Binding Numbers for all Fractional \((a, b, k)\)-Critical Graphs

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Abstract. Let \(G\) be a graph of order \(n\), and let \(a, b, k\) be nonnegative integers with \(2 \leq a \leq b\). A graph \(G\) is called all fractional \((a, b, k)\)-critical if after deleting any \(k\) vertices of \(G\) the remaining graph of \(G\) has all fractional \([a, b]\)-factors. In this paper, it is proved that \(G\) is all fractional \((a, b, k)\)-critical if \(n \geq \frac{a}{a + b - 1}(a + b - 3) + \frac{k}{a + b} + \frac{ak}{(a + b - 1)(n - 1)} + \frac{a}{a - 1}\) and \(\text{bind}(G) > \frac{an - ak - (a + b) + 2}{2}\). Furthermore, it is shown that this result is best possible in some sense.

1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let \(G\) be a graph with a vertex set \(V(G)\) and an edge set \(E(G)\). For \(x \in V(G)\), the set of vertices adjacent to \(x\) in \(G\) is said to be the neighborhood of \(x\), denoted by \(N_G(x)\). For any \(X \subseteq V(G)\), we write \(N_G(X) = \bigcup_{x \in X} N_G(x)\). For two disjoint subsets \(S\) and \(T\) of \(V(G)\), we denote by \(b_{xy}(S, T)\) the number of edges with one end in \(S\) and the other end in \(T\). Thus \(b_{xy}(x, V(G) \setminus \{x\}) = d(x)\) is the degree of \(x\) and \(\delta(G) = \min\{d(x) : x \in V(G)\}\) is the minimum degree of \(G\). For \(S \subseteq V(G)\), we use \(G[S]\) to denote the subgraph of \(G\) induced by \(S\), and \(G - S\) to denote the subgraph obtained from \(G\) by deleting vertices in \(S\) together with the edges incident to vertices in \(S\). A vertex set \(S \subseteq V(G)\) is called independent if \(G[S]\) has no edges. The binding number of \(G\) is defined as

\[
\text{bind}(G) = \min \frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G).
\]

Let \(g\) and \(f\) be two integer-valued functions defined on \(V(G)\) with \(0 \leq g(x) \leq f(x)\) for each \(x \in V(G)\). A \((g, f)\)-factor of a graph \(G\) is defined as a spanning subgraph \(F\) of \(G\) such that \(g(x) \leq d_F(x) \leq f(x)\) for each \(x \in V(G)\). We say that \(G\) has all \((g, f)\)-factors if \(G\) has an \(r\)-factor for every \(r : V(G) \rightarrow \mathbb{Z}^+\) such that \(g(x) \leq r(x) \leq f(x)\) for each \(x \in V(G)\) and \(r(V(G))\) is even.

A fractional \((g, f)\)-indicator function is a function \(h\) that assigns to each edge of a graph \(G\) a real number in the interval \([0,1]\) so that for each vertex \(x\) we have \(g(x) \leq h(E_x) \leq f(x)\), where \(E_x = \{e : e = xy \in E(G)\}\) and \(h(E_x) = \sum_{e \in E_x} h(e)\). Let \(b\) be a fractional \((g, f)\)-indicator function of a graph \(G\). Set \(E_b = \{e : e \in E(G), h(e) > 0\}\). If \(G_b\) is a spanning subgraph of \(G\) such that \(E(G_b) = E_b\), then \(G_b\) is called a fractional \((g, f)\)-factor of \(G\).
is also called the indicator function of $G$. If $h(e) \in \{0, 1\}$ for every edge $e$, then $G$ is just a $(g,f)$-factor of $G$. A fractional $(g,f)$-factor is a fractional $f$-factor if $g(x) = f(x)$ for each vertex $x$ in $G$. A fractional $(g,f)$-factor is a fractional $[a,b]$-factor if $g(x) = a$ and $f(x) = b$ for each vertex $x$ in $G$. We say that $G$ has all fractional $(g,f)$-factors if $G$ has a fractional $r$-factor for every $r : V(G) \rightarrow \mathbb{Z}^+$ such that $g(x) \leq r(x) \leq f(x)$ for each vertex $x$ in $G$. All fractional $(g,f)$-factors are said to be all fractional $[a,b]$-factors if $g(x) = a$ and $f(x) = b$ for each vertex $x$ in $G$. A graph $G$ is all fractional $(a,b,k)$-critical if after deleting any $k$ vertices of $G$ the remaining graph of $G$ has all fractional $[a,b]$-factors.

