Filomat 29:3 (2015), 593–597
DOI 10.2298/FIL1503593M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Locally Conformal Kaehler Space Forms

Pegah Mutlu[a], Zerrin Şentürk[a]

[a] *Istanbul Technical University, Faculty of Science and Letters, Mathematics Engineering Department, Maslak, TR-34469, Istanbul, Turkey*

Abstract. The notion of a locally conformal Kaehler manifold (an l.c.K-manifold) in a Hermitian manifold has been introduced by I. Vaisman in 1976. In [2], K. Matsumoto introduced some results with the tensor P_{ij} is hybrid. In this work, we give a generalisation about the results of an l.c.K-space form with the tensor P_{ij} is not hybrid. Moreover, the Sato's form of the holomorphic curvature tensor in almost Hermitian manifolds and l.c.K-manifolds are presented.

1. Preliminaries

Let (M, g, J) be a real 2n-dimensional Hermitian manifold with the structure (J, g) , where J is the almost complex structure and g is the Hermitian metric. Then

$$J^2 = -Id. \quad , \qquad\qquad g(JX, JY) = g(X, Y)$$

for any vector fields X and Y tangent to M. The fundamental 2-form Ω is defined by

$$\Omega(X, Y) = g(JX, Y) = -\Omega(Y, X).$$

The manifold M is called a *locally conformal Kaehler manifold (an l.c.K-manifold)* if each point x in M has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that

$$g^* = e^{-2\rho} g \mid_{\mathrm{U}}$$

is a Kaehlerian metric on U. Especially, if we can take $U = M$, then the manifold M is said to be *globally conformal Kaehler*.

A Hermitian manifold whose metric is locally conformal to a Kaehler metric is called an l.c.K-manifold. I. Vaisman gives its characterization as follows [6] :
A Hermitian manifold M is an l.c.K-manifold if and only if there exists on M a global closed 1-form α such that

$$d\Omega = 2\alpha \wedge \Omega \ ,$$

2010 *Mathematics Subject Classification*. 53 C 40

Keywords. Locally conformal Kaehler manifold, Lee form, Locally conformal Kaehler space form, hybrid, holomorphic curvature tensor.

Received: 21 Oktober 2014; Accepted: 19 November 2014

Communicated by Ljubica Velimirović and Mića Stanković

Email addresses: sariaslani@itu.edu.tr (Pegah Mutlu), senturk@itu.edu.tr (Zerrin Şentürk)

where α is called the Lee form.

A Hermitian manifold (M, g, J) is an l.c.K-manifold if and only if

$$\nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki} \ , \tag{1}$$

where

$$\beta_j = -\alpha_r J_j^r.$$

From (1), we obtain

$$\begin{aligned} \nabla_k \nabla_h J_{ij} - \nabla_h \nabla_k J_{ij} &= P_{kr} J_j^r g_{hi} - P_{kr} J_i^r g_{hj} - P_{hr} J_j^r g_{ki} + P_{hr} J_i^r g_{kj} \\ &- P_{kj} J_{hi} + P_{ki} J_{hj} + P_{hj} J_{ki} - P_{hi} J_{kj}, \end{aligned} \tag{2}$$

where

$$P_{ij} = -\nabla_i \alpha_j - \alpha_i \alpha_j + \frac{\|\alpha\|^2}{2} g_{ij}. \tag{3}$$

We note that $P_{ri} = P_{ir}$ and $\|\alpha\|$ denotes the length of the Lee form.

