pi}{2}\mu\right)\right|\right\} > 0 \qquad (z\in\mathbb{U};\ 0<\mu\leqq 1).$$

If $f(z)\in\mathcal{A}_n$ satisfies the following conditions:

$$\frac{zf'(z)}{f(z)} \neq 1 \qquad (z\in\mathbb{U})$$

and

$$\Re\left(\frac{zf'(z)}{f(z)}+g(z)\frac{zf'(z)}{f(z)}\left[1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right]\right) > \frac{1}{2nA}\left\{\left[\sin\left(\frac{\pi}{2}\mu\right)+2nA\right]\cos^2\left(\frac{\pi}{2}\mu\right)-n^2A^2\sin\left(\frac{\pi}{2}\mu\right)\right\} \qquad (z\in\mathbb{U}),$$

then

$$f(z)\in\mathcal{ST}\mathcal{S}(\mu).$$

Corollary 5. *Let the function $g(z)$ be analytic in $\mathbb{U}$ with*

$$A := \inf_{z\in\mathbb{U}} \left\{\Re\Big(g(z)\Big) + \Im\Big(g(z)\Big)\tan\beta\right\} > 0 \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \beta \leqq \frac{\pi}{2}\right).$$

If $f(z) \in \mathcal{A}_n$ satisfies the following conditions:

$$\frac{zf'(z)}{f(z)} \neq 1 \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{zf'(z)}{f(z)} + g(z)\frac{zf'(z)}{f(z)}\left[1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right]\right) > \frac{1}{2nA}\sec^2\beta - nA\cos^2\beta + \sin^2\beta \qquad (z \in \mathbb{U}),$$

then

$$f(z) \in \hat{\mathcal{S}}_\beta.$$

Proof. If we set $\mu = 1$ in Corollary 3, we are led easily to Corollary 5. □

4. Concluding Remarks and Observations

In the present investigation, we have derived sufficient conditions for functions to be in a certain general class of Carathéodory functions in the open unit disk $\mathbb{U}$ by using the Miller-Mocanu lemma (see the Lemma in Section 1). As an application of our main result (Theorem 2 above), we have deduced sufficient conditions for functions belonging to the class $\mathcal{ST}\hat{\mathcal{S}}^{\mu}_{\beta}$ which we have introduced in this paper. By suitably specializing the parameters involved in the general result (Theorem 2), several interesting corollaries and consequences have also been derived (see Corollaries 1 to 5 in Section 3). The various results presented here are shown to generalize and extend many known results.

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