Many authors have investigated factors [1,2,8] and fractional factors [3,4,7,10] of graphs. The following results on all $(g,f)$-factors, all fractional $[a,b]$-factors and all fractional $(a,b,k)$-critical graphs are known.

**Theorem 1.1.** (Niessen [6]). $G$ has all $(g,f)$-factors if and only if

$$g(S) + \sum_{x \in T} d_{G-S}(x) - f(T) - h_C(S,T,g,f) = \begin{cases} -1, & \text{if } f \neq g \\ 0, & \text{if } f = g \end{cases}$$

for all disjoint subsets $S,T \subseteq V(G)$, where $h_C(S,T,g,f)$ denotes the number of components $C$ of $G -(S \cup T)$ such that there exists a vertex $v \in V(C)$ with $g(v) < f(v)$ or $e_C(V(C),T) + f(V(C)) \equiv 1 \pmod{2}$.

**Theorem 1.2.** (Lu [5]). Let $a \leq b$ be two positive integers. Let $G$ be a graph with order $n \geq \frac{2(a+b)(a+b-1)}{a}$ and minimum degree $\delta(G) \geq \frac{(a+b-1)^2 + 4b}{4a}$. If $|N_G(x)| \cup |N_G(y)| \geq \frac{bn}{a+b}$ for any two nonadjacent vertices $x$ and $y$ in $G$, then $G$ has all fractional $[a,b]$-factors.

**Theorem 1.3.** (Zhou [9]). Let $a,b$ and $k$ be nonnegative integers with $1 \leq a \leq b$, and let $G$ be a graph of order $n$ with $n \geq a + k + 1$. Then $G$ is all fractional $(a,b,k)$-critical if and only if for any $S \subseteq V(G)$ with $|S| \geq k$

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \geq ak,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}$.

Using Theorem 3, Zhou [9] obtained a neighborhood condition for graphs to be all fractional $(a,b,k)$-critical graphs.

**Theorem 1.4.** (Zhou [9]). Let $a,b,k,r$ be nonnegative integers with $1 \leq a \leq b$ and $r \geq 2$. Let $G$ be a graph of order $n$ with $n > \frac{(a+b)(r(a+b) - 2) + ak}{a}$. If $\delta(G) \geq \frac{(r-1)b^2}{a} + k$, and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_r)| \geq \frac{bn + ak}{a+b}$ for any independent subset $\{x_1,x_2,\cdots,x_r\}$ in $G$, then $G$ is all fractional $(a,b,k)$-critical.

2. Main Result and Its Proof

In this paper, we proceed to study the existence of all fractional $(a,b,k)$-critical graphs and obtain a binding number condition for graphs to be all fractional $(a,b,k)$-critical. Our main result is the following theorem.

**Theorem 2.1.** Let $a$, $b$ and $k$ be nonnegative integers with $2 \leq a \leq b$, and let $G$ be a graph of order $n$ with $n \geq \frac{(a+b-1)(a+b-3) + ak}{a}$. If $\text{bind}(G) > \frac{(a+b-1)(n-1)}{an-ak-(a+b)+2}$, then $G$ is all fractional $(a,b,k)$-critical.

**Proof.** Suppose that $G$ satisfies the assumption of Theorem 2.1, but it is not all fractional $(a,b,k)$-critical. Then by Theorem 1.3, there exists some subset $S$ of $V(G)$ with $|S| \geq k$ such that

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \leq ak - 1,$$

(1)
where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}$. Clearly, $T \neq \emptyset$ by (1). Define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

In terms of the definition of $T$, we obtain $0 \leq h \leq b - 1$.

Now in order to prove the correctness of Theorem 2.1, we shall deduce some contradictions according to the following two cases.