Using the Ricci identity, in (2) we get [1]

$$\begin{aligned} -R_{hkir} J_j^r + R_{hkjr} J_i^r &= P_{kr} J_j^r g_{hi} - P_{kr} J_i^r g_{hj} - P_{hr} J_j^r g_{ki} + P_{hr} J_i^r g_{kj} \\ &- P_{kj} J_{hi} + P_{ki} J_{hj} + P_{hj} J_{ki} - P_{hi} J_{kj} \end{aligned} \tag{4}$$

and then

$$\begin{aligned} R_{hkrs} J_j^r J_i^s &= R_{hkji} + P_{ki} g_{hj} - P_{kj} g_{hi} + P_{hj} g_{ki} - P_{hi} g_{kj} \\ &+ P_{kr} J_i^r J_{hj} - P_{kr} J_j^r J_{hi} + P_{hr} J_j^r J_{ki} - P_{hr} J_i^r J_{kj}. \end{aligned} \tag{5}$$

Moreover, we have

$$R_{ir} J_j^r + R_{jr} J_i^r = 2(n-1)(P_{jr} J_i^r + P_{ir} J_j^r). \tag{6}$$

If the tensor P_{ij} is hybrid, i.e. $P_{ir} J_j^r + P_{jr} J_i^r = 0$, using (6) , the Ricci tensor is hybrid. The converse statement is also true.

In an almost Hermitian manifold (M, g, J), the tensor

$$\begin{aligned} (HR)_{ijhk} &= \frac{1}{16}\Big\{3[R_{ijhk} + R_{rshk} J_i^r J_j^s + R_{ijrs} J_h^r J_k^s + R_{rspq} J_i^r J_j^s J_h^p J_k^q] \\ &- R_{ihrs} J_k^r J_j^s - R_{rskj} J_i^r J_h^s - R_{ikrs} J_j^r J_h^s - R_{rsjh} J_i^r J_k^s \\ &+ R_{rhsj} J_i^r J_k^s + R_{irks} J_h^r J_j^s + R_{rkjs} J_i^r J_h^s + R_{irsh} J_k^r J_j^s\Big\} \end{aligned} \tag{7}$$

is called the *holomorphic curvature tensor* of Kaehler type [3].

2. Locally conformal Kaehler space form

An l.c.K-manifold $M(J, g, \alpha)$ is called *an l.c.K-space form* if it has a constant holomorphic sectional curvature. Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c, then the Riemannian curvature tensor R_{ijhk} with respect to g_{ij} can be expressed in the form [4]

$$\begin{aligned} R_{ijhk} &= \frac{c}{4}[g_{ik}g_{jh} - g_{ih}g_{jk} + J_{ik}J_{jh} - j_{ih}J_{jk} - 2J_{ij}J_{hk}] \\ &+ \frac{1}{8}\Big\{g_{ik}(7P_{jh} - P_{rs}J_j^rJ_h^s) - g_{ih}(7P_{jk} - P_{rs}J_j^rJ_k^s) \\ &+ g_{jh}(7P_{ik} - P_{rs}J_i^rJ_k^s) - g_{jk}(7P_{ih} - P_{rs}J_i^rJ_h^s) \\ &+ J_{ik}(P_{jr}J_h^r - P_{hr}J_j^r) - J_{ih}(P_{jr}J_k^r - P_{kr}J_j^r) \\ &+ J_{jh}(P_{ir}J_k^r - P_{kr}J_i^r) - J_{jk}(P_{ir}J_h^r - P_{hr}J_i^r) \\ &- 2J_{ij}(P_{hr}J_k^r - P_{kr}J_h^r) - 2J_{hk}(P_{ir}J_j^r - P_{jr}J_i^r)\Big\} \ . \end{aligned} \tag{8}$$

Contracting (8) with g^{ik}, we have

$$4R_{jh} = \{2(n+1)c + 3P\}g_{jh} + (7n-10)P_{jh} - (n+2)P_{rs}J_j^rJ_h^s \tag{9}$$

and the scalar field P is given by

$$P = P_{ij}g^{ij} = -\nabla_r\alpha^r + (n-1)||\alpha||^2 \ . \tag{10}$$

Contracting (9) with g^{jh} , the scalar curvature has the form

$$\kappa = n(n+1)c + 3(n-1)P \ . \tag{11}$$

Theorem 2.1. *If the tensor field P_{ij} is proportional to g_{ij} and the scalar field P is constant, then a real 2n-dimensional l.c.K-space form $M(c)$ is Einstein.*

Proof . If the tensor P_{ij} is proportional to g_{ij} and P is constant, then P_{ij} is written by

$$P_{ij} = \frac{P}{2n}g_{ij} \ . \tag{12}$$

Substituting the above equation into (9) , we obtain

$$4R_{jh} = \Big\{2(n+1)c + \frac{6(n-1)}{n}P\Big\}g_{jh} \ , \tag{13}$$

which means that the l.c.K-space form is Einstein.

Corollary 2.2. *A real 2n-dimensional Einstein l.c.K-space form $M(c)$ is a complex space form if $P = 0$.*

Theorem 2.3. *Let $M(c)$ be an l.c.K-space form. If κ is constant and $||\alpha||$ is non-zero constant, then*

$$\Big\{(\nabla_j\nabla_r\alpha_s)\alpha^r + 2(\nabla_j\alpha_s)||\alpha||^2\Big\}J^{sj} - (\nabla_j\alpha_r)\beta^r\beta^j = 0. \tag{14}$$

Proof . Let $M(c)$ be an l.c.K-space form with constant holomorphic sectional curvature c. If we assume that the scalar curvature κ is constant, then by virtue of (11), P is constant. Under this assumption, differentiating (9), we get

$$\begin{aligned} 4\nabla_kR_{jh} &= (7n-10)\nabla_kP_{jh} - (n+2)\Big[(\nabla_kP_{rs})J_j^rJ_h^s + (\nabla_kJ_j^r)P_{rs}J_h^s \\ &+ (\nabla_kJ_h^s)P_{rs}J_j^r\Big]. \end{aligned} \tag{15}$$

Substituting (3) into (15), using the Ricci identity and the equality $\nabla_j\alpha_i = \nabla_i\alpha_j$, we have

$$\begin{aligned}
4(\nabla_k R_{jh} - \nabla_j R_{kh}) &= (7n-10)\Big[R^r_{kjh}\alpha_r + (\nabla_j \alpha_h)\alpha_k - (\nabla_k \alpha_h)\alpha_j \\
&+ \frac{1}{2}\{(\nabla_k \|\alpha\|^2) g_{jh} - (\nabla_j \|\alpha\|^2) g_{kh}\}\Big] \\
&- (n+2)\Big[(\nabla_k P_{rs}) J^r_j J^s_h - (\nabla_j P_{rs}) J^r_k J^s_h + (\nabla_k J^r_j) P_{rs} J^s_h \\
&+ (\nabla_k J^s_h) P_{rs} J^r_j + (\nabla_k J^s_h) P_{rs} J^r_j - (\nabla_j J^r_k) P_{rs} J^s_h \\
&- (\nabla_j J^s_h) P_{rs} J^r_k\Big]. \qquad (16)
\end{aligned}$$

Contracting (16) with g^{jh} and taking into account $2\nabla_r R^r_k = \nabla_k \kappa$ [7], we obtain

$$\begin{aligned}
(7n & -10)\Big[R^r_k \alpha_r + (\nabla_j \alpha^j)\alpha_k - (n+2)\Big[-(\nabla_j P_{rs}) J^r_k J^s_h g^{jh} \\
+ & (\nabla_k J^r_j) P_{rs} J^s_h g^{jh} + (\nabla_k J^s_h) P_{rs} J^r_j g^{jh} - (\nabla_j J^r_k) P_{rs} J^s_h g^{jh} \\
- & (\nabla_j J^s_h) P_{rs} J^r_k g^{jh}\Big] = 0, \qquad (17)
\end{aligned}$$

where

$$\nabla_k J^r_j = -\beta_j \delta^r_k + \beta^r g_{kj} - \alpha_j J^r_k + \alpha^r J_{kj} \ . \qquad (18)$$

Now contracting (9) with g^{hr} and transvecting with α_r, we get

$$4R^r_k \alpha_r = \{2(n+1)c + 3P\}\alpha_k + 6(n-2)P_{kh}\alpha^h \ . \qquad (19)$$

From (10) , we have

$$3P\alpha_k = -3(\nabla_r \alpha^r)\alpha_k + 3(n-1)\|\alpha\|^2 \alpha_k \qquad (20)$$

and transvecting (3) with α^h , we obtain

$$P_{kh}\alpha^h = -\frac{1}{2}\nabla_k \|\alpha\|^2 - \frac{1}{2}\|\alpha\|^2 \alpha_k \ . \qquad (21)$$

Substituting (19) , (20) and (21) into (17) , and transvecting with β^k, we find (14).