**Case 1.** $h = 0$.

Let $X = \{x : x \in T, d_{G-S}(x) = 0\}$. Obviously, $X \neq \emptyset$ and $N_G(V(G) \setminus S) \cap X = \emptyset$, and so $|N_G(V(G) \setminus S)| \leq n - |X|$. According to the definition of $\delta(G)$ and the condition of Theorem 2.1, we have

$$\frac{(a + b - 1)(n - 1)}{an - ak - (a + b) + 2} < \delta(G) \leq \frac{|N_G(V(G) \setminus S)|}{|V(G) \setminus S|} \leq \frac{n - |X|}{n - |S|},$$

which implies

$$(a + b - 1)(n - 1)|S| > (a + b - 1)(n - 1)n - (an - ak - (a + b) + 2)n + (an - ak - (a + b) + 2)|X|$$

$$= (b - 1)(n - 1)n + (b - 2)n + an + (an - ak - (a + b) + 2)|X|$$

$$\geq (b - 1)(n - 1)n + akn + (an - ak - (a + b) + 2)|X|$$

$$= (b - 1)(n - 1)n + akn + [(n - 1) + (a - 1)n - ak - (a + b) + 3]|X|$$

$$\geq (b - 1)(n - 1)n + akn + [(n - 1) + (a - 1)n - ak - (a + b) + 3][X]$$

$$= (b - 1)(n - 1)n + ak(n - 1) + (n - 1)|X|.$$}

Thus, we obtain

$$|S| > \frac{(b - 1)n + ak + |X|}{a + b - 1}. \quad (2)$$

Using (1), (2) and $|S| + |T| \leq n$, we have

$$ak - 1 \geq a|S| + \sum_{x \in T} d_{G-S}(x) - b|T|$$

$$\geq a|S| + |T| - |X| - b|T|$$

$$= a|S| - (b - 1)|T| - |X|$$

$$\geq a|S| - (b - 1)(n - |S|) - |X|$$

$$= (a + b - 1)|S| - (b - 1)n - |X|$$

$$> (b - 1)n + ak + |X| - (b - 1)n - |X|$$

$$= ak,$$

which is a contradiction.

**Case 2.** $1 \leq h \leq b - 1$.

**Claim 1.** $\delta(G) > \frac{(b - 1)n + ak + a + b - 2}{a + b - 1}$.

Let $v$ be a vertex of $G$ with degree $\delta(G)$. Set $Y = V(G) \setminus N_G(v)$. Obviously, $Y \neq \emptyset$ and $v \notin N_G(Y)$. In terms of the definition of $\delta(G)$, we have

$$\frac{(a + b - 1)(n - 1)}{an - ak - (a + b) + 2} < \delta(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{n - 1}{n - \delta(G)}.$$

which implies

$$\delta(G) > \frac{(b - 1)n + ak + a + b - 2}{a + b - 1}.$$
This completes the proof of Claim 1.

Note that \( \delta(G) \leq |S| + h \). Then using Claim 1, we have

\[
|S| \geq \delta(G) - h > \frac{(b-1)n + ak + a + b - 2}{a + b - 1} - h. \tag{3}
\]

**Claim 2.** \(|\mathcal{T}| \leq \frac{an - ak - (a + b) + 1}{a + b - 1} + h\).

Assume that \(|\mathcal{T}| \geq \frac{an - ak - (a + b) + 2}{a + b - 1} + h\). We choose \( u \in \mathcal{T} \) such that \( d_{G-s}(u) = h \) and let \( Y = \mathcal{T} \setminus N_{G-s}(u) \).

Note that \(|N_{G-s}(u)| = d_{G-s}(u) = h\). Thus, we obtain

\[
|Y| \geq |\mathcal{T}| - d_{G-s}(u) = |\mathcal{T}| - h \geq \frac{an - ak - (a + b) + 2}{a + b - 1} > 0
\]

and

\[ N_{C}(Y) \neq V(G). \]

Combining these with the definition of \( \text{bind}(G) \), we have

\[
\text{bind}(G) \leq \frac{|N_{C}(Y)|}{|Y|} \leq \frac{n - 1}{|\mathcal{T}| - h} \leq \frac{(a + b - 1)(n - 1)}{an - ak - (a + b) + 2},
\]

which contradicts that the condition of Theorem 2.1. The proof of Claim 2 is completed.

According to (1), (3) and Claim 2, we obtain

\[
ak - 1 > a|S| + \sum_{v \in \mathcal{T}} d_{G-s}(v) - b|\mathcal{T}| \geq a|S| - (b - h)|\mathcal{T}|
\]

\[
> a \cdot \left( \frac{(b-1)n + ak + a + b - 2}{a + b - 1} - h \right) - (b - h) \cdot \left( \frac{an - ak - (a + b) + 1}{a + b - 1} + h \right)
\]

\[
= \frac{(h-1)ak + (a + b - h)ak - a}{a + b - 1} - (h-1)(a + b - h),
\]

that is,

\[
ak - 1 > \frac{(h-1)ak + (a + b - h)ak - a}{a + b - 1} - (h-1)(a + b - h). \tag{4}
\]

Let \( f(h) = \frac{(h-1)ak + (a + b - h)ak - a}{a + b - 1} - (h-1)(a + b - h) \). If \( h = 1 \), then by (4) we have \( ak - 1 > f(h) = f(1) = \frac{a}{a + b - 1} > ak - 1 \), which is a contradiction. In the following, we assume that \( 2 \leq h \leq b - 1 \).

In view of \( 2 \leq h \leq b - 1 \) and \( n \geq \frac{(a + b - 1)(a + b - 3) + a}{a} + \frac{ak}{a - 1} \), we have

\[
f'(h) = \frac{an - ak}{a + b - 1} - (a + b - h) + (h - 1)
\]

\[
= \frac{2h + \frac{an - ak}{a + b - 1} - (a + b + 1) \cdot \frac{(a + b - 1)(a + b - 3) + a}{a + b - 1}}{a + b - 1}
\]

\[
> 4 + \frac{(a - 1)(a + b - 3) + a}{a + b - 1} - (a + b + 1)
\]

\[
= \frac{a}{a + b - 1} > 0.
\]

Thus, we obtain

\[
f(h) \geq f(2). \tag{5}
\]
From (4), (5) and $n \geq \frac{(a + b - 1)(a + b - 3) + a}{a} + \frac{ak}{a - 1}$, we obtain

$$ak - 1 > f(h) \geq f(2) = \frac{an + (a + b - 2)ak - a}{a + b - 1} - (a + b - 2) \geq \frac{(a + b - 1)(a + b - 3) + a + ak + (a + b - 2)ak - a}{a + b - 1} - (a + b - 2) = ak - 1,$$

which is a contradiction. This completes the proof of Theorem 2.1.

**Remark.** In Theorem 2.1, the lower bound on the condition $bind(G)$ is best possible in the sense since we cannot replace $bind(G) > \frac{(a + b - 1)(n - 1)}{an - ak - (a + b) + 2}$ with $bind(G) \geq \frac{(a + b - 1)(n - 1)}{an - ak - (a + b) + 2}$, which is shown in the following example.

Let $b \geq a \geq 2, k \geq 0$ be three integers such that $a + b + k - 1$ is even and $\frac{a(b - 1) + b(b - 2) + (a + b - 1)k}{a}$ is a positive integer. Set $l = \frac{a + b + k - 1}{2}$ and $m = \frac{a(b - 1) + b(b - 2) + (a + b - 1)k}{a}$. We construct a graph $G = K_m \lor K_{2l}$. Then $n = m + 2l = \frac{a(b - 1) + b(b - 2) + (a + b - 1)k}{a} + a + b + k - 1$. Let $X = V(K_2)$, for any $x \in X$, then $|N_G(X \setminus x)| = n - 1$. According to the definition of $bind(G)$, we obtain

$$bind(G) = \frac{|N_G(X \setminus x)|}{|X \setminus x|} = \frac{n - 1}{2l - 1} = \frac{n - 1}{a + b + k - 2} = \frac{(a + b - 1)(n - 1)}{an - ak - (a + b) + 2}.$$

Let $S = V(K_m), T = V(K_{2l})$. Then $|S| = m \geq k, |T| = 2l$ and $\sum_{x \in T} d_{G-5}(x) = 2l$. Thus, we have

$$a[S] + \sum_{x \in T} d_{G-5}(x) - b[T] = am - 2l(b - 1)$$

$$= (a(b - 1) + b(b - 2) + (a + b - 1)k - (b - 1)(a + b + k - 1))$$

$$= ak - 1 < ak.$$

In terms of Theorem 1.3, $G$ is not all fractional $(a, b, k)$-critical.

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**References**