3. Sato's form of the holomorphic curvature tensor

The curvature tensor of an almost Hermitian manifold of constant holomorphic sectional curvature c is given by [5]

$$\begin{aligned}
R(X,Y,Z,W) &= \frac{c}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
&+ J(X,W)J(Y,Z) - J(X,Z)J(Y,W) \\
&- 2J(X,Y)J(Z,W)] \\
&= \frac{1}{96}\Big\{26[G(X,Y,Z,W) - G(Z,W,X,Y)] \\
&- 6[G(JX,JY,JZ,JW) + G(JZ,JW,JX,JY)] \\
&+ 13[G(X,Z,Y,W) + G(Y,W,X,Z) \\
&- G(X,W,Y,Z) - G(Y,Z,X,W)] \\
&- 3[G(JX,JZ,JY,JW) + G(JY,JW,JX,JZ) \\
&- G(JX,JW,JY,JZ) - G(JY,JZ,JX,JW)] \\
&+ 4[G(X,JY,Z,JW) + G(JX,Y,JZ,W)] \\
&+ 2[G(X,JZ,Y,JW) + G(JX,Z,JY,W) \\
&- G(X,JW,Y,JZ) - G(JX,W,JY,Z)]\Big\}, \qquad (22)
\end{aligned}$$

where

$$G(X,Y,Z,W) = R(X,Y,Z,W) - R(X,Y,JZ,JW).$$

Substituting the above equality into (22) , using (7) and the Bianchi identity we obtain

$$(HR)(X,Y,Z,W) = \frac{1}{24}\Big\{13[-R(X,Y,Z,W) + R(JX,JY,Z,W)]\Big\}. \tag{23}$$

The tensor (23) is said to be the Sato's form of the holomorphic curvature tensor.
Now substituting (5) into (23) , we get

$$\begin{aligned}(HR)_{ijhk} &= \frac{13}{24}[P_{kj}g_{hi} - P_{ki}g_{hj} + P_{hi}g_{kj} - P_{hj}g_{ki} \\ &+ P_{kr}J_j^r J_{hi} - P_{kr}J_i^r J_{hj} + P_{hr}J_i^r J_{kj} - P_{hr}J_j^r J_{ki}].\end{aligned} \tag{24}$$

Hence we get

Theorem 3.1. *The Sato's form of the holomorphic curvature tensor of an l.c.K-manifold has the form* (24).

References

[1] T. Kashiwada, Some Properties of Locally Conformal Kaehler Manifolds, Hokkaido Math. J. 8 (1978) 191–198.
[2] K. Matsumoto, Locally Conformal Kaehler Manifolds And Their Submanifolds, Memoriile Secţiilor Ştiinţifice XIV (1991) 1–49.
[3] M. Prvanovic, On a curvature tensor of Kaehler type in an almost Hermitian and almost para-Hermitian manifold, Mat. Vesnik 50 No. 1-2 (1998) 57–64.
[4] M. Prvanovic, Some Properties of the Locally Conformal Kaehler Manifold, SRC PBull. Cl. Sci. Math. Nat. Sci. Math. 35 (2010) 9–23.
[5] T. Sato, On some almost Hermitian manifolds with constant holomorphic sectional curvature, Kyungpook Math. J. 29 No. 1 (1989) 11–25.
[6] I. Vaisman, On Locally Conformal Almost Kaehler Manifolds, Israel J. of Math. 24 (1976) 338–351.
[7] K. Yano, Differential Geometry On Complex And Almost Complex Spaces, Pergamon Press., 1965.